Connected M_2-equicoverable graphs with circumference at most 5 *†

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Abstract

A graph G is called H-*equicoverable* if every minimal H-covering in G is also a minimum H-covering in G. In this paper, we give the characterization of connected M_2-equicoverable graphs with circumference at most 5.

Keywords: H-covering, coverable, H-equicoverable.

1 Introduction and preliminaries

The problem that we study stems from the research of H-decomposable graphs, randomly packable graphs and equipackable graphs. For further definitions and results refer to [1], [3], [4], [5].

Let H be a subgraph of G. By $G - H$, we denote the graph remaining after we delete from G the edges of H and any resulting isolated vertices. Let $L = \{H_1, H_2, \cdots, H_k\}$ be a collection of copies of H. If $H_1, H_2, \cdots, H_k$ are edge-disjoint, L is called an H-*packing* in G; if every edge of G appears in at least one member of L, then L is called an H - *covering* of G. If G has an H-covering, G is called H - *coverable*. A graph G is called H - *decomposable* if it has an H-packing which is also an H-covering. A matching with t edges is denoted by M_t. Caro ([2],[3]) characterized M_2-decomposable graphs:

Theorem 1.1. *Let G be a graph of size $2m > 0$ and without isolated vertices. Then G is M_2-decomposable if and only if $\Delta(G) \le m$ and G is not isomorphic to $K_3 \cup K_2$.*

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An H−covering of G with k copies $H_1, H_2, \cdots, H_k$ of H is called *minimal* if, for any H_j, $\bigcup_{i=1}^{k} H_i - H_j$ is not an H−covering of G. An H−covering of G with k copies $H_1, H_2, \cdots, H_k$ of H is called *minimum* if there exists no H−covering with less than k copies of H. A graph G is called H − *equicoverable* if every minimal H-covering in G is also a minimum H−covering in G.

All P_3−equicoverable graphs are characterized in [6]. In [7], we obtain some results of M_2−equicoverable graphs. In this paper, we give the characterization of connected graphs with circumference at most 5.

Obviously, if G contains an edge e which is adjacent to all the other edges, e belongs to no copy of M_2. Consequently, e can not be covered by any M_2, and G is not M_2−coverable. It is easy to see that a graph G is M_2−coverable if and only if there exists no edge in G which is adjacent to all the other edges.

The following observation is crucial to our work:

Observation: If a graph G is not M_2−coverable, there are three possibilities:

(1) There exists only one edge e_i in G which is adjacent to all the other edges; that is, $G - e_i$ is M_2−coverable.

(2) There exist exactly two adjacent edges e_i and e_j in G each of which is adjacent to all the other edges, respectively; that is, $G - e_i - e_j$ is M_2−coverable.

(3) There exist at least three edges in G each of which is adjacent to all the other edges, respectively; that is, G is $K_{1,k} (k \geq 3)$ or K_3.

If G is not M_2−coverable, G can not be M_2−equicoverable. So the graphs that we'll characterize are all M_2−coverable.

The non-adjacent edge-degree of an edge e in a graph G, written by $d_1(e)$, is the number of edges which are not adjacent to e. Denote by $N(e)$ the set of all the adjacent edges of e and denote by $c_0(e)$ the number of M_2 in the minimum M_2−covering of $N(e)$(if $N(e)$ is not M_2−coverable, $c_0(e) = 0$).

Lemma 1.2. *[7] If there exists an edge e in G such that $d_1(e) + c_0(e) > c(G)$, G is M_2−equicoverable.*

If there exists an edge e in G such that $d_1(e) + c_0(e) = c(G)$ and the neighbor set $N(e)$ contains two edges which are non-adjacent to the same edge of $G - N(e)$, G is not M_2−equicoverable.

Lemma 1.3. *Let F be a subgraph in G which is not M_2-equicoverable. If $G-F$ is M_2-coverable, G is not M_2-equicoverable.*

Proof. Take a minimal M_2-covering of F which is not a minimum M_2-covering, then take any minimal M_2-covering of $G-F$. Their union is a minimal M_2-covering of G which is not minimum. By the definition, G is not M_2-equicoverable. □

2 Connected M_2-equicoverable graphs with circumference 5

Theorem 2.1. *Let G be a connected graph with circumference* 5 *and girth* 5. *Then G is M_2-equicoverable if and only if G is C_5.*

Proof. Let G satisfy $c(G)=g(G)=5$, then G only contains 5−cycles. Let $C=v_1v_2v_3v_4v_5v_1$ be a 5−cycle in G. Let $v_1v_2=e_1$, $v_2v_3=e_2$, $v_3v_4=e_3$, $v_4v_5=e_4$, $v_5v_1=e_5$.

If $G=C$, it is easy to verify that G is M_2-equicoverable.

If G is not a cycle, there must exist an edge $e_0 \in E(G)$ such that e_0 is incident to some vertex of C(assume it is v_1). Let $C\cup e_0=G_0$. Then $\{\{e_0,e_3\},\{e_1,e_3\},\{e_5,e_3\},\{e_2,e_4\}\}$ is a minimal M_2-covering of G_0 which is not minimum. So G_0 is not M_2-equicoverable. If $G-G_0$ is M_2-coverable, G is not M_2-equicoverable by Lemma 1.3; if $G-G_0$ is not M_2-coverable, there are three possibilities:

Case 1: There exists only one edge e in $G-G_0$ such that $G-G_0-e$ is M_2-coverable. Since G contains no 3−cycles, e can not be adjacent to both e_2 and e_4. Assume that e is non-adjacent to e_2, then $\{\{e_0,e_3\},\{e_1,e_3\},\{e_5,e_3\},\{e,e_2\},\{e_2,e_4\}\}$ is a minimal M_2-covering of $G_0\cup e$ which is not minimum, so $G_0\cup e$ is not M_2-equicoverable. By Lemma 1.3, G is not M_2-equicoverable.

Case 2: There exist exactly two adjacent edges e_i and e_j in $G-G_0$ such that $G-G_0-e_i-e_j$ is M_2-coverable.

Subcase 1: Neither e_i nor e_j is adjacent to e_3. Then $\{\{e_0,e_3\},\{e_1,e_3\},\{e_5,e_3\},\{e_i,e_3\},\{e_j,e_3\},\{e_2,e_4\}\}$ is a minimal M_2-covering of $G_0\cup e_i\cup e_j$ which is not minimum. So $G_0\cup e_i\cup e_j$ is not M_2-equicoverable.

Subcase 2: At least one edge of e_i,e_j is adjacent to e_3. Denote the common vertex of e_i and e_j by v.

(i) v is $v_3(v_4)$. Since G contains no 3−cycle, neither e_i nor e_j is adjacent to e_4(or e_2). Then $\{\{e_0,e_3\},\{e_1,e_3\},\{e_5,e_3\},\{e_i,e_4\},\{e_j,e_4\},\{e_2,e_4\}\}$

$(\{\{e_0, e_3\}, \{e_1, e_3\}, \{e_5, e_3\}, \{e_i, e_2\}, \{e_j, e_2\}, \{e_2, e_4\}\})$ is a minimal M_2-covering of $G_0 \cup e_i \cup e_j$ which is not minimum, so $G_0 \cup e_i \cup e_j$ is not M_2-equicoverable.

