On chromatic number of graphs without certain induced subgraphs

Fang Duan and Baoyindureng Wu*
College of Mathematic and System Sciences, Xinjiang University,
Urumqi, Xinjiang 830046, P. R. China
Email: baoyin@xju.edu.cn
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Abstract

Gyárfás conjectured that for a given forest F, there exists an integer function $f(F, \omega(G))$ such that $\chi(G) \leq f(F, \omega(G))$ for any F-free graph G, where $\chi(G)$ and $\omega(G)$ are respectively, the chromatic number and the clique number of G. Let G be a C_5-free graph and k be a positive integer. We show that if G is $(kP_1 + P_2)$-free for $k \geq 2$, then $\chi(G) \leq 2\omega^{k-1}\sqrt{\omega}$; if G is $(kP_1 + P_3)$-free for $k \geq 1$, then $\chi(G) \leq \omega^k\sqrt{\omega}$. A graph G is k-divisible if for each induced subgraph H of G with at least one edge, there is a partition of the vertex set of H into k sets $V_1, \cdots, V_k$ such that no V_i contains a clique of size $\omega(G)$. We show that a $(2P_1+P_2)$-free and C_5-free graph is 2-divisible.

Keywords: F-free graph, Perfect graph, Divisibility.

1 Introduction

All graphs considered here are finite, undirected and simple. We refer to [1] for unexplained terminology and notation. Let $G = (V(G), E(G))$ be a graph, and let S be a nonempty subset of $V(G)$. The subgraph of G induced by S, denoted $G[S]$, is the subgraph of G with vertex set S, in

*Corresponding author.

which two vertices are adjacent if and only if they are adjacent in G. If $G[S]$ has no edge, S is called an independent set; if $G[S]$ is a complete graph, then S is called a clique. The maximum cardinality of an independent set is called the independence number of G, denoted $\alpha(G)$; the maximum cardinality of a clique is called the clique number of G, denoted $\omega(G)$. The chromatic number of G, denoted $\chi(G)$, is the minimum number k such that the vertices of G can be partitioned into k independent sets. In general, there is no upper bound on the chromatic number of a graph in terms of its clique number, since there are graphs containing no triangle, but having arbitrarily large chromatic number.

A graph G is called perfect if $\chi(H) = \omega(H)$ for each induced subgraph H of G. There are graph classes which can be characterized by forbidden induced subgraphs, e.g. cographs (i.e. P_4-free graphs), chordal graphs, split graphs, threshold graphs. Berge conjectured that a graph G is perfect if and only if neither G nor its complement $\overline{G}$ contains an induced odd cycle of order at least five. This famous conjecture, known as Strong Perfect Graph Conjecture, has recently been solved by Chudnovsky, Robertson, Seymour and Thomas [2].

Gyárfás [4] has introduced the concept of χ-bound functions. Here, a family $\mathbf{G}$ of graphs is called χ-bound with χ-binding function f, if $\chi(H) \leq f(\omega(H))$ holds whenever H is an induced subgraph of $G \in \mathbf{G}$. For a given graph F, a graph G is F-free if it does not contain an induced subgraph isomorphic to F. Gyárfás proposed a conjecture: if F is a forest, there exist an integer $f(F, \omega)$ such that every F-free graph with maximum clique size ω is $f(F, \omega)$-colorable. This conjecture is only proved in special cases.

Hoàng and McDiarmid [5] recently introduced the notion of k-divisible graphs. A k-division of a graph $G = (V, E)$ is a partition of the vertex set V into k sets $V_1, \cdots, V_k$ such that no V_i contains a clique of size $\omega(G)$. A graph G is k-divisible if each induced subgraph of G with at least one edge has a k-division. The least such k is the divisibility number $div(G)$. A strong k-division of a graph G is a partition of the vertex set V into k sets $V_1, \cdots, V_k$ such that no V_i contains a maximal clique of G. We shall say that a graph is strongly k-divisible if each induced subgraph with no isolated vertices has a strong k-division. Obviously, every strongly k-divisible graph is k-divisible.

We denote the path on k vertices by P_k. The graph with vertices a, b, c, d

and edges ab,ac will be called a co-paw. The graph with vertices a, b, c, d and edges ab and cd is called $2K_2$. The graph with vertices a, b, c, d and edge ab will be called a $2P_1 + P_2$. Wagon [6] proved that for any $2K_2$-free graph G, $\chi(G) \leq \frac{1}{2}(\omega(G)+1)\omega(G)$. Hoàng and McDiarmid [5] showed that for any C_5-free and co-paw free graph G, $\chi(G) \leq \omega(G)^{\frac{3}{2}}$.

