On product cordial graphs

M. A. SEOUD [1] and E. F. HELMI [2]
Department of Mathematics, Faculty of Science, Ain Shams University,
Abbassia, Cairo, Egypt.

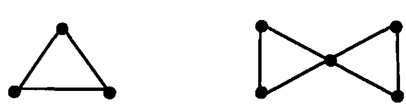
Abstract
Here, we determine all graphs of order less than 7 which are not product cordial. Also, we give some families of graphs which are product cordial.

0. Introduction

Sundaram, Ponraj and Somasundaram [3] introduced the notion of product cordial labelings. A product cordial labeling of a graph G with vertex set V is a function f from V to $\{0,1\}$, such that if each edge uv is assigned the label $f(u)f(v)$, the number of vertices labeled with 0 and the number of vertices labeled with 1 differ by at most 1, and the number of edges labeled with 0 and the number of edges labeled with 1 differ by at most 1.

A graph with a product cordial labeling is called a product cordial graph. In [3] and [4] Sundaram, Ponraj and Somasundaram prove that the following graphs are product cordial. Trees; unicyclic graphs of odd order; triangular snakes; helms; $P_m \cup P_n$; $C_m \cup P_n$; $P_m \cup K_{1,n}$; $W_m \cup F_n$ (F_n is the fan P_n+K_1); $K_{1,m} \cup K_{1,n}$; $W_m \cup K_{1,n}$; $W_m \cup P_n$; $W_m \cup C_n$; the total graph of P_n (the total graph of P_n has vertex set $V(P_n) \cup E(P_n)$ with two vertices adjacent whenever they are neighbors in P_n); $K_2 + mK_1$ if and only if m is odd; $C_m \cup P_n$ if and only if $m+n$ is odd; $K_{m,n} \cup P_s$ if $s \geq mn$; $C_{n+2} \cup K_{1,n}$; $K_n \cup K_{n,(n-1)/2}$ when n is odd; $K_n \cup K_{n-1,n/2}$ when n is even; and P^2_n if and only if n is odd. They also prove that $K_{m,n}$, $m,n > 2$; $P_m \times P_n$, $m,n > 2$, and wheels are not product cordial.

They also claim that the following graphs are product cordial: dragons; C_n if and only if $n < 4$; and if a (p,q)-graph is a product cordial graph, then $q < (p-1)(p+1)/4$. In the text we show that the first and the second claims are not true. For the third we have the following counter examples, which are product cordial, where $p = 3$, $q = 3$, and $p = 5$, $q = 6$.



Here, we show some short theorems and give a survey of all graphs of order ≤ 6 which are not product cordial. Also, we show that, the complete graphs K_n are not product cordial for all $n \geq 4$, and the following graphs are product cordial:

the cycle C_n if and only if n is odd ; the gear graph G_m if and only if m is odd; the web $\overline{W_n}$ for all n; the corona $C_n \odot \overline{K_w}$ for all $w \geq 1$, $n \geq 3$; the conjunctions : $P_m \wedge P_n$ for $n,m \geq 2$; $P_n \wedge S_m$, $n,m \geq 2$; the corona graph $T_n \odot K_1$ for $n \geq 2$ and the C_4−snake if and only if n is odd .

Throughout this paper , we use the standard notations and conventions as in [1] and [2].

1.General Theorems

Theorem 1.1. Let G be a connected graph of even order p, $p \geq 6$ and size q .If G contains exactly one cycle of order $p/2$, then G is product cordial if $q = p-1$, p or $p+1$.

Proof. Let G be a graph of even order p , $p \geq 6$ and size q , G contain a cycle of order $p/2$, and let the number of vertices of G labeled with 1 be $p/2$, and also the number of vertices of G labeled with 0 be $p/2$, such that the vertices of the cycle are labeled 1. Since the number of edges labeled with 1of the cycle is $p/2$, the number of the remaining edges of G labeled with 0 is $q - (p/2)$. If $q = p-1, p$ or $p+1$, then G is product cordial .

Theorem 1.2. Let G be a connected graph of odd order p, $p \geq 5$ and size q .If G contains exactly one cycle of order $(p+1)/2$,then G is product cordial if $q = p$, $p+1$ or $p+2$.

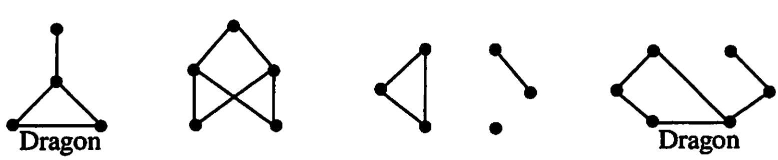
Proof. Let G be a graph of odd order p , $p \geq 5$ and size q , G contain a cycle of order $(p+1)/2$ and let the number of vertices of G labeled with 1 be $(p+1)/2$, and the number of vertices of G labeled with 0 be $(p-1)/2$,such that the vertices of the cycle are labeled 1. Since the number of edges labeled with 1 of the cycle is $(p+1)/2$, the number of the remaining edges of G labeled with 0 is $q-[(p+1)/2]$. If $q = p$, $p+1$ or $p+2$,then G is product cordial .

