Note on the Rainbow k-Connectivity of Regular Complete Bipartite Graphs*

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Abstract

A path in an edge-colored graph G, where adjacent edges may be colored the same, is called a rainbow path if no two edges of the path are colored the same. For a κ-connected graph G and an integer k with $1 \leq k \leq \kappa$, the rainbow k-connectivity $rc_k(G)$ of G is defined as the minimum integer j for which there exists a j-edge-coloring of G such that any two distinct vertices of G are connected by k internally disjoint rainbow paths. Denote by $K_{r,r}$ an r-regular complete bipartite graph. Chartrand et al. in "G. Chartrand, G.L. Johns, K.A. McKeon, P. Zhang, The rainbow connectivity of a graph, Networks 54(2009), 75-81" left an open question of determining an integer $g(k)$ for which the rainbow k-connectivity of $K_{r,r}$ is 3 for every integer $r \geq g(k)$. This short note is to solve this question by showing that $rc_k(K_{r,r}) = 3$ for every integer $r \geq 2k\lceil \frac{k}{2} \rceil$, where $k \geq 2$ is a positive integer.

Keywords: edge-colored graph, rainbow path, rainbow k-connectivity, regular complete bipartite graph

AMS Subject Classification 2000: 05C15, 05C40

All graphs considered in this paper are simple, finite and undirected. Let G be a nontrivial connected graph with an edge coloring $c : E(G) \rightarrow \{1, 2, \cdots, k\}$, $k \in \mathbb{N}$, where adjacent edges may be colored the same. A path of G is called *rainbow* if no two edges of it are colored the same. A well-known result shows that in every κ-connected graph G with $\kappa \geq 1$, there are k internally disjoint $u - v$ paths connecting any two distinct vertices u and v for every integer k with $1 \leq k \leq \kappa$. Chartrand et al. [2] defined the *rainbow k-connectivity* $rc_k(G)$ of G, which is the minimum integer j for which there exists a j-edge-coloring of G such that for any two distinct vertices u and v of G, there exist at least k internally disjoint $u - v$ rainbow paths.

The concept of rainbow k-connectivity has applications in transferring information of high security in communication networks. For details we refer to [2] and [3].

In [2], Chartrand et al. studied the rainbow k-connectivity of the complete graph K_n for various pairs k, n of integers. It was shown in [2] that for every integer $k \geq 2$, there exists an integer $f(k)$ such that $rc_k(K_n) = 2$ for every integer $n \geq f(k)$. In [4], we improved the upper bound of $f(k)$ from $(k + 1)^2$ to $ck^{\frac{3}{2}} + C$ (here $0 < c < 1$ and $C = o(k^{\frac{3}{2}})$), i.e., from $O(k^2)$ to $O(k^{\frac{3}{2}})$. Chartrand et al. in [2] also investigated the rainbow k-connectivity of r-regular complete bipartite graphs for some pairs k, r of integers with $2 \leq k \leq r$, and they obtained the following results.

Proposition 1. For each integer $r \geq 2$,

$$rc_2(K_{r,r}) = \begin{cases} 4 & \text{if } r = 2 \\ 3 & \text{if } r \geq 3. \end{cases}$$

Proposition 2. For each integer $r \geq 3$, $rc_3(K_{r,r}) = 3$.

Theorem 3. For every integer $k \geq 2$, there exists an integer r such that $rc_k(K_{r,r}) = 3$.

Moreover, they showed that $r = 2k\lceil \frac{k}{2} \rceil$ is a desired integer for Theorem 3. However, they could not show a similar result as for complete graphs, and therefore they left an open question: For every integer $k \geq 2$, determine an integer (function) $g(k)$, for which $rc_k(K_{r,r}) = 3$ for every integer $r \geq g(k)$, that is, the rainbow k-connectivity of the complete bipartite graph $K_{r,r}$ is essentially 3. This short note is to solve this question by showing that $rc_k(K_{r,r}) = 3$ for every integer $r \geq 2k\lceil \frac{k}{2} \rceil$. We use a method similar to but more complicated than the proof of Theorem 3 in [2]. For notation and terminology not defined here, we refer to [1].

Theorem 4. For every integer $k \geq 2$, there exists an integer $g(k)$ such that $rc_k(K_{r,r}) = 3$ for any $r \geq g(k)$.

Proof. Let $g(k) = 2k\lceil \frac{k}{2} \rceil$. We will show that $rc_k(K_{r,r}) = 3$ for every $k \geq 2$, where $r \geq 2k\lceil \frac{k}{2} \rceil$ is an integer. By Propositions 1 and 2, we know that the conclusion holds for $k = 2, 3$. So we assume $k \geq 4$.