Let G be a C_5-free graph and k be a positive integer. We show that if G is $(kP_1 + P_2)$-free for $k \geq 2$, then $\chi(G) \leq 2\omega^{k-1}\sqrt{\omega}$; if G is $(kP_1 + P_3)$-free for $k \geq 1$, then $\chi(G) \leq \omega^k\sqrt{\omega}$. Gravier, Hoàng and Maffray [3] showed that any $(2P_1 + P_2)$-free graph is 3-divisible. But we are interested in the 2-divisible case. Accordingly, we show that a $(2P_1 + P_2)$-free and C_5-free graph is 2-divisible in this paper.

2 Colorability

Let $R(p, q)$ be the Ramsey function, that is the smallest $m = m(p, q)$ such that all graphs on m vertices contain either an independent set of p vertices or a clique of q vertices. It was pointed out in [4] that for a $(2P_1 + P_2)$-free graph G, $\chi(G) \geq \frac{R(3,\omega+1)-1}{2}$. Accordingly, $(2P_1 + P_2)$-free graphs have not linear χ-binding function.

Let G be a graph. For a vertex $x \in V(G)$, $N(x)$ denote the set of neighbors of x and $M(x) = V(G) \setminus (\{x\} \cup N(x))$.

Theorem 2.1. *Suppose a graph G contains no induced 5-cycle and no induced $2P_1 + P_2$. Then*

(1) if $\omega(G) = 2$, $\chi(G) = \omega(G)$,

(2) if $\omega(G) \geq 3$, $\chi(G) \leq 2\omega(G)^{\frac{3}{2}}$.

Proof. We can assume that G is connected. First, we prove (1). Since $\omega(G) = 2$, G contains no triangle. Note that the existence of an induced odd cycle of length greater than five would imply that of $2P_1 + P_2$. Combining these to the assumption that G has no odd cycle of length five, G is bipartite, and thus $\chi(G) = 2$.

We shall prove (2) by induction on the number of vertices of G. Suppose there is a function $g(\omega)$ such that $\chi(H) \leq g(\omega(H))$ for every proper induced subgraph H of G. The function g will be defined later.

We may assume that $\alpha \geq 2$, for otherwise the theorem holds trivially. If for every edge $xy \in E(G)$, $N(x)\setminus(N(y)\cup\{y\}) = \emptyset$, $N(y)\setminus(N(x)\cup\{x\}) = \emptyset$

and $M(xy) = \emptyset$, then G must be a complete graph. If it is not, then there are two non-adjacent vertices, say u and v, in G. Since G is connected, we can take a neighbor, say x, of u. We consider the edge xu. If $xv \in E(G)$, then $N(x) \setminus (N(u) \cup \{u\}) \neq \emptyset$ since it contains v. If $xv \notin E(G)$, then $M(xu) \neq \emptyset$ since it contains v. A contradiction. So in this case, the result trivially holds.

Now assume that there is an edge xy in G such that at least one of $N(x) \setminus (N(y) \cup \{y\})$, $N(y) \setminus (N(x) \cup \{x\})$ and $M(xy)$ is not empty. If $(N(x) \setminus (N(y) \cup \{y\})) \neq \emptyset$, put $A = (N(x) \setminus (N(y) \cup \{y\})) \cup M(xy)$ and $B = \{y\} \cup (N(y) \setminus N(x))$. Otherwise, put $A = (N(y) \setminus (N(x) \cup \{x\})) \cup M(xy)$ and $B = \{x\} \cup (N(x) \setminus N(y))$. Obviously, A, B are not empty. Without loss of generality, it suffices to consider the case that $A = (N(x) \setminus (N(y) \cup \{y\})) \cup M(xy)$ and $B = \{y\} \cup (N(y) \setminus N(x))$.

Claim 1. $G[A]$ and $G[B]$ are cographs.