(ii) v is neither v_3 nor v_4. Then at most one of e_i or e_j can be adjacent to e_3(Otherwise, G contains a 6−cycle). So exactly one of e_i, e_j (assume it is e_i) is adjacent to e_3. Obviously, e_i can not be adjacent to e_2 and e_4 at the same time. Assume e_i is non-adjacent to e_2. Then $\{\{e_0, e_3\}, \{e_1, e_3\}, \{e_5, e_3\}, \{e_i, e_2\}, \{e_j, e_3\}, \{e_2, e_4\}\}$ is a minimal M_2-covering of $G_0 \cup e_i \cup e_j$ which is not minimum, so $G_0 \cup e_i \cup e_j$ is not M_2-equicoverable.

By Lemma 1.3, G is not M_2-equicoverable.

Case 3: $G - G_0$ is $K_{1,k}(k \geq 3)$. Denote the k edges of $K_{1,k}(k \geq 3)$ by $e_{01}, e_{02}, \cdots, e_{0k}$ and denote the center by v.

Subcase 1: None of the k edges of the star is incident to v_3 or v_4. That is, none of the k edges is adjacent to e_3. Then $\{\{e_0, e_3\}, \{e_1, e_3\}, \{\{e_5, e_3\}, \{e_2, e_4\}, \{e_{01}, e_3\}, \{e_{02}, e_3\}, \cdots, \{e_{0k}, e_3\}\}$ is a minimal M_2- covering of G which is not minimum. So G is not M_2- equicoverable.

Subcase 2: At least one edge e_{0i} of the star is incident to v_3 or v_4(that is, e_{0i} is adjacent to e_3). Assume that e_{0i} is incident to v_3. Since G only contains 5−cycles, no edge of the star is adjacent to e_4. Then $\{\{e_0, e_3\}, \{e_1, e_3\}, \{e_5, e_3\}, \{e_2, e_4\}, \{e_{01}, e_4\}, \{e_{02}, e_4\}, \cdots, \{e_{0k}, e_4\}\}$ is a minimal M_2- covering of G which is not minimum. So G is not M_2- equicoverable.

From the above, only the 5−cycle C_5 is the connected M_2-equicoverable graph with circumference 5 and girth 5. □

Theorem 2.2. *If G is a connected graph with circumference* 5 *and girth* 4, *G is not M_2-equicoverable.*

Proof. Since $c(G) = 5$, $g(G) = 4$, G can contain only 5−cycles and 4−cycles. Let $C = v_1v_2v_3v_4v_5v_1$ be a 5−cycle of G and $v_1v_2 = e_1$, $v_2v_3 = e_2$, $v_3v_4 = e_3$, $v_4v_5 = e_4$, $v_5v_1 = e_5$. Let $C' = v_{01}v_{02}v_{03}v_{04}v_{01}$ be a 4−cycle of G and $v_{01}v_{02} = e_{01}$, $v_{02}v_{03} = e_{02}$, $v_{03}v_{04} = e_{03}$, $v_{04}v_{01} = e_{04}$. Consider $E(C) \cap E(C')$. If $E(C) \cap E(C')$ has one edge or three edges, G must contain a 3−cycle or an $n-$cycle($n = 6, 7$), which is a contradiction. So there are only two cases:

Case 1: $E(C) \cap E(C')$ is an empty set. There must exist an edge(suppose it is e_{04}) in C' which is non-adjacent to all the edges of C. Then the subgraph G_0 induced by $E(C) \cup e_{04}$ is not M_2-coverable. Since G contains no 3−cycle, there exists no edge e in $G - G_0$ such that e is adjacent to all of e_{01}, e_{02}, e_{03}. So $G - G_0$ must be M_2-coverable. By Lemma 1.3, G is not M_2-equicoverable.

Case 2: $E(C) \cap E(C')$ has exactly two edges. Up to isomorphism, there is only one possibility, without loss of generality, suppose that $e_{01} = e_1$, $e_{02} = e_2$, shown as Figure 1:

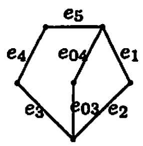


Figure 1

Denote by G_0 the subgraph induced by the edge set $\{e_{03}, e_{04}, e_1, e_2, e_3, e_5\}$. Then $\{\{e_2, e_5\}, \{e_2, e_{04}\}, \{e_1, e_{03}\}, \{e_1, e_3\}\}$ is a minimal M_2–covering of G_0 which is not minimum. So G_0 is not M_2–equicoverable. If $G - G_0$ is M_2–coverable, G is not M_2–equicoverable by Lemma 1.3; if $G - G_0$ is not M_2–coverable, since G contains no 3–cycles, there are three possibilities:

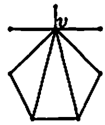
Subcase 1: There exists only one edge e_i in $G - G_0$ such that $G - G_0 - e_i$ is M_2–coverable. Since $e_4 \in E(G - G_0)$, e_i is e_4 or e_i must be adjacent to e_4. So e_i is not adjacent to e_2. Then $\{\{e_2, e_5\}, \{e_2, e_{04}\}, \{e_2, e_i\}, \{e_1, e_{03}\}, \{e_1, e_3\}\}$ is a minimal M_2–covering of $G_0 \cup e_i$ which is not minimum. So $G_0 \cup e_i$ is not M_2–equicoverable. By Lemma 1.3, G is not M_2–equicoverable;

Subcase 2: There exist exactly two adjacent edges e_i and e_j in $G - G_0$ such that $G - G_0 - e_i - e_j$ is M_2–coverable. Both e_i and e_j must be adjacent to e_4(one of them can be e_4). Since G contains no 3–cycle, neither e_i nor e_j can be adjacent to e_1 or e_2. Then $\{\{e_2, e_5\}, \{e_2, e_{04}\}, \{e_2, e_i\}, \{e_2, e_j\}, \{e_1, e_{03}\}, \{e_1, e_3\}\}$ is a minimal M_2–covering of $G_0 \cup e_i \cup e_j$ which is not minimum. So $G_0 \cup e_i \cup e_j$ is not M_2–equicoverable. By Lemma 1.3, G is not M_2–equicoverable;

Subcase 3: $G - G_0$ is $K_{1,k} (k \geq 3)$. Denote the k edges of the star by $e_{11}, e_{12}, \cdots, e_{1k}$. Then e_4 must be an edge of the star and the center of the star must be an endpoint of e_4. Since G contains no 3–cycle, no edge of the star can be adjacent to e_1 and e_2. Then $\{\{e_2, e_5\}, \{e_2, e_{04}\}, \{e_2, e_4\}, \{e_1, e_{03}\}, \{e_1, e_3\}, \{e_2, e_{11}\}, \{e_2, e_{12}\}, \cdots, \{e_2, e_{1k}\}\}$ is a minimal M_2–covering of G which is not minimum. So G is not M_2–equicoverable.