Theorem 1.3. All graphs of order ≤ 6 are product cordial, except the following 100 graphs

(i) C_4, C_6, K_4, $K_3 \cup K_2$,P_4^2.

(ii) If $p = 5$ or $p = 6$ and $q \geq 8$, where p is the order and q is the size of the graph.

(iii) the following graphs of order ≤ 6 :



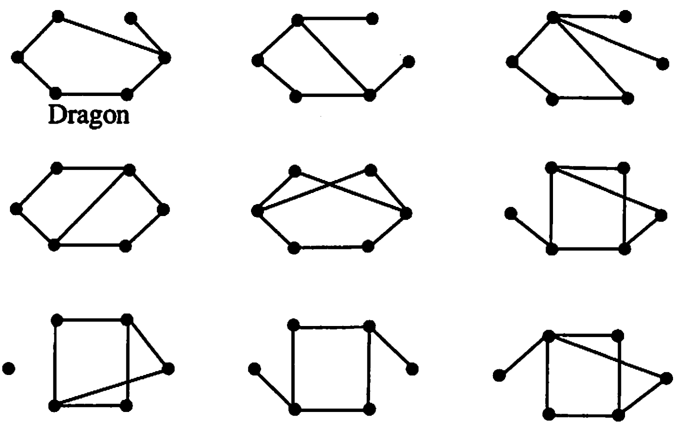


Figure 1.

Proof.

(i) By Theorem 2.2 , C_4 , C_6 are not product cordial ; by Theorem 2.1, K_4 is not product cordial ; it is immediate that $K_3 \cup K_2$ is not product cordial; and by theorem due to Sundaram ,Ponraj and Somasundaram [3],[4], P_4^2 is not product cordial.

(ii) Since $p = 5$ or $p = 6$, the maximum number of vertices labeled 1 is 3. These vertices induce, a maximum number of edges labeled 1, which is 3. But $q \geq 8$, consequently the graphs cannot be product cordial.

(iii) These graphs are shown to be not product cordial by giving all possible product cordial labelings assigned to the vertices.

2. Families of product cordial graphs

Theorem 2.1. The complete graphs K_n are not product cordial for all $n \geq 4$.

Proof.

<u>If *n* is even</u>

The number of vertices labeled with 1and also the number of vertices labeled with 0 is $n/2$.Since every pair of distinct vertices are adjacent the number of edges labeled with 1 is $\binom{n/2}{2}$ and the number of edges labeled with 0 is

$\binom{n}{2} - \binom{n/2}{2} > \binom{n/2}{2} + 1$, $n \geq 4$, then the graphs K_n are not product cordial for n is even, $n \geq 4$.

<u>If *n* is odd</u>

Let the number of vertices labeled with 1 be $(n+1)/2$ and the number of vertices labeled with 0 is $(n-1)/2$. The same argument as if n is even. Labeling $(n+1)/2$ vertices with 0 does not alter the result.

Theorem 2.2. The cycle graph C_m is product cordial if and only if m is odd.
Proof. Let C_m be described as in Figure 2.

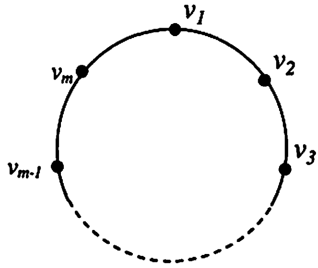


Figure 2.

We define the function $f: V(C_m) \to \{0,1\}$ as follows:

<u>**If m is odd**</u>

$$f(v_i) = \begin{cases} 1 & , \ i = 1,2,3,\ldots,(m+1)/2 \\ 0 & , \ (m+3)/2 \leq i \leq m \end{cases}$$

The number of vertices labeled with 1 is $(m+1)/2$, the number of vertices labeled with 0 is $(m-1)/2$, the number of edges labeled with 1 is $(m-1)/2$ and the number of edges labeled with 0 is $(m+1)/2$.

<u>**If m is even**</u>

The number of vertices labeled with 1and also the number of vertices labeled with 0 is $m/2$, the maximum number of edges labeled with 1 is $(m-2)/2$, the minimum number of edges labeled with 0 is $(m+2)/2$. Since the number of edges labeled with 0 and the number of edges labeled with 1 differ by at least 2, the graph C_m is not product cordial if m is even.

