Global Domination and Packing Numbers

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Abstract

For a graph $G = (V, E)$, $X \subseteq V$ is a global dominating set if X dominates both G and the complement graph $\overline{G}$. A set $X \subseteq V$ is a packing if its pairwise members are distance at least 3 apart. The minimum number of vertices in any global dominating set is $\gamma_g(G)$, and the maximum number in any packing is $\rho(G)$. We establish relationships between these and other graphical invariants, and characterize graphs for which $\rho(G) = \rho(\overline{G})$. Except for the two self-complementary graphs on 5 vertices and when G or $\overline{G}$ has isolated vertices, we show $\gamma_g(G) \le \lfloor n/2 \rfloor$, where $n = |V|$.

1. Introduction

In a graph $G = (V, E)$, $X \subseteq W \subseteq V$ is said to dominate W when every vertex in W–X is adjacent to a vertex (a neighbor) in X. When W = V, we simply say X dominates G. A global dominating set is a set of vertices that dominates both G and the complement graph $\overline{G}$. The number of vertices in a smallest global dominating set is denoted by $\gamma_g(G)$. The following is an investigation of relationships between $\gamma_g(G)$ and the 2–packing number $\rho(G)$, described in the next section.

We adopt the following notation: the *order* of a graph G is $n = |V|$; $\delta(G)$ is the *minimum degree* of the vertices in V while $\Delta(G)$ is the *maximum degree*; diam(G) is the *diameter* and r(G) is the *radius*; $\gamma(G)$ is the *domination number*; and $\gamma_c(G)$ is the *connected domination number*. For any vertex $v \in V$, the *open neighborhood* of v in G is $N_G(v)$ and is the set of vertices adjacent to v, and $N_G[v] = N_G(v) \cup \{v\}$ is the *closed neighborhood* of v. The subscript in the neighborhood notation will be omitted unless referring specifically to a graph other than G. For example, $N_{\overline{G}}(v) = V - N[v]$ is the open neighborhood of v in the complement graph $\overline{G}$. For $W \subseteq V$, $N(W)$ and $N[W]$ are the unions of the open and closed neighborhoods, respectively, for every $v \in W$. Finally, K_n is the complete graph on $n \geq 1$ vertices, C_n is the cycle on $n \geq 3$ vertices, P_n is a path on $n \geq 2$ vertices, and H (a P_5 with an edge added between the two distance two, degree two, vertices) and C_5 are the two self–complementary graphs on five vertices.

2. Packings

For positive integer k, a k–packing of a graph G is a set of vertices that are pairwise distance at least k+1. An early reference is Meir and Moon [8]. The number of vertices in a maximum k–packing is the k–packing number of G and is denoted by $\rho_k(G)$. The 1–packing number $\rho_1(G)$ is also known as the independence number of G. The 2–packing number, $\rho_2(G)$, is the only packing invariant studied in this paper. Therefore, for notational simplicity, we will omit the subscript and simply refer to a 2–packing as a packing.

Packing number results have mainly focused on special classes of graphs. For example Meir and Moon [8] show $\rho(G) = \gamma(G)$ for trees. Several authors have studied packings in grid graphs, including Fisher [5], Hare and Hare [6], and Hartnell [7]. The following observations are straightforward.

Observations:

(a) If G has k connected components $G_1, G_2, \ldots, G_k$, then

$$\rho(G) = \rho(G_1)+\rho(G_2)+\ldots+\rho(G_k),$$

(b) $\rho(G) \leq \gamma(G)$, and

(c) $\rho(G) = 1$ if and only if $diam(G) \leq 2$.

Theorem 1. If G is connected, $\rho(G) \geq \lceil (diam(G)+1)/3 \rceil$.

Proof: Let x_0 and $x_{diam(G)}$ be maximum distance vertices and let $x_0, x_1, \ldots, x_{diam(G)}$ be the vertices on a shortest path joining them. Then, $\{x_{3i} \mid 0 \leq i \leq \lfloor diam(G)/3 \rfloor\}$ is a packing of G and has $\lceil (diam(G)+1)/3 \rceil$ vertices. ■

To relate the packing number and the global domination number, it is helpful to first obtain relationships between packings in G and $\overline{G}$.