□

Denote by $F_5 \cdot S_k$ the graph obtained from a fan F_5 and a k–star $K_{1,k} (k \geq 0)$ by identifying one vertex of the fan F_5 with the center of $K_{1,k}$. See Figure 2 for $k = 3$:



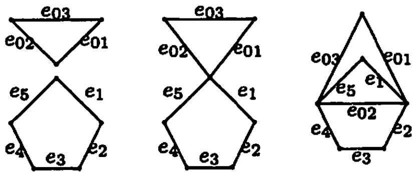
Fiure 2: $F_5 \cdot S_3$

Theorem 2.3. *Let G be a connected graph with circumference* 5 *and girth* 3. *Then G is M_2-equicoverable if and only if G is the graph $F_5 \cdot S_k (k \geq 0)$.*

Proof. It is easy to verify that $F_5 \cdot S_k (k \geq 0)$ is M_2-equicoverable.

Conversely, let G be an M_2-equicoverable connected graph which satisfies $c(G) = 5$ and $g(G) = 3$. So G must contain a 5−cycle and a 3−cycle. Let $C = v_1v_2v_3v_4v_5v_1$ be a 5−cycle of G and $v_1v_2 = e_1$, $v_2v_3 = e_2$, $v_3v_4 = e_3$, $v_4v_5 = e_4$, $v_5v_1 = e_5$. Let $C' = v_{01}v_{02}v_{03}v_{01}$ be a 3−cycle of G and $v_{01}v_{02} = e_{01}$, $v_{02}v_{03} = e_{02}$, $v_{03}v_{01} = e_{03}$. For $E(C) \cap E(C')$, there are three cases:

Case 1: $E(C) \cap E(C')$ is an empty set.



(a) (b) (c)

Figure 3

Subcase 1: C and C' have no common vertex, shown as Figure 3(a). Denote by G_0 the subgraph induced by $\{e_{01}, e_{03}, e_1, e_2, e_4, e_5\}$. Then $\{\{e_1, e_{01}\}, \{e_1, e_{03}\}, \{e_2, e_4\}, \{e_2, e_5\}\}$ is a minimal M_2-covering of G_0 which is not minimum. So G_0 is not M_2-equicoverable. If $G - G_0$ is M_2-coverable, G is not M_2-equicoverable by Lemma 1.3. If $G - G_0$ is not M_2-coverable, $G - G_0$ can not be a star because $G - G_0$ contains the matching $\{e_{02}, e_3\}$. There are two possibilities:

(i) There exists only one edge e_i in $G - G_0$ such that $G - G_0 - e_i$ is M_2-coverable. Since $G - G_0$ contains $\{e_{02}, e_3\}$, e_i must be adjacent to both e_{02} and e_3. Then e_i can not be adjacent to e_1. $\{\{e_1, e_{01}\}, \{e_1, e_{03}\}, \{e_1, e_i\}, \{e_2, e_4\}, \{e_2, e_5\}\}$ is a minimal M_2-covering of $G_0 \cup e_i$ which is not minimum. So $G_0 \cup e_i$ is not M_2-equicoverable.

(ii) There exist two edges e_i and e_j in $G - G_0$ such that $G - G_0 - e_i - e_j$ is M_2-coverable. In the same way, both e_i and e_j must be adjacent to e_{02} and e_3. So neither of them can be adjacent to e_1. Then $\{\{e_1, e_{01}\},$

$\{e_1, e_{03}\}$, $\{e_1, e_i\}$, $\{e_1, e_j\}$, $\{e_2, e_4\}$, $\{e_2, e_5\}\}$ is a minimal M_2−covering of $G_0 \cup e_i \cup e_j$ which is not minimum. So $G_0 \cup e_i \cup e_j$ is not M_2−equicoverable.

By Lemma 1.3, G is not M_2−equicoverable.

Subcase 2: C and C' have only one common vertex, shown as Figure 3(b). Denote by G_0 the subgraph induced by $\{e_{01}, e_{03}, e_1, e_3, e_4, e_5\}$. It is easy to see that G_0 is not M_2−equicoverable. If $G - G_0$ is M_2−coverable, G is not M_2−equicoverable by Lemma 1.3; if $G - G_0$ is not M_2−coverable, $G - G_0$ can not be a star since $G - G_0$ contains the matching $\{e_{02}, e_2\}$. There are two possibilities:

(i) There exists only one edge e_i in $G - G_0$ such that $G - G_0 - e_i$ is M_2−coverable. Since $G - G_0$ contains the matching $\{e_{02}, e_2\}$, e_i must be adjacent to both e_{02} and e_2. So e_i can not be adjacent to e_4. Then $\{\{e_4, e_{01}\}$, $\{e_4, e_{03}\}$, $\{e_4, e_i\}$, $\{e_1, e_4\}$, $\{e_3, e_5\}\}$ is a minimal M_2−covering of $G_0 \cup e_i$ which is not minimum. $G_0 \cup e_i$ is not M_2−equicoverable. By Lemma 1.3, G is not M_2−equicoverable.

(ii) There exists exactly two edges e_i and e_j in $G - G_0$ such that $G - G_0 - e_i - e_j$ is M_2−coverable. In the same way, both e_i and e_j must be adjacent to e_{02}, e_2, and neither of them can be adjacent to e_4. Then $\{\{e_4, e_{01}\}$, $\{e_4, e_{03}\}$, $\{e_4, e_i\}$, $\{e_4, e_j\}$, $\{e_1, e_4\}$, $\{e_3, e_5\}\}$ is a minimal M_2−covering of $G_0 \cup e_i \cup e_j$ which is not minimum. $G_0 \cup e_i \cup e_j$ is not M_2−equicoverable. By Lemma 1.3, G is not M_2−equicoverable.

Subcase 3: C and C' have two common vertices, shown as Figure 3(c). Denote by G_0 the subgraph induced by $\{e_{01}, e_{03}, e_1, e_2, e_4, e_5\}$. Then $\{\{e_1, e_{03}\}$, $\{e_2, e_{03}\}$, $\{e_{01}, e_4\}$, $\{e_{01}, e_5\}\}$ is a minimal M_2−covering of G_0 which is not minimum. So G_0 is not M_2−equicoverable. If $G - G_0$ is M_2−coverable, G is not M_2−equicoverable by Lemma 1.3; if $G - G_0$ is not M_2−coverable, since $G - G_0$ contains $\{e_{02}, e_3\}$, $G - G_0$ can not be a star. Neither e_2 nor e_4 is the edge of $G - G_0$. So there can not exist two adjacent edges in $G - G_0$ which are respectively adjacent to both e_{02} and e_3. We have only one case: there exists exactly one edge e_i in $G - G_0$ such that $G - G_0 - e_i$ is M_2−coverable and e_i is adjacent to both e_{02} and e_3. Only one edge of e_{01} and e_{03}(assume it is e_{01}) is non-adjacent to e_i. Then $\{\{e_1, e_{03}\}$, $\{e_2, e_{03}\}$, $\{e_{01}, e_4\}$,$\{e_{01}, e_i\}$, $\{e_{01}, e_5\}\}$ is a minimal M_2−covering of $G_0 \cup e_i$ which is not minimum. So $G_0 \cup e_i$ is not M_2−coverable. By Lemma 1.3, G is not M_2−equicoverable.