Proof. By contradiction, suppose *abcd* is an induced subgraph of $G[A]$ which is isomorphic to P_4. Then $G[\{a, b, d, y\}]$ is a $2P_1 + P_2$, a contradiction. So A is P_4-free. Similarly, $N(y) \setminus (N(x) \cup \{x\})$ is P_4-free, and since $B \setminus \{y\} \subseteq N(y)$, $G[B]$ is P_4-free. So $G[A]$ and $G[B]$ are cographs. □

Let W' be a maximum clique in $G[A]$, and $|W'| = s$. For $i = 0, 1, \cdots s$, put $X_i = \{u \in N(xy) : |N(u) \cap W'| = i\}$.

Claim 2. $X_0 \cup X_1 \cdots \cup X_{s-2}$ is a clique.

Proof. Suppose, on the contrary, that u and v are two non-adjacent vertices in $G[X_0 \cup X_1 \cdots \cup X_{s-2}]$ with $u \in X_i$ and $v \in X_j$.

First assume that $i = j$. By the definition of X_i, both u and v have exactly i neighbors in W'. If u and v are adjacent to same i vertices of W', then for $i \leq s - 2$, there are two vertices in W' adjacent to neither u nor v, which contradicts the assumption that G is $(2P_1 + P_2)$-free. If u, v are not adjacent to same i vertices of W', then, there exist two vertices u' and v' of W' such that u' is adjacent to u and is not adjacent to v, and v' is adjacent to v and is not adjacent to u. However, $G[\{y, u, v, u', v'\}] \cong C_5$, a contradiction. So $G[X_i]$ is a clique for any $i \in \{0, 1, 2, \cdots, s - 2\}$.

Now we consider the case $i \neq j$, and without loss of generality, let $i < j$. If $N(u) \cap W' \subset N(v) \cap W'$, and since $i < j \leq s - 2$, then there exists an edge

ab in W' such that both a and b are adjacent to neither u nor v. Hence, $G[\{a,b,u,v\}] \cong 2P_1 + P_2$, a contradiction. If $N(u) \cap W' \subsetneq N(v) \cap W'$, there exists an edge ab such that $G[\{y,u,v,a,b\}] \cong C_5$, a contradiction. So $X_0 \cup X_1 \cdots \cup X_{s-2}$ is a clique. □

Claim 3. $G[X_{s-1}]$ is a perfect graph.

Proof. Obviously, $G[X_{s-1}]$ contains no induced 5-cycles by assumption, and no induced odd cycle of length greater than five, otherwise, it must contain an induced subgraph isomorphic to $2P_1 + P_2$. Next we prove that the complement of $G[X_{s-1}]$ does not contain an induced cycle of length at least five. Suppose, on the contrary, it does such one. Then clearly, $G[X_{s-1}]$ contains an induced subgraph H isomorphic to $P_1 + P_2$, and let $V(H) = \{a,b,c\}$, $bc \in E(H)$. If $N(a) \cap W' \neq N(b) \cap W'$, and since $|N(a) \cap W'| = |N(b) \cap W'| = s-1$, $(N(a) \setminus N(b)) \cap W' \neq \emptyset$, $(N(b) \setminus N(a)) \cap W' \neq \emptyset$. Let $a' \in (N(a) \setminus N(b)) \cap W'$, $b' \in (N(b) \setminus N(a)) \cap W'$. Then $G[\{a,a',b',b,y\}] \cong C_5$. It is not possible. Hence, $N(a) \cap W' = N(b) \cap W'$. Similarly, $N(a) \cap W' = N(c) \cap W'$. However, if let $d \in W' \setminus N(a)$, $G[\{d,a,b,c\}] \cong 2P_1 + P_2$, a contradiction. So neither $G[X_{s-1}]$ nor its complement contains an odd cycle of length greater than three, $G[X_{s-1}]$ is a perfect graph by strong perfect graph theorem. □

Let $\omega(G) = \omega$. Since y is not adjacent to any vertex in A, and by Claim 1, $\chi(G[A \cup \{y\}]) = \omega(G[A \cup \{y\}]) = s$. Now, we give two different colorings of G. First, we have

$$\chi(G) \leq g_1 = \chi(G[N(xy) \cup (N(y) \setminus N(x))]) + \chi(G[A \cup \{y\}]).$$

Note that $\omega(G[N(xy) \cup (N(y) \setminus N(x))]) \leq \omega - 1$, so by the induction hypothesis,

$$g_1 \leq g(\omega - 1) + s.$$

On the other hand, since $G[A]$ is perfect, $\chi(G[A]) = \omega(G[A]) \leq \omega$. Since $\omega(G[X_s]) \leq \omega(G) - s$,

$$\begin{aligned} \chi(G) \leq g_2 &= \chi(G[X_0 \cup \cdots \cup X_{s-2}]) + \chi(G[A]) + \chi(G[B]) \\ &\quad + \chi(G[X_s]) + \chi(G[X_{s-1}]) \\ &\leq g(\omega - s) + 4\omega. \end{aligned}$$