Theorem 2. For any graph G and its complement $\overline{G}$,

(1) if $\rho(G) \geq 3$, then $\rho(\overline{G}) = 1$, and

(2) if $\rho(G) = 2$, then $\rho(\overline{G}) \leq 2$.

Proof: Let $X \subseteq V$ be a maximum packing of G. If $\{u, v, w\} \subseteq X$ then, in $\overline{G}$, every pair of vertices has at least one of u, v, or w as a common neighbor. Thus, $\overline{G}$ has no pair of distance three vertices and, hence, $\rho(\overline{G}) = 1$ by Observation (c).

For (2), assume $\rho(G) = 2$ and $\rho(\overline{G}) > 2$. Then, from (1), $\rho(G) = 1$. This contradiction shows $\rho(\overline{G}) \leq 2$. ■

A Nordhaus–Gaddum type result for the packing number easily follows.

Corollary 3. For any graph G,

$$\rho(G) + \rho(\overline{G}) = \begin{cases} 4 & \text{if } \rho(G) = \rho(\overline{G}) = 2 \\ \max\{\rho(G), \rho(\overline{G})\} + 1 & \text{otherwise} \end{cases}$$

Theorem 2, Theorem 4 (next) and Theorem 6 (later) will together characterize graphs for which $\rho(G) = \rho(\overline{G})$. From Theorem 2, it is sufficient to characterize graphs for which $\rho(G) = \rho(\overline{G}) \leq 2$.

Theorem 4. $\rho(G) = \rho(\overline{G}) = 1$ if and only if $\text{diam}(G) = \text{diam}(\overline{G}) = 2$ or $G = K_1$.

Proof: The claim holds for $G = K_1$. Therefore, we assume $n \geq 2$. From Observation (c), when $\rho(G) = \rho(\overline{G}) = 1$, $\text{diam}(G) \leq 2$ and $\text{diam}(\overline{G}) \leq 2$. We may assume $\text{diam}(G) \leq \text{diam}(\overline{G})$ and suppose $\text{diam}(G) = 1$. Then, G is complete and, since $G \neq K_1$, $\overline{G}$ is disconnected and $\rho(\overline{G}) = n \geq 2$, a contradiction. Therefore, $diam(G) = diam(\overline{G}) = 2$.

Conversely, when, say, $\rho(G) > 1$, G has two vertices distance at least three apart. Therefore, $diam(G) \geq 3$ and $G \neq K_1$. ■

An additional equivalent condition for graphs to satisfy $\rho(G) = \rho(\overline{G}) = 1$ will be given by Theorem 16 in the next section. We first examine properties of graphs with $\rho(G) \geq 2$. From Observation (c), $\rho(G) \geq 2$ if and only if $diam(G) \geq 3$. Lemma 5 provides another equivalent condition for graphs to have $\rho(G) \geq 2$

Lemma 5. $\rho(G) \geq 2$ if and only if $\gamma_c(\overline{G}) \leq 2$ and $G \neq K_1$.

Proof: When $\rho(G) \geq 2$, G can not be a K_1. Further, any two vertices of any non trivial packing of G is a connected dominating set of $\overline{G}$. Thus, $\gamma_c(\overline{G}) \leq 2$.

Next, suppose $\rho(G) = 1$. If $\gamma_c(\overline{G}) = 1$, then G has an isolated vertex and either $G = K_1$ or, by Observation (a), $\rho(G) \geq 2$, a contradiction. If $\gamma_c(\overline{G}) = 2$, G must have two non adjacent vertices x and y with no common neighbor. Thus, the distance between x and y is at least three and, hence, from Observation (c), $\rho(G) \geq 2$, again a contradiction. Therefore, either $\gamma_c(\overline{G}) \geq 3$ or $G = K_1$, and completes the proof. ■

It follows immediately from Lemma 5 and Observation (c) that

$$\gamma_c(\overline{G}) \geq 3 \text{ or } G = K_1 \text{ if and only if } diam(G) \leq 2.$$

A characterization of graphs for which $\rho(G) = \rho(\overline{G}) = 2$ is now possible.

Theorem 6. The following are equivalent statements for a graph G:

(1) $\rho(G) = \rho(\overline{G}) = 2$;

(2) $diam(G) = diam(\overline{G}) = 3$; and

(3) $\gamma_c(G) = \gamma_c(\overline{G}) = 2$.