This is a contradiction, so Case 1 is impossible.

Case 2: $E(C) \cap E(C')$ has two edges. Up to isomorphism, there is only one possibility, without loss of generality, let $e_{03} = e_3$, shown as Figure 4, denote this graph by G_0.

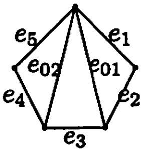


Figure 4

Similar to the proof of Case 1, we can prove: if G is M_2−equicoverable, none of the 5 graphs shown as Figure 5 is a subgraph of G.

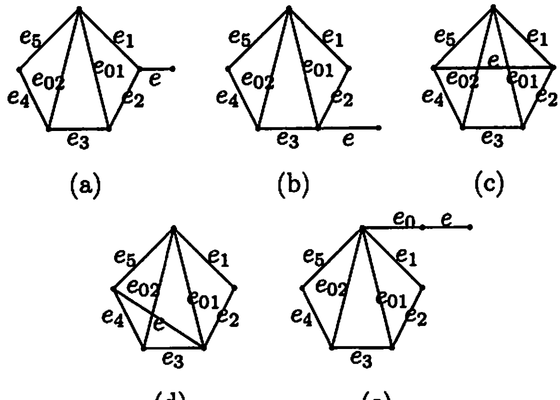


Figure 5

So none of v_2,v_3,v_4,v_5 has any incident edge outside G_0 and all the paths beginning with v_1 have length no more than 1. G must be $F_5 \cdot S_k (k \geq 0)$.

Case 3: $E(C) \cap E(C')$ has only one edge. Since G contains no 6−cycle, up to isomorphism, there is only one possibility. Suppose that $e_{01} = e_1$, $e_{02} = e_2$, shown as Figure 6:

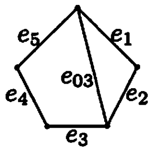


Figure 6

Denote by G_0 the subgraph induced by $\{e_{03}, e_1, e_2, e_3, e_4, e_5\}$. It's easy to see that G_0 is not M_2−equicoverable. If $G - G_0$ is M_2−coverable, G is not M_2−equicoverable by Lemma 1.3; if $G - G_0$ is not M_2−coverable, we first prove there must exist an edge e in $G - G_0$ such that e is v_1v_4 or v_3v_5.

We prove by contradiction. Suppose that neither v_1v_4 nor v_3v_5 is an edge of $G - G_0$.

Subcase 1: In $G - G_0$, there is only one edge e_i such that $G - G_0 - e_i$ is M_2-coverable. Since v_1v_4 and v_3v_5 are not the edges of $G - G_0$, e_i can not be adjacent to both e_3 and e_5. Assume that e_i is non-adjacent to e_3. Then $\{\{e_3, e_1\}, \{e_3, e_5\}, \{e_3, e_i\}, \{e_4, e_{03}\}, \{e_4, e_2\}\}$ is a minimal M_2-covering of $G_0 \cup e_i$ which is not minimum. So $G_0 \cup e_i$ is not M_2-equicoverable. By Lemma 1.3, G is not M_2-equicoverable.

Subcase 2: There exist exactly two adjacent edges e_i and e_j in $G - G_0$ such that $G - G_0 - e_i - e_j$ is M_2-coverable. In the same way, neither e_i nor e_j can be adjacent to both e_3 and e_5. Assume that neither e_i nor e_j is adjacent to e_3. Then $\{\{e_3, e_1\}, \{e_3, e_5\}, \{e_3, e_i\}, \{e_3, e_j\}, \{e_4, e_{03}\}, \{e_4, e_2\}\}$ is a minimal M_2-covering of $G_0 \cup e_i \cup e_j$ which is not minimum, so $G_0 \cup e_i \cup e_j$ is not M_2-equicoverable. By Lemma 1.3, G is not M_2-equicoverable.

Subcase 3: $G - G_0$ is $K_{1,k}(k \geq 3)$. Since neither v_1v_4 nor v_3v_5 is an edge of $G - G_0$, no edge of the star can be adjacent to e_3 and e_5 at the same time. In the same way, we can get a minimal M_2-covering of G which is not minimum. So G is not M_2-equicoverable.

This is a contradiction. So there must exist an edge e in $G - G_0$ such that e is v_1v_4 or v_3v_5. Without loss of generality, suppose $v_1v_4 \in E(G - G_0)$, and take $C' = v_1v_3v_4v_1$. Then $E(C) \cap E(C')$ has two edges. By the proof of Case 2, G must be $F_5 \cdot S_k$. □

3 Connected M_2-equicoverable graphs with circumference 4

Denote by $C_4 \cdot S_k$ the graph obtained from a cycle C_4 and a k-star $K_{1,k}(k \geq 0)$ by identifying one vertex of the cycle C_4 with the center of $K_{1,k}$. See the first figure shown as Figure 7 for $k = 3$.

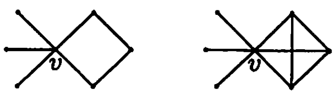


Figure 7

Theorem 3.1. *Let G be a connected graph with circumference 4 and girth 4. Then G is M_2-equicoverable if and only if G is $C_4 \cdot S_k(k \geq 0)$.*

Proof. It is easy to verify that $C_4 \cdot S_k(k \geq 0)$ is M_2-equicoverable.

Conversely, let G be an M_2−equicoverable connected graph with $c(G) = g(G) = 4$. So G can only contain 4−cycles. Let $C = v_1v_2v_3v_4v_1$ be a 4−cycle of G and $v_1v_2 = e_1$, $v_2v_3 = e_2$, $v_3v_4 = e_3$, $v_4v_1 = e_4$. If G is C, it obviously satisfies the condition; if G is not a cycle, there must exist an edge $e_0 \in E(G - C)$ such that e_0 is incident to some vertex of C. Without loss of generality, let $e_0 = v_1v_0$ and $C \cup e_0 = G_0$. Then the following holds:

(1) In $G - G_0$, v_0 has no incident edge.

Otherwise, if v_0 has an incident edge $e = v_0u_0$ outside G_0, let $G_0 \cup e = G_1$, shown as Figure 8(a).

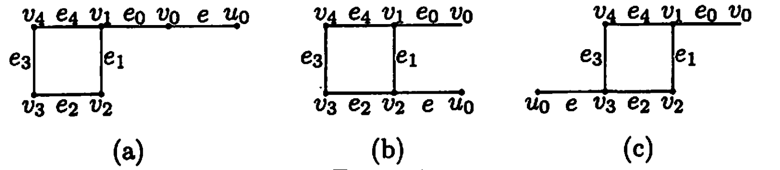


Figure 8

Obviously, G_1 is not M_2−equicoverable. If $G - G_1$ is M_2−coverable, G is not M_2−equicoverable by Lemma 1.3; if $G - G_1$ is not M_2−coverable, since G contains no 3−cycle, there are three possibilities:

Case 1: In $G - G_1$, there is only one edge e_i such that $G - G_1 - e_i$ is M_2−coverable. Since e_i can not be adjacent to all of e, e_1, e_2, e_3 at the same time, suppose that e_i is non-adjacent to e_2. Then $\{\{e_2, e_0\}, \{e_2, e_4\}, \{e_2, e_i\}, \{e_1, e_3\}, \{e_1, e\}\}$ is a minimal M_2−covering of $G_1 \cup e_i$ which is not minimum. So $G_1 \cup e_i$ is not M_2−equicoverable. By Lemma 1.3, G is not M_2−equicoverable.