By setting $g(\omega) = 2\omega^{\frac{3}{2}}$, we have $min(g_1, g_2) \le g(\omega) = 2\omega\sqrt{\omega}$. Indeed, if $s \le 2\sqrt{\omega}$ then $g_1 \le 2(\omega-1)\sqrt{\omega} + 2\sqrt{\omega} \le g(\omega)$ and if $s > 2\sqrt{\omega}$ then $g_2 \le 2(\omega - s)\sqrt{\omega - s} + 4\omega \le 2(\omega - 2\sqrt{\omega})\sqrt{\omega} + 4\omega \le g(\omega)$. □

Corollary 2.2. *Suppose that the graph G contains no induced 5-cycle. We have*

(1) If G contains no induced $kP_1 + P_2$ for an integer $k \ge 2$, then $\chi(G) \le 2\omega^{k-1}\sqrt{\omega}$,

(2) If G contains no induced $kP_1 + P_3$ for an integer $k \ge 1$, then $\chi(G) \le \omega^k\sqrt{\omega}$.

Proof. The proof is made by induction on $\omega + k$.

First we prove (1). If $k = 2$, then $kP_1 + P_2 = 2P_1 + P_2$, and by Theorem 2.1, $\chi(G) \le 2\omega\sqrt{\omega}$. Now assume that $k \ge 3$ and G contains no induced 5-cycle and no $kP_1 + P_2$. Pick a vertex x from G. Then it is clear that $G[M(x)]$ contains no induced 5-cycle and no $(k-1)P_1 + P_2$. So by induction hypothesis that $\chi(G[\{x\} \cup M(x)]) \le 2\omega^{k-2}\sqrt{\omega}$. On the other hand, $\omega(G[N(x)]) \le \omega - 1$, and thus $\chi(G[N(x)]) \le 2(\omega-1)^{k-1}\sqrt{\omega - 1}$. This gives

$$\begin{aligned}
\chi(G) &= \chi(G[N(x)] + \chi(G[\{x\} \cup M(x)]) \\
&\le 2(\omega-1)^{k-1}\sqrt{\omega-1} + 2\omega^{k-2}\sqrt{\omega} \\
&\le 2(\omega-1)\omega^{k-2}\sqrt{\omega} + 2\omega^{k-2}\sqrt{\omega} \\
&\le 2\omega^{k-1}\sqrt{\omega}.
\end{aligned}$$

Now we show (2). If $k = 1$, $kP_1 + P_3$ is known as the co-paw, and by [5], $\chi(G) \le \omega\sqrt{\omega}$. Now assume $k \ge 2$, and G contains no induced 5-cycle and no $kP_1 + P_3$. Then clearly, for any vertex $x \in V(G)$, $G[M(x)]$ contains no induced $(k-1)P_1 + P_3$ and no induced 5-cycle, and thus by induction hypothesis, $\chi(G) \le \chi(G[N(x)]) + \omega^{k-1}\sqrt{\omega}$. On the other hand, since $\omega(G[N(x)]) \le \omega - 1$, and by induction hypothesis, $\chi(G[N(x)]) \le (\omega-1)^k\sqrt{\omega-1}$.

$$\begin{aligned}
\chi(G) &= \chi(G[N(x)] + \chi(G[\{x\} \cup M(x)]) \\
&\le (\omega-1)^{k}\sqrt{\omega-1} + \omega^{k-1}\sqrt{\omega} \\
&\le (\omega-1)\omega^{k-1}\sqrt{\omega} + \omega^{k-1}\sqrt{\omega} \\
&\le \omega^{k}\sqrt{\omega}.
\end{aligned}$$

The proof is complete. □

3 Divisibility

Lemma 3.1. [5] *Every C_5-free non-complete graph is strongly α-divisible.*

Observe that for a subset S of the vertices of a graph G, if there exists a subset $T \subseteq V(G)$ such that $S \cap T = \emptyset$ and each vertex of T is adjacent to each vertex of S in G, then S does not contain any maximal clique of G. In the proof of the following theorem, we frequently use this fact.