Proof: (1) → (2). When $\rho(G) = \rho(\overline{G}) = 2$, Observation (c) shows that G and $\overline{G}$ each have distance 3 vertices. Thus, $diam(G) \geq 3$ and $diam(\overline{G}) \geq 3$. For any graph G, $diam(G) \geq 3$ implies $diam(\overline{G}) \leq 3$, thus, we have $diam(G) = diam(\overline{G}) = 3$.

(2) → (3). Let v and w be distance three vertices in any graph with diameter 3. Then, in in the complement graph, v and w form a connected dominating set. Therefore, $\gamma_c(G) \leq 2$ and $\gamma_c(\overline{G}) \leq 2$. If, say, $\gamma_c(G) = 1$, G must have a vertex of degree $n-1$. Then, $diam(G) \leq 2$ and contradicts (2). Hence, $\gamma_c(G) = \gamma_c(\overline{G}) = 2$.

(3) → (1). Since $\gamma_c(G) > 1$, $G \neq K_1$. Hence, by Lemma 5, $\rho(G) \geq 2$ and $\rho(\overline{G}) \geq 2$. Equality follows from Theorem 2. ■

As with Theorem 4, another condition equivalent to $\rho(G) = \rho(\overline{G}) = 2$ will be presented in Theorem 16. Theorems 2, 4, and 6 provide a characterization of graphs for which $\rho(G) = \rho(\overline{G})$.

Theorem 7. $\rho(G) = \rho(\overline{G})$ if and only if $diam(G) = diam(\overline{G})$.

It is interesting to note that when $G \neq K_1$, $\rho(G) = \rho(\overline{G})$ is equivalent to $diam(G) = diam(\overline{G}) = p(G)+1$.

3. Packings and Global Domination

Global domination was introduced by Sampathkumar [10], and independently by Brigham and Dutton [2] as a special case of factor domination of a graph G. The special case, when G is complete and the number of factors is two, is global domination. Further results on factor domination appear in Dankelman and Laskar [3]. A survey of global domination, as of 1998, was given by Brigham and Carrington [1]. Additional global domination results are given by Dutton and Brigham [4]. The following three theorems appear in [2] and also in the survey article [1].

Theorem A. For any graph G and its complement $\overline{G}$,

$$\max\{\gamma(G), \gamma(\overline{G})\} \le \gamma_g(G) = \gamma_g(\overline{G}) \le \gamma(G) + \gamma(\overline{G}).$$

Theorem B. If G and $\overline{G}$ are connected and $\max\{r(G), r(\overline{G})\} \ge 3$, then $\gamma_g(G) = \max\{\gamma(G), \gamma(\overline{G})\}$.

Theorem C. If G is triangle free, $\gamma(G) \le \gamma_g(G) \le \gamma(G)+1$.

The conditions for Theorem B are overly restrictive and a stronger result can be obtained. When G is disconnected, we stipulate that r(G) is infinite. Furthermore, Ore [9] shows that $\gamma(G) \le \lfloor n/2 \rfloor$ for any graph without isolated vertices. Hence, when G has k isolated vertices, $\gamma(G) \le \lfloor (n-k)/2 \rfloor + k = \lfloor (n+k)/2 \rfloor$.

Theorem 8. If $r(G) \ge 3$, then $2-\lceil k/n \rceil = \gamma(\overline{G}) = \gamma_c(\overline{G}) \le 2 \le \gamma_g(G) = \gamma(G) \le \lfloor (n+k)/2 \rfloor$, where k is the number of degree zero vertices in G.

Proof: When $r(G) \ge 3$, $G \ne K_1$ and $\rho(G) \ge 2$. Hence, by Lemma 5, $\gamma(\overline{G}) \le \gamma_c(\overline{G}) \le 2 \le \gamma(G)$. Since $\gamma(\overline{G}) = 1$ if and only if G has a degree zero vertex, it follows that $2-\lceil k/n \rceil = \gamma(\overline{G}) = \gamma_c(\overline{G}) \le 2$. Now, let D be any γ–set of G. If D does not dominate $\overline{G}$, then V–D contains a vertex x which dominates D in G. Then every vertex in V is distance at most two from x, contradicting the assumption $r(G) \ge 3$. Therefore, $\gamma_g(G) = \gamma(G)$. The upper bound on $\gamma(G)$ follows from the comments preceding the statement of the theorem. ■

Notice that a direct result of Theorem 8 is that if $\max\{r(G), r(\overline{G})\} \ge 3$, then $\gamma_g(G) = \max\{\gamma(G), \gamma(\overline{G})\}$, and supercedes Theorem B.