Case 2: In $G - G_1$, there exist exactly two adjacent edges e_i and e_j such that $G - G_1 - e_i - e_j$ is M_2−coverable. Since G only contains 4−cycles, there is at least one edge (let it be e) among e, e_1, e_2, e_3 which is non-adjacent to both e_i and e_j. Then $\{\{e, e_1\}, \{e, e_3\}, \{e, e_i\}, \{e, e_j\}, \{e_2, e_0\}, \{e_2, e_4\}\}$ is a minimal M_2−covering of $G_1 \cup e_i \cup e_j$ which is not minimum. So $G_1 \cup e_i \cup e_j$ is not M_2−equicoverable. By Lemma 1.3, G is not M_2−equicoverable.

Case 3: $G - G_1$ is $K_{1,k} (k \geq 3)$. Denote the k edges of the star by $e_{11}, e_{12}, \cdots, e_{1k}$. Since G contains only 4−cycles, there exists at least one edge(assume it is e_3) among e, e_1, e_2, and e_3 which is non-adjacent to all the edges of the star. Then $\{\{e_2, e_4\}, \{e_2, e_0\}, \{e_3, e\}, \{e_3, e_1\}, \{e_3, e_{11}\}, \{e_3, e_{12}\}, \cdots, \{e_3, e_{1k}\}\}$ is a minimal M_2−covering of G which is not minimum. So G is not M_2−equicoverable.

This is a contradiction.

(2) In $G-G_0$, v_2 has no incident edge. By symmetry, v_4 has no incident edge, either.

Otherwise, if v_2 has an incident edge $e = v_2u_0$ outside G_0, let $G_0 \cup e = G_1$, shown as Figure 8(b). It is easy to see that G_1 is not M_2-equicoverable. If $G - G_1$ is M_2-coverable, G is not M_2-equicoverable by Lemma 1.3; if $G - G_1$ is not M_2-coverable, since G contains no 3−cycle, there are three possibilities:

Case 1: In $G - G_1$, there is only one edge e_i such that $G - G_1 - e_i$ is M_2-coverable. Since G contains no 3−cycle, e_i can not be adjacent to e_1 and e_3 at the same time. Assume that e_i is non-adjacent to e_1. Then $\{\{e_2, e_0\}, \{e_2, e_4\}, \{e_1, e_i\}, \{e_1, e_3\}, \{e_3, e\}\}$ is a minimal M_2-covering of $G_1 \cup e_i$ which is not minimum(whose minimum M_2-covering uses 4 copies of M_2). So $G_1 \cup e_i$ is not M_2-equicoverable. By Lemma 1.3, G is not M_2-equicoverable.

Case 2: In $G - G_1$, there are exactly two edges e_i and e_j such that $G - G_1 - e_i - e_j$ is M_2-coverable. Since G contains no 3−cycle, e_i can not be adjacent to both e_1 and e_3 at the same time, e_j can not be adjacent to e_1 and e_3 at the same time. Suppose that e_i is non-adjacent to e_1 and e_j is non-adjacent to e_3. Then $\{\{e_2, e_0\}, \{e_2, e_4\}, \{e_1, e_i\}, \{e_1, e_3\}, \{e_3, e\}, \{e_3, e_j\}\}$ is a minimal M_2-covering of $G_1 \cup e_i \cup e_j$ which is not minimum. So $G_1 \cup e_i \cup e_j$ is not M_2-equicoverable. By Lemma 1.3, G is not M_2-equicoverable.

Case 3: $G - G_1$ is $K_{1,k} (k \geq 3)$. Denote the k edges of the star by $e_{11}, e_{12}, \cdots, e_{1k}$.

Subcase 1: The center of the star is not a vertex of G_1. Since G contains only 4−cycles, at least one vertex of v_1, v_2(let it be v_2) is not a leaf of the star. Then there exists an edge e such that $d_1(e) = k + 3$, $c_0(e) = 0$, $c(G) = \max\{k, 4, \lceil \frac{k+6}{2} \rceil\}$. When $k \geq 3$, $d_1(e) + c_0(e) > c(G)$. By Lemma 1.2, G is not M_2-equicoverable.

Subcase 2: The center of the star is v_1. Since G contains no 3−cycle, no edge of the star can be adjacent to e_2. Then $\{\{e_2, e_4\}, \{e_2, e_0\}, \{e_3, e\}, \{e_3, e_1\}, \{e_2, e_{11}\}, \{e_2, e_{12}\}, \cdots, \{e_2, e_{1k}\}\}$ is a minimal M_2-covering of G which is not minimum. So G is not M_2-equicoverable.

By symmetry, if the center of the star is v_2, G is not M_2-equicoverable, either.

Subcase 3: The center of the star is v_3. Since G contains no 3−cycle, no edge of the star is adjacent to e_4. Then $\{\{e_4, e_2\}, \{e_4, e\}, \{e_3, e_0\}, \{e_3, e_1\}, \{e_4, e_{11}\}, \{e_4, e_{12}\}, \cdots, \{e_4, e_{1k}\}\}$ is a minimal M_2-covering of G which is not minimum. And G is not M_2-equicoverable.

By symmetry, if the center of the star is v_4, G is not M_2–equicoverable.

Subcase 4: The center of the star is v_0. No edge of the star can be adjacent to e_4. Then $\{\{e_4, e_2\}, \{e_4, e\}, \{e_3, e_0\}, \{e_3, e_1\}, \{e_4, e_{11}\}, \{e_4, e_{12}\}, \cdots, \{e_4, e_{1k}\}\}$ is a minimal M_2–covering of G which is not minimum. G is not M_2–equicoverable.

By symmetry, if the center of the star is u_0, G is not M_2–equicoverable.

(3) In $G - G_0$, v_3 has no incident edges.

Otherwise, if v_3 has an incident edge $e = v_3u_0$ outside G_0, let $G_0 \cup e = G_1$, shown as Figure 8(c). It is easy to see that G_1 is not M_2–equicoverable. Similar to the proof of (2), we can prove that G is not M_2–equicoverable.

From the above, only v_1 can have incident edges in $G - G_0$. From the proof of (1), the paths beginning with v_1 in $G - G_0$ have length no more than 1. So if G is M_2–equicoverable, G must be $C_4 \cdot S_k$. □

We denote a graph obtained from a complete graph K_4 and a k–star $K_{1,k}(k \geq 0)$ by identifying one vertex of K_4 with the center of $K_{1,k}$ by $K_4 \cdot S_k$. See the second graph shown as Figure 7 for $k = 3$.