Theorem 3.2. *Suppose that a graph G contains no induced 5-cycle and no induced $2P_1 + P_2$. Then G is strongly 2-divisible.*

Proof. We assume G is connected, and S is a maximum independent set of G. Take a vertex x from S and let y be a neighbor of x. We only need prove that G has a strong 2-division. Let us consider $M(xy)$.

If $M(xy) = \emptyset$, it is easy to see that (A, B) is a strong 2-division of G, where $A = N(y) \setminus N(x)$ and $B = N(x)$.

Now suppose $|M(xy)| = 1$ and let $M(xy) = \{a\}$. We claim that at most one of $\{a\} \cup (N(x) \setminus N(y))$ and $\{a\} \cup (N(y) \setminus N(x))$ contains a maximal clique of G. Otherwise, let X and Y be maximal cliques of $\{a\} \cup (N(x) \setminus N(y))$ and $\{a\} \cup (N(y) \setminus N(x))$, respectively. Clearly, both X and Y contain $\{a\}$. Pick a vertex x' from X. Then it must be adjacent to each vertex of Y, since a vertex $y' \in Y$ is not adjacent to x', $G[\{x, x', a, y', y\}] \cong C_5$, a contradiction. So, without loss of generality, assume that $\{a\} \cup (N(x) \setminus N(y))$ does not contain a maximal clique of G. Then G has a strong 2-division A, B with $A = \{a\} \cup (N(x) \setminus N(y))$, $B = N(y)$.

If $\alpha \leq 2$, G is strongly 2-divisible by Lemma 3.1. Next we consider the case when $\alpha \geq 3$ and $|M(xy)| \geq 2$.

Claim 1. The following statements are true.

(1) Each vertex of $(M(xy) \cup (N(y) \setminus N(x))) \setminus S$ is adjacent to each vertex of $S \setminus \{x\}$ in G.

(2) $M(xy)$ is a clique of G.

(3) Each vertex of $N(x) \setminus (\{y\} \cup N(y))$ is not adjacent to at most one vertex of $M(xy)$ in G.

Proof. We show (1) by contradiction. Suppose a vertex $u \in (M(xy) \cup (N(y) \setminus N(x))) \setminus S$ is not adjacent to a vertex, say v, of $S \setminus \{x\}$. Since S is a maximum independent set of G, v is adjacent to a vertex, say w, of S. It is obvious that $w \neq x$, and v, w, x are not mutually adjacent since they are elements of S. Thus $G[\{u, v, w, x\}] \cong 2P_1 + P_2$, a contradiction. To see $M(xy)$ is a clique, if two vertices u and v of $M(xy)$ are not adjacent in G, then $G[\{u, v, x, y\}] \cong 2P_1 + P_2$, a contradiction. Now suppose that two vertices $u, v \in M(xy)$ which are not adjacent to a vertex $w \in N(x) \setminus (\{y\} \cup N(y))$. Then $G[\{u, v, w, y\}] \cong 2P_1 + P_2$, a contradiction. □

Accordingly, $S \subseteq N(y) \setminus N(x)$, or $|S \cap M(xy)| = 1$ and $S \setminus M(xy) \subseteq N(y) \setminus N(x)$.

Claim 2. If $S \subseteq N(y) \setminus N(x)$, then

(1) Each vertex of $N(x) \setminus N(y)$ is adjacent to each vertex of S in G.

(2) (A, B) is a strong 2-division of G, where $A = S \cup N(xy)$ and $B = M(xy) \cup (N(x) \setminus N(y)) \cup (N(y) \setminus S)$.

Proof. We prove (1) by contradiction. Suppose a vertex $u \in N(x) \setminus N(y)$ and $v \in S$ are not adjacent in G. Clearly, $u \neq y$ and $x \neq v$. By Claim 1, there exists a vertex, say w, of $M(xy)$, which is adjacent to both u and v in G. Then $G[\{x, y, v, w, u\}] \cong C_5$, a contradiction.

Let us consider (2). Since $A \subseteq N(y)$, it contains no maximal clique of G. By (1) of Claim 1 and (1) of Claim 2, each vertex of B is adjacent to each vertex of $S \setminus \{x\}$, and thus B does not contain any maximal clique of G either. □

Now assume that $|S \cap M(xy)| = 1$ and $S \setminus M(xy) \subseteq N(y) \setminus N(x)$. Let $M(xy) \cap S = \{a\}$.