Three graphs, K_1, H, and C_5 merit special attention. For graphs G in this group, it is easily checked that $\gamma_g(G) = \lceil n/2 \rceil > \lfloor n/2 \rfloor$. We show in the remainder of this section that these, along with graphs in which G or $\overline{G}$ has a sufficiently large number of isolated vertices as covered in Theorem 8, are the only graphs for which $\gamma_g(G) > \lfloor n/2 \rfloor$. The following is straightforward.

Theorem 9. $\gamma_g(G) = 1$ if and only if $G = K_1$.

A set $S \subseteq V$ is a *perfect* dominating set if every vertex in V–S has exactly one neighbor in S. If S also is independent, S is a packing and $\rho(G) = |S|$.

Theorem 10. $\gamma_g(G) = 2$ if and only if G has a perfect dominating set of two vertices.

Proof: Let X = {v, w} be a γ_g–set of G. Since X dominates both G and $\overline{G}$, v and w can have no common neighbors in G. That is, X is a perfect dominating set of G.

Now, suppose X = {v, w} is a perfect dominating set of G. Then, X also dominates $\overline{G}$. Therefore, $\gamma_g(G) \leq 2$. Since $G \neq K_1$, $\gamma_g(G) = 2$, by Theorem 9. ■

The set X in the proof of Theorem 10 is a packing for one of G or $\overline{G}$, and a connected dominating set for the other. The next three lemmas establish a stronger relationship between global dominating sets and packings.

Lemma 11. For any graph G = (V, E) and any vertex $v \in V$, either N[v] or V–N(v) is a global dominating set.

Proof: Since N[v] always dominates $\overline{G}$ and V–N(v) always dominates G, assume, by way of contradiction, that (1) N[v] does not dominate G and (2) V–N(v) does not dominate $\overline{G}$. Then, from (1), there exists a vertex $w \in$ V–N[v] for which $N[w] \subseteq$ V–N[v] and, from (2), a vertex $u \in N(v)$ for which $N(u) \supseteq$ V–N(v). Therefore, (2) implies u and w must be adjacent, while (1) implies u and w are not adjacent, a contradiction that establishes the result. ■

Lemma 12. For any graph G with $\delta(G) \geq 1$ and any maximal packing X, N(X) $\cup$ {v} is a global dominating set of G, for any $v \in X$.

Proof: Assume G is a graph with $\delta(G) \geq 1$ and a maximal packing X. Let Z = N(X) $\cup$ {v}, for any $v \in X$. Then, for every vertex $w \in V-Z$, w is either in X–{v} or is distance 2 from some vertex in X. In either case, w has a neighbor in N(X) and is, thus, dominated in G by Z. In $\overline{G}$, w is dominated by v. It follows that Z is a global dominating set. ■

Lemma 13. For any graph G with $\rho(G) \geq 2$ and any maximum packing X, V–N(X) is a global dominating set of G.

Proof: In G, $w \in N(X)$ is dominated by exactly one vertex in $X \subseteq V-N(X)$. Thus, V–N(X) dominates G. If $\rho(G) \geq 2$, w is not adjacent to $\rho(G)-1 \geq 1$ vertices in X. Thus, w is dominated in $\overline{G}$ and V–N(X) is a global dominating set. ■

Lemma 12 implies, when $\rho(G) = 1$, that N[v] is a global dominating set for every $v \in V$. Theorem 14 shows this is also a sufficient condition for $\rho(G) = 1$.

Theorem 14. $\rho(G) = 1$ if and only if N[v] is a global dominating set for every $v \in V$.