We first assume that k is even. Then, $g(k) = 2k \cdot \frac{k}{2}$. Since $r \geq g(k)$, then $r = k_1 \cdot (2k) + r_1$, where $k_1 \geq \frac{k}{2}, 1 \leq r_1 \leq 2k - 1$. Let the bipartite sets of $G = K_{r,r} = K_{k_1 \cdot (2k) + r_1, k_1 \cdot (2k) + r_1}$ be U and W. Let U', W' be the set of first $k_1 \cdot (2k)$ vertices of U, W, respectively. $U \setminus U' = \{u_1, \ldots, u_{r_1}\}$ and $W \setminus W' = \{w_1, \ldots, w_{r_1}\}$. Suppose that

$$U' = U'_1 \cup \ldots \cup U'_{2k}, W' = W'_1 \cup \ldots \cup W'_{2k},$$

where $U'_i = \{u_{i,1}, \ldots, u_{i,k_1}\}$ and $W'_i = \{w_{j,1}, \ldots, w_{j,k_1}\}$ for $1 \leq i, j \leq 2k$. Let G' be an induced subgraph of G with bipartite sets U' and W'. Suppose that

$$U = U_1 \cup \ldots \cup U_{2k}, W = W_1 \cup \ldots \cup W_{2k},$$

where $U_i = U'_i \cup \{u_i\}$, $W_j = W'_j \cup \{w_j\}$ for $1 \leq i, j \leq r_1$ and $U_i = U'_i$, $W_j = W'_j$ for $r_1 + 1 \leq i, j \leq 2k$.

We now give G a 3-edge coloring as follows: Let G'_1 be the spanning subgraph of G' such that $E(G'_1) = \{u_{i,p} w_{j,p} : 1 \leq i, j \leq 2k, 1 \leq p \leq k_1, i$ and j are of the same parity$\}$. Let G_1 be the spanning subgraph of G such that $E(G_1) = E(G'_1) \cup \{u_i w_j : 1 \leq i, j \leq r_1, i$ and j are of the same parity$\}$. Let G_2 be the spanning of subgraph of G such that

$$G_2 = H_1 \cup \ldots \cup H_{2k},$$

where H_1 has bipartite sets U_1 and W_{2k}, H_i $(2 \leq i \leq 2k)$ has bipartite sets U_i and W_{i-1}. So, $H_i = K_{m,n}(\{m, n\} = \{k_1, k_1 + 1\})$. See Figure 0.1 for the case $r = 18$, $k = 4$, $r_1 = 2$. Finally, let

$$G_3 = G - (E(G_1) \cup E(G_2)).$$

Assign each edge of $G_i (1 \leq i \leq 3)$ the color i.

Next we will show that the above edge-coloring is a k-rainbow coloring, that is, there are at least k internally disjoint rainbow paths connecting any two distinct vertices u, v of G. We will consider the following two cases:

Case 1. $u \in V(G')$. Without loss of generality, let $u = u_{1,1}$.

Subcase 1.1. u and v belong to the same bipartite set of G.

Subsubcase 1.1.1. $v \in U_1$. Then G contains the k internally disjoint $u_{1,1} - v$ rainbow paths $u_{1,1}, w_{i,1}, v$ where $1 \leq i \leq 2k - 1$ and i is odd.
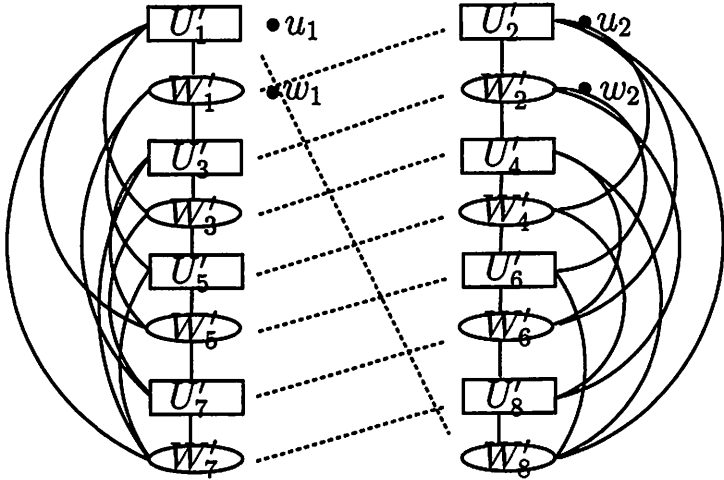
Figure 0.1 The figure for the case $r = 18$, $k = 4$, $r_1 = 2$.

Subsubcase 1.1.2. $v \in U_i$, $3 \le i \le 2k - 1$, and i is odd, say $v \in U_3$. Then G contains the $2k_1 \ge k$ internally disjoint $u - v$ rainbow paths $u_{1,1}, w_{2,j}, v$ and $u_{1,1}, w_{2k,j}, v$, where $1 \le j \le k_1$.