Claim 3. Set $C_1 = N(xy) \cap N(a)$, $C_2 = N(xy) \setminus C_1$. Let $D_1 = \{u \in C_1 \colon S \subseteq N(u)\}$. $D_2 = C_1 \setminus D_1$. If $|M(xy)| \geq 3$, then

(1) Each vertex of $N(x) \setminus N(y)$ is adjacent to each vertex of $S \setminus \{a\}$.

(2) Each vertex of C_2 is adjacent to each vertex of $S \setminus \{x, a\}$

(3) Each vertex of D_2 is adjacent to each vertex of $M(xy)$.

(4) (A, B) is a strong 2-division of G, where $A = S \cup D_2$ and $B = (N(x) \setminus N(y)) \cup (N(y) \setminus S) \cup (M(xy) \setminus \{a\}) \cup C_2 \cup D_1$.

Proof. We show (1) by contradiction. Suppose that a vertex $u \in N(x) \setminus (N(y) \cup \{y\})$ is not adjacent to a vertex $v \in S \setminus \{x, a\}$. Since $|M(xy)| \geq 3$ and Claim 1, there exists a vertex $w \in M(xy) \setminus \{a\}$, which is adjacent to u and v. But, $G[\{x, u, w, v, y\}] \cong C_5$, a contradiction.

To show (2), suppose that a vertex $p \in C_2$ is not adjacent to a vertex $q \in S \setminus \{a, x\}$, then $G[\{a, q, x, p\}] \cong 2P_1 + P_2$, a contradiction.

Now we show (3) by contradiction. Suppose a vertex $u \in D_2$ is not adjacent to a vertex $w \in M(xy)$. By definition of D_2, there is a vertex $v \in S \setminus \{a, x\}$ which is not adjacent to u. Then $G[\{y, u, w, a, v\}] \cong C_5$, a contradiction.

Now let us prove (4). Note that all vertices of B are adjacent to each vertex of $S \setminus \{x, a\}$. Hence B contains no any maximal clique of G. $(S \setminus \{a\}) \cup D_2$ contains no maximal clique of G, since each vertex of $(S \setminus \{a\}) \cup D_2 \subseteq N(y)$. It remains to see that $\{a\} \cup D_2$ contains no maximal clique, since each vertex of $M(xy) \setminus \{a\}$ is adjacent to $\{a\} \cup D_2$. □

In what follows, assume that $|M(xy)| = 2$, and let b be the other element of $M(xy)$ different from a.

Claim 4. Assume that $N(y) \setminus (S \cup N(x) \cup \{x\}) = \emptyset$. Let $B_1 = N(xy) \cap N(b)$ and $B_2 = N(xy) \setminus B_1$. Then (A, B) is a strong 2-division of G, where $A = S \cup B_1$ and $B = \{b\} \cup (N(x) \setminus N(y)) \cup B_2$.

Proof. First, since $(S \setminus \{a\}) \cup B_1 \subseteq N(y)$ and $\{a\} \cup B_1 \subseteq N(b)$, A contains no maximal clique of G. To prove B contains no maximal clique of G, by the definition of B_2, it suffices to show that both $(N(x) \setminus N(y)) \cup B_2$ and $\{b\} \cup (N(x) \setminus N(y))$ does not contain any maximal cliques of G. It is easy to see that $(N(x) \setminus N(y)) \cup B_2$ contains no maximal clique of G, since $(N(x) \setminus N(y)) \cup B_2 \subseteq N(x)$. On the other hand, by Claim 1, $S \setminus \{x\} \subseteq N(b)$, and since G contains no induced C_5, $N(b) \cap (N(x) \setminus N(y)) \subseteq N(z)$ for any vertex $z \in S \setminus \{a, x\}$. It follows that $\{b\} \cup (N(x) \setminus N(y))$ does not contain any maximal cliques of G. □

Now assume that $N(y) \setminus (S \cup N(x) \cup \{x\}) \neq \emptyset$ and we consider two cases based on $M(xy) = \{a, b\}$.

First assume that $\{a, b\}$ is a maximal clique of G. Let $A_1 = (N(x) \setminus (N(y) \cup \{y\})) \cap N(a)$, $A_2 = (N(x) \setminus (N(y) \cup \{y\})) \setminus A_1$.

Claim 5. If $A_2 \neq \emptyset$, then the following statements hold:

(1) each vertex of A_2 is adjacent to each vertex of $S \setminus \{a\}$.