Proof: The theorem holds for $G = K_1$. Otherwise, if $\rho(G) = 1$, then $\delta(G) \geq 1$, and the conclusion follows from Lemma 12. When $\rho(G) \geq 2$, there are two vertices v and w that are at least distance 3 apart. Then, N[v] can not dominate w. Hence, N[v] can not be a global dominating set of G. ■

Corollary 15. If $\rho(G) = 1$, then $\gamma_g(G) \leq \delta(G)+1$. If, further, $\rho(G) = \rho(\overline{G}) = 1$, then $\gamma_g(G) \leq \min\{\delta(G), \delta(\overline{G})\}+1$.

When X is any maximum packing, N[X], and hence V, contains at least $\rho(G)(\delta(G)+1)$ vertices. If $\rho(G) > 1$, for every $v \in X$, N[v] can not dominate X and, therefore, can not be a global dominating set of G. Thus, there are at least $\rho(G)$ vertices v for which N[v] is not a global dominating set. These comments are the basis of the following.

Theorem 16. Let M and M' be the sets of vertices for which N[v] and V–N(v), respectively, are not global dominating sets. Assume, without loss of generality, that $0 \le m' = |M'| \le m = |M|$. Then, M and M' are disjoint, and either

(a) $\rho(G) = \rho(\overline{G}) = 1$ and $m' = m = 0$, or

(b) $\rho(G) = \rho(\overline{G}) = 2 \le m' \le m$, or

(c) $m' = 0$, $\rho(\overline{G}) = 1 < \rho(G) \le \min\{m, \lfloor n/(\delta(G)+1) \rfloor\}$.

Proof: The sets M and M' are disjoint by Lemma 11. Part (a) follows immediately from Theorem 14, which also shows, for Part (b), that $\rho(G) \ge 2$ and $\rho(\overline{G}) \ge 2$. Equality holds by Theorem 2. Notice that neither m nor m' can equal one, since the existence of one vertex v for which N[v], or V–N(v), is not a global dominating set implies the existence of another. Finally, for Part (c), Theorem 14 again shows that, if $m' = 0$, $\rho(\overline{G}) = 1$, and when $m > 0$, $\rho(G) > 1$. The upper bound on $\rho(G)$ follows from the comments preceding the statement of Theorem 16. ■

The sets M and M' can be determined easily in polynomial time, for example by computing the distance matrices for G and $\overline{G}$. This will decide the packing number for at least one of G or $\overline{G}$. The other, say G, is also determined if $m = 2$ or $\delta(G) \ge \lceil (n-2)/3 \rceil$. Case (b) can also be confirmed by the existence of any two vertices v and w where neither N[v] nor V–N(w) is a global dominating set, since that eliminates cases (a) and (c).

The upper bound m in Theorem 16 (c) can be replaced by the packing number of the subgraph of G induced by the set M. That is, $\rho(G) \le \rho(\langle M \rangle)$.

Interestingly, when m = n, N[v] is not a global dominating set for any $v \in V$. Thus, $r(G) \geq 3$ and, by Theorem 8, $\gamma_g(G) = \gamma(G)$.

Theorem 17. If $\rho(G) = \rho(\overline{G}) = 1$, then $\gamma_g(G) \leq \lfloor n/2 \rfloor$ or $G \in \{K_1, C_5\}$.

Proof: It is easily checked, when $G \in \{K_1, C_5\}$, that $\rho(G) = \rho(\overline{G}) = 1$ and $\gamma_g(G) = \lceil n/2 \rceil > \lfloor n/2 \rfloor$. Thus, in the following, we may assume $G \notin \{K_1, C_5\}$ and that both G and $\overline{G}$ are connected. From Corollary 15, it follows that $\gamma_g(G)-1 \leq \delta(G) \leq \Delta(G) \leq n-\gamma_g(G)$. Therefore, $\gamma_g(G) \leq \lfloor (n+1)/2 \rfloor = \lceil n/2 \rceil$ and the conclusion follows when n is even.

Now, suppose n is odd and $\gamma_g(G) = \lceil n/2 \rceil$. Then, from the chain of inequalities in the last paragraph, G and $\overline{G}$ are both $\lfloor n/2 \rfloor$–regular. Hence, by Theorem 16, for any vertex $x \in V$, $D = N[x]$ is a global dominating set. Since G is $\lfloor n/2 \rfloor$–regular, $|D| = \lceil n/2 \rceil$ and it follows that D is a γ_g–set.