Subsubcase 1.1.3. $v \in U_i$, $2 \le i \le 2k$, and i is even, say $v \in U_2$. Then G contains the $2k_1 \ge k$ internally disjoint $u - v$ rainbow paths $u_{1,1}, w_{1,j}, v$ and $u_{1,1}, w_{2k,j}, v$, where $1 \le j \le k_1$.

Subcase 1.2. u and v belong to different bipartite sets, and so $v \in W$.

Subsubcase 1.2.1. $v \in W_i$, where $1 \le i \le 2k - 1$ and i is odd, say $v \in W_1$. Then G contains the $2k_1 \ge k$ internally disjoint $u - v$ rainbow paths $u_{1,1}, w_{2,j}, u_{2,j}, v$ and $u_{1,1}, w_{2k,j}, u_{2k,j}, v$, where $1 \le j \le k_1$.

Subsubcase 1.2.2. $v \in W_i$, where $2 \le i \le 2k$ and i is even, say $v \in W_2$. If $v \in W_2'$, without loss of generality, let $v = w_{2,1}$, then G contains the $u_{1,1} - v$ path $u_{1,1}, v$ together with the $u_{1,1} - v$ rainbow paths $u_{1,1}, w_{3,j}, u_{3,j}, v$; $u_{1,1}, w_{3,1}, u_{4,j}, v$ and $u_{1,1}, w_{2k,j}, u_{2k,j}, v$, where $2 \le j \le k_1$. The cases for $v = w_2$ and $v \in W_{2k}$ are similar.

Case 2. $u \in V(G) \setminus V(G')$, that is, $u \in \{u_1, \ldots, u_{r_1}; w_1, \ldots, w_{r_1}\}$. Without loss of generality, let $u = u_1$. By Case 1, we only need to show that there are at least k internally disjoint rainbow paths connecting u and v for every $v \in V(G) \setminus V(G')$.

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Subcase 2.1. u and v belong to the same bipartite set of G.

Subsubcase 2.1.1. $v = u_i$, $3 \leq i \leq 2k - 1$ and i is odd, say $v = u_3$. Then G contains the $2k_1 \geq k$ internally disjoint $u - v$ rainbow paths $u_1, w_{2,j}, u_3$ and $u_1, w_{2k,j}, u_3$, where $1 \leq j \leq k_1$.

Subsubcase 2.1.2. $v = u_i$, $2 \leq i \leq 2k$ and i is even, say $v = u_2$. Then G contains the $2k_1 \geq k$ internally disjoint $u - v$ rainbow paths $u_1, w_{1,j}, u_2$ and $u_1, w_{2k,j}, u_2$, where $1 \leq j \leq k_1$.

Subcase 2.2. u and v belong to different bipartite sets of G.

Subsubcase 2.2.1. $v = w_i$, $1 \leq i \leq 2k - 1$ and i is odd, say $v = w_1$. Then G contains the $2k_1 \geq k$ internally disjoint $u - v$ rainbow paths $u_1, w_{2,j}, u_{2,j}, w_1$ and $u_1, w_{2k,j}, u_{2k,j}, w_1$ where $1 \leq j \leq k_1$.

Subsubcase 2.2.2. $v = w_i$, $2 \leq i \leq 2k$ and i is even, say $v = w_2$. Then G contains the $2k_1 \geq k$ internally disjoint $u - v$ rainbow paths $u_1, w_{1,j}, u_{3,j}, w_2$ and $u_1, w_{2k,j}, u_{2k,j}, w_2$ where $1 \leq j \leq k_1$.

So the conclusion holds for the case that k is even.

Next we assume that k is odd. Then $g(k) = 2k \cdot \frac{k+1}{2}$. Since $r \geq g(k)$, then $r = k_2 \cdot (2k) + r_2$, where $k_2 \geq \frac{k+1}{2}, 1 \leq r_2 \leq 2k-1$. Then with a similar argument to the case that k is even, we can show that the conclusion also holds when k is odd. ∎

Remark 2.5. In [4] we showed that for every pair of integers $k \geq 2$ and $r \geq 1$, there is an integer $f(k, r)$ such that if $\ell \geq f(k, r)$, then the rainbow k-connectivity of an r-regular complete ℓ-partite graph is 2, where r-regular means that every partite set has the same number r of elements. That is, for sufficiently many number ℓ of partite sets, the rainbow k-connectivity of an r-regular complete ℓ-partite graph is 2. Theorem 4 of this note implies that for sufficiently large size r of every partite set, the rainbow k-connectivity of an r-regular complete ℓ-partite graph is at most 3. So, an interesting question is to think about the question of determining some bounds on k, r, ℓ that tell us the rainbow k-connectivity of an r-regular complete ℓ-partite graph is 2 or 3.

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