(2) each vertex of $N(xy)$ is adjacent to each vertex of $S \setminus \{a\}$.

(3) each vertex of A_1 is adjacent to each vertex $N(y) \setminus (S \cup N(x) \cup \{x\})$.

(4) each vertex of A_2 is adjacent to each vertex of A_1.

(5) (A, B) is a strong 2-division of G with $A = A_2 \cup (N(y) \setminus S) \cup \{y, b\}$ and $B = S \cup A_1$.

Proof. By contradiction, suppose a vertex $u \in A_2$ is not adjacent to a vertex $v \in S \setminus \{a\}$. Then $G[\{x, u, v, a\}] \cong 2P_1 + P_2$, a contradiction. This proves (1).

To show (2), suppose that a vertex $u \in N(xy)$ is not adjacent to a vertex $v \in S \setminus \{a, x\}$. Then u must be adjacent to a, for otherwise, $G[\{u, a, v, y\}] \cong 2P_1 + P_2$. Moreover, since $\{a, b\}$ is a maximal clique, u is not adjacent to b. So $G[\{y, u, v, a, b\}] \cong C_5$, a contradiction.

Now we prove (3). In fact, if a vertex $u \in A_1$ is not adjacent to a vertex $v \in N(y) \setminus (S \cup N(x) \cup \{x\})$, then by the definition of A_1, $ua \in E(G)$, and by Claim 1, $va \in E(G)$. Hence $G[\{u, v, x, y, a\}] \cong C_5$, a contradiction.

Suppose (4) is not true, and a vertex $u \in A_1$ is not adjacent to a vertex $v \in A_2$. Then, since $\{a, b\}$ is a maximal clique of G, u is not adjacent to b and by the definition A_2 and Claim 1, v is not adjacent to a. In this case, $G[\{x, u, v, a, b\}] \cong C_5$, a contradiction.

Finally we prove (5). By Claim 1 and (1-2) of Claim 4, each vertex of A is adjacent to each vertex of $S \setminus \{a, x\}$, A does not contain any maximal clique of G. Since S is an independent set, to see that B has not maximal clique it suffices to show that $A_1 \cup \{z\}$ has not a maximal clique of G for each $z \in S$. At first, by (4) of Claim 5, $\{x\} \cup A_1$ does not contain a maximal clique. Secondly, by (3), each vertex of $A_1 \cup (S \setminus \{x\})$ is adjacent to each vertex of $N(y) \setminus (S \cup N(x) \cup \{x\})$, $A_1 \cup (S \setminus \{x\})$ does not contain any maximal cliques of G. □

Claim 6. Let $C_1 = N(xy) \cap N(a)$, $C_2 = N(xy) \setminus C_1$. If $A_2 = \emptyset$, then the following statements holds:

(1) Each vertex of C_2 is adjacent to every vertex of $S \setminus \{a\}$.

(2) Each vertex of C_1 is adjacent of each vertex of $S \setminus \{a\}$.

(3) (A, B) is a strong 2-division of G with $A = (N(x) \setminus N(y)) \cup (N(y) \setminus$

$S) \cup \{b\} \cup C_1$ and $B = S \cup C_2$.

Proof. Suppose (1) is not true, and let a vertex $u \in C_2$ is not adjacent to a vertex $v \in S \setminus \{a\}$. Then $G[\{x, u, v, a\}] \cong 2P_1 + P_2$, a contradiction.

We show (2) by contradiction. Suppose a vertex $u \in C_1$ is not adjacent to a vertex $v \in S \setminus \{a\}$. By the definition of C_1 and the assumption that $\{a, b\}$ is a maximal clique of G, $ua \in E(G)$ and $ub \notin E(G)$. By Claim 1, $vb \in E(G)$. Hence $G[\{y, u, v, a, b\}] \cong C_5$, a contradiction. Now we conclude that each vertex of $(N(y) \setminus S) \cup \{y\} \cup C_1$ is adjacent to all of $S \setminus \{a\}$.