Suppose there is a vertex $v \in N(x)$ with a private neighbor w in V–D. Then, w must dominate V–D and v can not dominate D, since v has at least one neighbor in V–D. Thus, x and w dominate G, and {x, v, w} dominates $\overline{G}$. Hence, {x, v, w} is a global dominating set. That is, $\lfloor n/2 \rfloor < \gamma_g(G) \leq 3$ and, hence, $n \in \{1, 3, 5\}$. Since $G \neq K_1$ and all graphs on 3 vertices have either G or $\overline{G}$ disconnected, we must have n = 5. Then, since G is 2–regular, $G = C_5$, a contradiction. Therefore, since G is connected, there must be a vertex $v \in N(x)$ that has neighbors in V–D, but no private neighbors. Then, D–{v} is a global dominating set with $\lceil n/2 \rceil - 1 < \gamma_g(G)$ vertices, a contradiction that completes the proof. ■

We now assume at least one of $\rho(G)$ or $\rho(\overline{G})$ is at least two. When $\rho(G) = \rho(\overline{G}) = 2$, Theorem 6 shows $\gamma_c(G) = \gamma_c(\overline{G}) = 2$. Therefore, since $\gamma(G) \leq \gamma_c(G)$ and $\gamma_g(G) \leq \gamma(G)+\gamma(\overline{G})$, the following holds.

Corollary 18. If $\rho(G) = \rho(\overline{G}) = 2$, then $\gamma_g(G) \leq 4$.

Theorem 19. If $\rho(G) \geq 2$ and $\delta(G) \geq 1$, then either $\gamma_g(G) \leq \lfloor n/2 \rfloor$ or $G = H$.

Proof: Let G be a graph for which $\rho(G) \geq 2$ and $\delta(G) \geq 1$. When $G = H$, it is easily checked that $\rho(H) = 2$, $\delta(H) = 1$ and $\gamma_g(H) = \lceil n/2 \rceil > \lfloor n/2 \rfloor$. Therefore, we may assume $G \neq H$ and, from Theorem 8, that G is connected. Suppose X is any maximum packing of G. From Lemmas 12 and 13, $\gamma_g(G) \leq \min\{1+|N(X)|, n-|N(X)|\}$. It follows that $\gamma_g(G) \leq 1+|N(X)| \leq 1+n-\gamma_g(G)$. Thus, $\gamma_g(G) \leq \lceil n/2 \rceil$, and the conclusion holds when n is even.

Assume n is odd and $\lfloor n/2 \rfloor < \gamma_g(G) = \lceil n/2 \rceil$. Since, $\gamma_g(G)-1 \leq |N(X)| \leq n-\gamma_g(G)$, $|N(X)| = \lfloor n/2 \rfloor$. Thus, N(X), which dominates G, can not dominate $\overline{G}$, since $|N(X)| < \gamma_g(G)$. It follows that there is a vertex z in V–N[X] that dominates N(X).

The set X contains $k \geq 0$ degree one vertices that, if $k > 0$, are labeled $x_1, x_2, \ldots, x_k$ with corresponding neighbors in N(X) labeled $y_1, y_2, \ldots, y_k$. The remaining vertices in N(X) are labeled $y_{k+1}, y_{k+2}, \ldots, y_{\lfloor n/2 \rfloor}$. Notice that $k < \rho(G)$ if and only if $\rho(G) < \lfloor n/2 \rfloor$. Suppose $k < \rho(G)$. Then, for any y_i, $k+1 \leq i \leq \lfloor n/2 \rfloor$, let x be its neighbor in X, and x' any member of X other than x. If y_i has a neighbor in N(X), let $D = N(X)-\{y_i\}+\{x'\}$, otherwise let $D = N(X)-\{y_i\}+\{x\}$. In either case, D is a dominating set of $\overline{G}$. Thus, since $|D| = \lfloor n/2 \rfloor < \gamma_g(G)$, D can not dominate G. Hence, for $k+1 \leq i \leq \lfloor n/2 \rfloor$, y_i must have at least one private neighbor in V–N[X]. Thus, V–N[X] must have at least $\lfloor n/2 \rfloor - k$ vertices that are private neighbors plus z which is not the private neighbor of any vertex in N(X). That is, since $|V-N[X]| = \lceil n/2 \rceil - \rho(G)$, $1+\lfloor n/2 \rfloor - k \leq \lceil n/2 \rceil - \rho(G)$, or $\rho(G) \leq k$, and contradicts the assumption that $k < \rho(G)$. Hence, $k = \rho(G) = \lfloor n/2 \rfloor$. Then, X must consist of $\lfloor n/2 \rfloor$ degree one vertices, each with a unique neighbor in N(X), and $V-N[X] = \{z\}$. Suppose there are non adjacent vertices y and y' in N(X). Let x and x' be their respective neighbors in X. Then, $X-\{x\}+\{y\}$ is a global dominating set, a contradiction, since this set has $\lfloor n/2 \rfloor < \gamma_g(G)$ vertices. Thus, V–X is complete with $\lceil n/2 \rceil$ vertices.