Theorem 3.2. *Let G be a connected graph with circumference* 4 *and girth* 3*. Then G is M_2–equicoverable if and only if G is the graph $K_4 \cdot S_k(k \geq 0)$ or belongs to one of the two families listed in Figure 9 (where v has k neighbors with degree* 1*. In the first family, $k \geq 0$; in the second family, $k \geq 1$).*

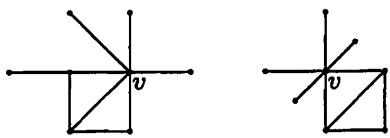


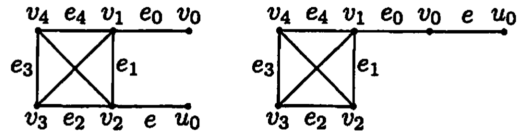
Figure 9

Proof. It is easy to verify that $K_4 \cdot S_k(k \geq 0)$ and the two families list in Figure 9 are all M_2–equicoverable.

Conversely, let G be an M_2–equicoverable connected graph with $c(G) = 4$ and $g(G) = 3$. So G must contain a 4–cycle and a 3–cycle. We call the edge joining two non-adjacent vertices of the cycle a diagonal. Let $C = v_1v_2v_3v_4v_1$ be a 4–cycle of G and $v_1v_2 = e_1$, $v_2v_3 = e_2$, $v_3v_4 = e_3$, $v_4v_1 = e_4$.

Case 1: C has diagonals.

Subcase 1: C has two diagonals. C and its two diagonals induce a complete graph K_4. It is easy to see that the complete graph K_4 is M_2–equicoverable. If G contains other edges besides the edges of K_4, by connection and symmetry, we can assume that G also contains an edge $e_0 = v_1 v_0$. If G has a subgraph shown as 10(a), let G_0 be the subgraph induced by $\{e_0, e_1, e_2, e_3, e_4, e\}$. It is easy to see G_0 is not M_2−equicoverable. Since there exists no edge in $G - G_0$ which is adjacent to both $v_1 v_3$ and $v_2 v_4$, there exists no edge in $G - G_0$ which is adjacent to all the other edges. So $G - G_0$ is M_2−coverable. By Lemma 1.3, G is not M_2−equicoverable which is a contradiction. So G doesn't contain a subgraph which is isomorphic to the graph shown as Figure 10(a). In a similar way, G doesn't contain a subgraph which is isomorphic to the graph shown as Figure 10(b).



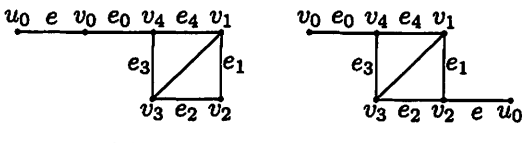
(a) (b)

Figure 10

So outside the complete graph K_4, neither $v_2(v_3, v_4)$ nor v_0 has incident edges and the paths beginning with v_1 have length no more than 1. And G must be the graph $K_4 \cdot S_k (k \geq 0)$.

Subcase 2: C has only one diagonal, and we can let it be $v_1 v_3$. Since G is M_2−coverable, v_2 or v_4 must have incident edges in $G - C - v_1 v_3$. Assume that v_4 has an incident edge $e_0 = v_4 v_0$. Denote by G_0 the subgraph induced by $\{e_0, e_1, e_2, e_3, e_4, v_1 v_3\}$. Then the following statements are true:

(1) G doesn't contain subgraphs isomorphic to the two graphs as shown in Figure 11; that is, in $G - G_0$, v_0 and v_2 have no other incident edges, and the paths beginning with v_4 in $G - G_0$ have length no more than 1.



(a) (b)

Figure 11

Otherwise, G contains a subgraph shown as Figure 11(a); that is, v_0 has an incident edge $e = v_0 u_0$. Denote by G_1 the subgraph induced by $\{e_0, e_1, e_2, e_3, e_4, e\}$. It is easy to see that G_1 is not M_2−equicoverable. If $G - G_1$ is M_2−coverable, G is not M_2−equicoverable by Lemma 1.3; if $G - G_1$ is not M_2−coverable, there are three possibilities.

(i) There exists only one edge e_i in $G - G_1$ such that $G - G_1 - e_i$ is M_2−coverable. If e_i is v_1v_3, $\{\{e_3, e\}, \{e_4, e\}, \{e, e_i\}, \{e_0, e_1\}, \{e_0, e_2\}\}$ is a minimal M_2−covering of $G_1 \cup e_i$ which is not minimum. If e_i is not v_1v_3, e_i must be adjacent to v_1v_3. Since $c(G) = 4$ and e_i is not adjacent to e, $\{\{e_3, e\}, \{e_4, e\}, \{e, e_i\}, \{e_0, e_1\}, \{e_0, e_2\}\}$ is a minimal M_2−covering of $G_1 \cup e_i$ which is not minimum. So $G_1 \cup e_i$ is not M_2−equicoverable, and G is not M_2−equicoverable.

(ii) There exist exactly two adjacent edges e_i and e_j in $G - G_1$ such that $G - G_1 - e_i - e_j$ is M_2−coverable. As (i), neither e_i nor e_j can be adjacent to e. Then $\{\{e_3, e\}, \{e_4, e\}, \{e, e_i\}, \{e, e_j\}, \{e_0, e_1\}, \{e_0, e_2\}\}$ is a minimal M_2−covering of $G_1 \cup e_i$ which is not minimum. Thus G is not M_2−equicoverable.

(iii) $G - G_1$ is $K_{1,k}(k \geq 3)$ (or K_3). In the same way, any edge of the star (or K_3) can not be adjacent to e. These edges and e form k(or 3) copies of M_2 which along with $\{e_3, e\}$, $\{\{e_4, e\}, \{e_0, e_1\}, \{e_0, e_2\}\}$ constitute a minimal M_2−covering of G which is not minimum. So G is not M_2−equicoverable.

In the same way, if G contains a subgraph shown as Figure 11(b), G is not M_2−equicoverable, which is a contradiction.

(2) G doesn't contain subgraphs isomorphic to the three graphs shown as Figure 12. In $G - G_0$, only one vertex of v_1, v_3 and v_4 can have incident edges, and the paths beginning with v_1 have length no more than 1.

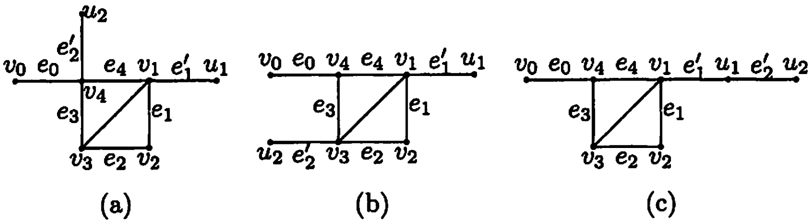


Figure 12

Otherwise, G contains a subgraph G'. We can take a non-M_2−equicoverable subgraph G_1 of G' with size 6 such that $G' - G_1$ is a copy of M_2(Thus $G - G_1$ can not be a star or K_3). In the same way as before, we can prove that G is not M_2−equicoverable, which is a contradiction.

From (1) and (2), G must be the two kinds of graphs of Figure 9.

Case 2: C has no diagonal.