Now we show (3). Firstly, B has not maximal clique of G, since $(S \setminus \{a\}) \cup C_2 \subseteq N(y)$. Next we prove that A does not contain a maximal clique. It is easy to see that $((N(x) \setminus N(y)) \setminus \{y\}) \cup (N(y) \setminus S) \cup \{b\} \cup C_1 \subseteq N(a)$, $((N(x) \setminus N(y)) \setminus \{y\}) \cup (N(y) \setminus S) \cup \{b\} \cup C_1$ does not contain a maximal clique of G. It remains to show that $(N(y) \setminus S) \cup \{y\} \cup C_1$ not contain a maximal clique of G. It suffices to prove that each vertex of C_1 is adjacent of each vertex of $S \setminus \{a\}$. By (2), Accordingly, A has not maximal clique. □

Claim 7. If $\{a, b\}$ is not a maximal clique of G, then the following statements holds.

(1) If a vertex $u \in N(x) \setminus (N(y) \cup \{y\})$ is adjacent to both a and b, it is adjacent to all of $N(y) \setminus (N(x) \cup \{x\})$. Furthermore, (A, B) is a strong 2-division of G, where $A = (N(y) \setminus N(x)) \cup \{a, b\}$ and $B = N(x)$.

(2) If each vertex of $N(x) \setminus (N(y) \cup \{y\})$ are adjacent to exactly one vertex of $\{a, b\}$, then (A, B) is a strong 2-division of G with $A = \{a, b\} \cup (N(x) \setminus N(y))$ and $B = N(y)$.

Proof. We prove (1) by contradiction. Suppose that there is a vertex $v \in N(y) \setminus (N(x) \cup \{x\})$ is not adjacent to u. We consider two cases. If $v \in S \setminus \{a, x\}$, then by Claim 1, $vb \in E(G)$. But, $G[\{x, y, u, v, b\}] \cong C_5$, a contradiction. If $v \in N(y) \setminus (N(x) \cup S)$, then by Claim 1, $va \in E(G)$. But, $G[\{x, y, u, v, a\}] \cong C_5$, a contradiction. It is clear that both A and B does not a maximal clique of G.

Let us show (2). Obviously, B has not a maximal clique of G. Next we show that A does not contain a maximal clique of G. Since y is adjacent to neither a nor b, it only need to prove that neither $N(x) \setminus N(y)$ nor $\{a, b\} \cup (N(x) \setminus (N(y) \cup \{y\}))$ contain a maximal clique of G. Since $N(x) \setminus$

$N(y) \subseteq N(x)$, $N(x) \setminus N(y)$ does not contain a maximal clique of G. To prove $N(x) \setminus (N(y) \cup \{y\}) \cup \{a, b\}$ has not maximal clique it suffices to prove both $N(x) \setminus (N(y) \cup \{y\}) \cup \{a\}$ and $N(x) \setminus (N(y) \cup \{y\}) \cup \{b\}$ does not contain a maximal clique of G, since each vertex of $N(x) \setminus (N(y) \cup \{y\})$ are adjacent to exactly one vertex of $\{a, b\}$. Suppose $N(x) \setminus (N(y) \cup \{y\}) \cup \{a\}$ contains a maximal clique D. Then for each vertex $z \in N(y) \setminus (N(x) \cup S)$, $D \subseteq N(z)$, otherwise, $G[\{x, y, d, a, z\}] \cong C_5$, where $d \in D \setminus \{a\}$.

Suppose $N(x) \setminus (N(y) \cup \{y\}) \cup \{b\}$ contains a maximal clique D'. Then for each vertex $w \in S \setminus \{x, a\}$, $D' \subseteq N(w)$, otherwise, $G[\{x, y, d', b, w\}] \cong C_5$, where $d' \in D' \setminus \{b\}$. Hence A does not contain a maximal clique of G. The proof is complete. □

References

[1] J.A. Bondy and U.S.R. Murty, Graph Thoery With Application, MacMillan Press Ltd., London, 1976.

[2] M. Chudnovsky, N. Robertson, P. Seymour and R. Thomas, The strong perfect graph theorem, Annals of Math. 164(2006) 51-229.

[3] S. Gravier, C.T. Hoàng, F. Maffray, Coloring the hypergraph of maximal cliques of a graph with no long path, Discrete Mathematics 272(2003) 285-290.

[4] A. Gyárfás, Problems from the world surrounding perfect graphs, Appl. Math. XIX (3-4)(1987) 413-441.

[5] C.T. Hoàng, C. McDiarmid, On the divisibility of graphs, Discrete Math. 242(2002) 145-156.

[6] S. Wagon, A bound on the chromatic number of graphs without certain induced subgraphs, Journal of Combin. Theory, series B 29(1980) 345-346.