Therefore, G consists of a $K_{\lceil n/2 \rceil}$ and $\lfloor n/2 \rfloor$ degree one vertices with each having a unique neighbor in the $K_{\lceil n/2 \rceil}$. If $\lfloor n/2 \rfloor \geq 3$, $X-\{x_1\}+\{y_1\}$ is a global dominating set of $\lfloor n/2 \rfloor$ vertices, a contradiction. Hence, we must have $n \leq 5$. That is, $G = H$. ■

Lemma 20. If $\rho(G) \neq \rho(\overline{G})$, then $\gamma_g(G) \leq \max\{\gamma(G), \gamma(\overline{G})\}+2$.

Proof: We may assume, without lose of generality, that $\rho(G) \geq 2$. Then, from Theorem 2 and the assumption that $\rho(G) \neq \rho(\overline{G})$, $\rho(\overline{G}) = 1$. From Lemma 5, $\gamma_c(\overline{G}) \leq 2$. Therefore, from Theorem A and the fact that $\gamma(\overline{G}) \leq \gamma_c(\overline{G})$, $\gamma_g(G) \leq \gamma(G)+\gamma(\overline{G}) \leq \max\{\gamma(G), \gamma(\overline{G})\}+2$. ■

4. Conclusion

In summary, we have the following result.

Theorem 21. For any graph G,

	$= 1$	$G = K_1$,
	$= 3$	$G \in \{C_5, H\}$,
	$= \max\{\gamma(G), \gamma(\overline{G})\} \leq \lfloor (n+k)/2 \rfloor$	$\max\{r(G), r(\overline{G})\} \geq 3$, k is the number of isolated vertices in G or $\overline{G}$,
$\gamma_g(G) \leq$	$\min\{\max\{\gamma(G), \gamma(\overline{G})\}+1, \lfloor n/2 \rfloor\}$	$\rho(G) \neq \rho(\overline{G})$, $r(G) = r(\overline{G}) = 2$, and one of G or $\overline{G}$ is triangle-free,
	$\leq \min\{\max\{\gamma(G), \gamma(\overline{G})\}+2, \lfloor n/2 \rfloor\}$	$\rho(G) \neq \rho(\overline{G})$, $r(G) = r(\overline{G}) = 2$, and neither is triangle-free,
	$\leq \min\{4, \lfloor n/2 \rfloor\}$	$\rho(G) = \rho(\overline{G}) = 2$, $G \neq H$,
	$\leq \min\{\delta(G)+1, \delta(\overline{G})+1, \lfloor n/2 \rfloor\}$	$\rho(G) = \rho(\overline{G}) = 1$, and $G \notin \{K_1, C_5\}$.

Proof: We treat each bound in turn and refer to them as line 1, line 2, etc. Lines 1 and 2 are easily checked. Line 3 follows from Theorem 8. Line 4 follows from Theorem C and Theorem 19, since $G \neq H$. Line 5 follows from Lemma 20, since $\rho(G) \neq \rho(\overline{G})$, and Theorem 19, since $G \neq H$. Line 6 follows from Corollary 18, Theorem 19, and the fact that $G \neq H$. Finally, Line 7 follows from Corollary 15, Theorem 17, and the assumption that $G \notin \{K_1, C_5\}$. ■

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