Let $C' = v_{01}v_{02}v_{03}v_{01}$ be a 3−cycle of G and $v_{01}v_{02} = e_{01}$, $v_{02}v_{03} = e_{02}$, $v_{03}v_{01} = e_{03}$. Since G contains no 5−cycle and C has no diagonal, C and

C' have no common edges. We have two subcases.

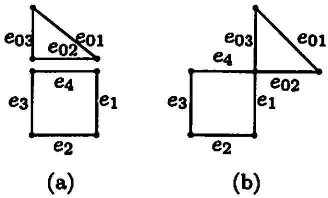


Figure 13

Subcase 1: C and C' have no common vertices, shown as Figure 13(a). Denote by G_0 the subgraph induced by $\{e_{01}, e_{02}, e_{03}, e_1, e_2, e_3\}$. It is easy to get that G_0 is not M_2−equicoverable. If $G - G_0$ is M_2−coverable, G is not M_2−equicoverable; if $G - G_0$ is not M_2−coverable, there are three possibilities:

(i) There exists only one edge e_i in $G - G_0$ such that $G - G_0 - e_i$ is M_2−coverable. Since $e_4 \in E(G - G_0)$, e_i is e_4 or e_i is adjacent to e_4. Because C has no diagonal, e_i can not be adjacent to e_2. Then $\{\{e_2, e_{01}\}, \{e_2, e_{02}\}, \{e_2, e_{03}\}, \{e_1, e_3\}, \{e_2, e_i\}\}$ is a minimal M_2−covering of $G_0 \cup e_i$. So $G_0 \cup e_i$ is not M_2−equicoverable. And G is not M_2−equicoverable.

(ii) There exist exactly two adjacent edges e_i and e_j in $G - G_0$ such that $G - G_0 - e_i - e_j$ is M_2−coverable. In the same way, neither e_i nor e_j can be adjacent to e_2. Then $\{\{e_2, e_{01}\},\{e_2, e_{02}\}, \{e_2, e_{03}\}, \{e_2, e_i\}, \{e_2, e_j\}, \{e_1, e_3\}\}$ is a minimal M_2−covering of $G_0 \cup e_i \cup e_j$ which is not minimum. $G_0 \cup e_i \cup e_j$ is not M_2−equicoverable. So G is not M_2−equicoverable.

(iii) $G - G_0$ is $K_{1,k}(k \geq 3)$(or K_3). In the same way, all the edges of the star(or K_3) are non-adjacent to e_2. All the k edges together with e_2 form k(or 3)copies of M_2, which along with $\{e_2, e_{01}\},\{e_2, e_{02}\}, \{e_2, e_{03}\}, \{e_1, e_3\}$ constitute a minimal M_2−covering of G which is not minimum. So G is not M_2−equicoverable.

Subcase 2: C and C' have exactly one common vertex, shown as Figure 13(b). Denote by G_0 the subgraph induced by $\{e_{01}, e_{02}, e_{03}, e_1, e_2, e_3\}$. It is easy to see that G_0 is not M_2−coverable. Similar to the proof of Subcase 1, we can get that G is not M_2−equicoverable.

So Case 2 is impossible.

From the above, if the graph G is M_2−equicoverable, G is the graph $K_4 \cdot S_k(k \geq 0)$ or belongs to one of the two families listed in Figure 9 (where v has k neighbors with degree 1. In the first family, $k \geq 0$; in the second family, $k \geq 1$). □

4 Connected M_2–equicoverable graphs with circumference 3

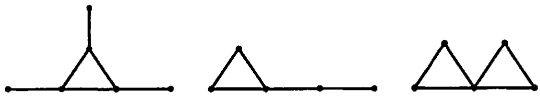


Figure 14

For the connected graphs with circumference 3, when the size is no more than 6, by method of exhaustion, we can easily verify that only the three graphs shown as Figure 14 are M_2–equicoverable; when the size is more than 6, to get the result, we first give the following lemma:

Lemma 4.1. *Let G be a connected graph with circumference 3 which is not a cycle. If G is M_2–equicoverable, G doesn't contain any subgraph shown as Figure 15.*

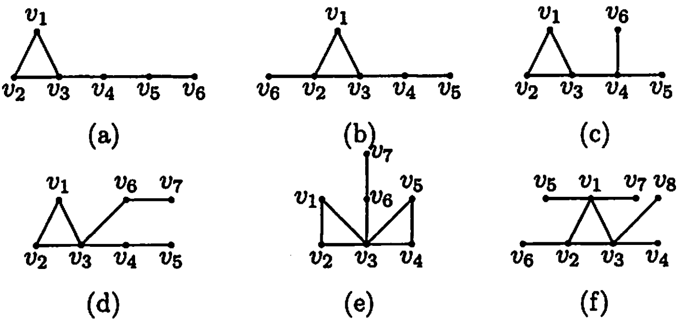


Figure 15

Proof. We prove by contradiction.

(1) If G contains a subgraph shown as Figure 15(a), we denote it by G_0. It is easy to see that G_0 is not M_2–equicoverable. If $G - G_0$ is M_2–coverable, G is not M_2–equicoverable by Lemma 1.3; if $G - G_0$ is not M_2–coverable, there are three possibilities:

(i) In $G - G_0$, there exists only one edge e_i such that $G - G_0 - e_i$ is M_2–coverable. Since G doesn't contain n–cycles($n \geq 4$), e_i can not be adjacent to both v_1v_2 and v_4v_5. Suppose that e_i is non-adjacent to v_1v_2. Then $\{\{v_1v_2, v_3v_4\}, \{v_1v_2, v_5v_6\}, \{v_4v_5, v_1v_3\}, \{v_4v_5, v_2v_3\}, \{v_1v_2, e_i\}\}$ is a minimal M_2–covering of $G_0 \cup e_i$ which is not minimum. $G_0 \cup e_i$ is not M_2–equicoverable. And G is not M_2–equicoverable.

(ii) In $G - G_0$, there exist two adjacent edges e_i and e_j such that $G - G_0 - e_i - e_j$ is M_2−coverable. In the same way, neither e_i nor e_j can be adjacent to both v_1v_2 and v_4v_5. Suppose that e_i is non-adjacent to v_1v_2 and e_i is non-adjacent to v_4v_5. Then $\{\{v_1v_2, v_3v_4\}, \{v_1v_2, v_5v_6\}, \{v_4v_5, v_1v_3\}, \{v_4v_5, v_2v_3\}, \{v_1v_2, e_i\}, \{v_4v_5, e_j\}\}$ is a minimal M_2−covering of $G_0 \cup e_i \cup e_j$ which is not minimum. Thus $G_0 \cup e_i \cup e_j$ is not M_2−equicoverable. And G is not M_2−equicoverable.

(iii) $G - G_0$ is $K_{1,k}(k \geq 3)$(or K_3). In the same way, no edge of the star or K_3 can be adjacent to both v_1v_2 and v_4v_5. All the edges of $K_{1,k}$(or K_3) using v_1v_2 or v_4v_5 form k(or 3) copies of M_2 which along with $\{v_1v_2, v_3v_4\}$, $\{v_1v_2, v_5v_6\}$, $\{v_4v_5, v_1v_3\}$, $\{v_4v_5, v_2v_3\}$ constitute a minimal M_2−covering of G which is not minimum. So G is not M_2−equicoverable.

In the same way, if G contains a subgraph shown as Figure 15(b) or (c), G is not M_2−equicoverable.

(2) If G contains a subgraph shown as Figure 15(d), denote by G_0 the subgraph induced by $\{v_1v_2, v_2v_3, v_3v_4, v_3v_1, v_4v_5, v_6v_7\}$. It is easy to see that G_0 is not M_2−coverable. If $G - G_0$ is M_2−coverable, G is not M_2−equicoverable by Lemma 1.3; if $G - G_0$ is not M_2−coverable, there are three possibilities:

(i) In $G - G_0$, there exists only one edge e_i such that $G - G_0 - e_i$ is M_2−coverable. Since $v_3v_6 \in E(G - G_0)$ and G doesn't contain cycles with length more than 3, e_i can not be adjacent to both v_1v_2 and v_4v_5 whether e_i is v_3v_6 or e_i is adjacent to v_3v_6. Suppose e_i is non-adjacent to v_1v_2. Then $\{\{v_1v_2, v_3v_4\}, \{v_1v_2, v_6v_7\}, \{v_4v_5, v_1v_3\}, \{v_4v_5, v_2v_3\}, \{v_1v_2, e_i\}\}$ is a minimal M_2−covering of $G_0 \cup e_i$ which is not minimum. So $G_0 \cup e_i$ is not M_2−equicoverable.

(ii) In $G - G_0$, there exist two edges e_i and e_j such that $G - G_0 - e_i - e_j$ is M_2−coverable. In the same way, neither e_i nor e_j can be adjacent to both v_1v_2 and v_4v_5. Suppose that e_i is non-adjacent to v_1v_2 and e_i is non-adjacent to v_4v_5. Then $\{\{v_1v_2, v_3v_4\}, \{v_1v_2, v_6v_7\}, \{v_4v_5, v_1v_3\}, \{v_4v_5, v_2v_3\}, \{v_1v_2, e_i\}, \{v_4v_5, e_j\}\}$ is a minimal M_2−covering of $G_0 \cup e_i \cup e_j$ which is not minimum. Thus $G_0 \cup e_i \cup e_j$ is not M_2−equicoverable.

(iii) $G - G_0$ is $K_{1,k}(k \geq 3)$(or K_3). Similarly, no edge of the star (or K_3) can be adjacent to both v_1v_2 and v_4v_5. All the edges of $K_{1,k}$(or K_3) using v_1v_2 or v_4v_5 form k(or 3) copies of M_2 which along with $\{v_1v_2, v_3v_4\}$, $\{v_1v_2, v_6v_7\}$, $\{v_4v_5, v_1v_3\}$, $\{v_4v_5, v_2v_3\}$ constitute a minimal M_2−covering of G that is not minimum.

So G is not M_2−equicoverable.

(3) If G contains a subgraph shown as Figure 15(e), denote by G_0 the

subgraph induced by $\{v_1v_2, v_2v_3, v_3v_4, v_3v_1, v_3v_5, v_6v_7\}$. It is easy to see that G_0 is not M_2–equicoverable. Since G doesn't contain cycles with length more than 3 and there exists no edge in $G - G_0$ which is adjacent to both v_3v_6 and v_4v_5, $v_3v_6 \in E(G - G_0)$, $v_4v_5 \in E(G - G_0)$, there exists no edge in $G - G_0$ which is adjacent to all the other edges. So $G - G_0$ is M_2–coverable. By Lemma 1.3, G is not M_2–equicoverable.

In the same way, if G contains a subgraph shown as Figure 15(f), G is not M_2– equicoverable.

So G doesn't contain any subgraph shown as Figure 15. □

Theorem 4.2. *Let G be a connected graph with circumference 3 and size more than 6. Then G is M_2–equicoverable if and only if G belongs to the three families listed in Figure 16, where v has $k(k \geq 1)$ neighbors of degree 1(In the first two families, $k \geq 1$; in the third family, $k \geq 0$).*

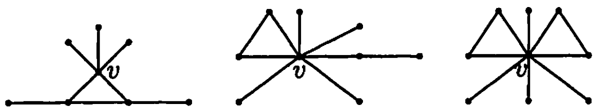


Figure 16

Proof. It is easy to verify the three families listed in Figure 16 are all connected M_2–equicoverable graphs with circumference 3.

Conversely, let G be a connected M_2–equicoverable graph with circumference 3 and size more than 6. Arbitrarily take a connected subgraph G_0 with size 6 from G. By Lemma 4.1, G_0 can not be any graph shown as Figure 15(a), (b), (c). So up to isomorphism, there are 5 possibilities for G_0 as shown in Figure 17.

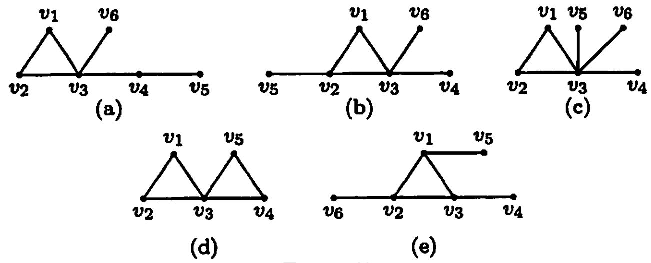


Figure 17

Case 1: G_0 is shown as Figure 17(a). By Lemma 4.1, none of v_1, v_2, v_4, v_5, v_6 has incident edges in $G - G_0$, and the paths beginning with v_3 have length no more than 1. So G belongs to the second family listed in Figure 16.

Case 2: G_0 is shown as Figure 17(b). By Lemma 4.1, none of v_4, v_5, v_6 has incident edges in $G - G_0$, and the paths beginning with v_1 and v_3 have length no more than 1. Since G is M_2−equicoverable, G_0 is not M_2−coverable, v_1 must have incident edges in $G - G_0$. By Lemma 4.1, v_1 must have only one incident edge in $G - G_0$. So G belongs to the first family listed in Figure 16.

Case 3: G_0 is shown as Figure 17(c). Since G is M_2−equicoverable, at least one of v_4, v_5, v_6 has incident edge in $G - G_0$. Suppose that v_4 has incident edge v_4v_7 in $G - G_0$. By Lemma 4.1, none of v_1, v_2, v_5, v_6, v_7 has incident edges in $G - G_0$ and the paths beginning with v_3 have length no more than 1. So G belongs to the second family listed in Figure 16.

Case 4: G_0 is shown as Figure 17(d). By Lemma 4.1, none of v_1, v_2, v_4, v_5 has incident edges in $G - G_0$, and the paths beginning with v_3 have length no more than 1. So G belongs to the third family listed in Figure 16.

Case 5: G_0 is shown as Figure 17(e). By Lemma 4.1, none of v_4, v_5, v_6 has incident edges in $G - G_0$, and all the paths beginning with v_1, v_2, v_3 have length no more than 1. So G belongs to the first family listed in Figure 16. □

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