Definition 2.3.
The **gear** graph G_m is obtained from a wheel by adding a vertex between every pair of adjacent vertices of the cycle of the wheel.

Theorem 2.4. The gear graph G_m is product cordial if and only if m is odd.
Proof. Let G_m be described as in Figure 3.

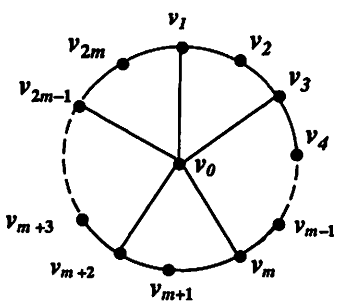


Figure 3.

G_m is a graph of size $3m$ and order $2m+1$.

If m is odd

We define the function $f: V(G_m) \to \{0,1\}$ as follows :

$$f(v_i) = \begin{cases} 1 & , \ i = 0,1,2,3,\ldots, m \\ 0 & , \ m+1 \le i \le 2m \end{cases}$$

The number of vertices labeled with 1 is $m+1$, the number of vertices labeled with 0 is m ,the number of edges labeled with 0 is $(3m+1)/2$ and the number of edges labeled with 1 is $(3m-1)/2$.

If m is even

It is easy to see that the maximum number of edges labeled by 1 is $(3m-2)/2$, while the minimum number of edges labeled by 0 is $(3m+2)/2$, hence the result.

Definition 2.5.

The **web graph** $\overline{W_n}$ is obtained by joining the pendant points of a helm to form a cycle and then adding a single pendant edge to each vertex of this outer cycle.

Theorem 2.6. The web graph $\overline{W_n}$ is product cordial for all $n \ge 3$.

Proof. Let $\overline{W_n}$ be described as in Figure 4.

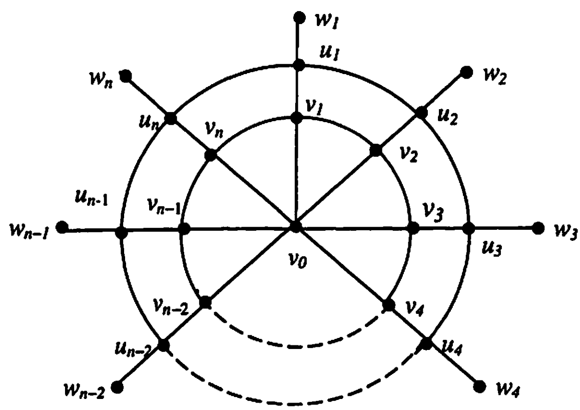


Figure 4.

$\overline{W_n}$ is a graph of size $5n$ and order $3n+1$. We define the function $f: V(\overline{W_n}) \to \{0,1\}$ as follows :

If n is even

$f(v_i) = 1 \quad , \ i = 0,1,2,3,\ldots, n \qquad , \qquad f(u_i) = \begin{cases} 1 & , \ i = 1,3,5,\ldots,n-1 \\ 0 & , \ i = 2,4,6,\ldots, n \end{cases}$

$f(w_i) = 0 \quad , \ i = 1,2,3,4,\ldots, n$

The number of vertices labeled with 1 is $(3n+2)/2$, the number of vertices labeled with 0 is $3n/2$, the number of edges labeled with 1 is $5n/2$, also the number of edges labeled with 0 is $5n/2$.

If n is odd

$f(u_n) = 0$,

$f(v_i) = 1$, $i = 0,1,2, \ldots, n$, $f(u_i) = \begin{cases} 1 & , \ i = 1,3,5,\ldots,n-2 \\ 0 & , \ i = 2,4,6,\ldots, n-1 \end{cases}$

$f(w_i) = 0$, $i = 1,2,3,\ldots, n$

The number of vertices labeled with 1 is $(3n+1)/2$, also the number of vertices labeled with 0 is $(3n+1)/2$,the number of edges labeled with 1 is $(5n-1)/2$, the number of edges labeled with 0 is $(5n+1)/2$.

Definition 2.7.

(i) A triangular snake T_n is obtained from from a path $u_1, u_2, \ldots, u_n$ by joining u_i and u_{i+1} to a new vertex v_i for $i = 1,2,\ldots,n-1$.

(ii) The corona $(G_1 \odot G_2)$ of two graphs G_1 and G_2 is the graph obtained by taking one copy of G_1 (which has p_1 vertices) and p_1 copies of G_2 , and then joining the j^{th} vertex of G_1 to every vertex in the j^{th} copy of G_2.

Theorem 2.8. The corona graph $T_n \odot K_1$ is product cordial for all $n \geq 2$.
Proof. Let $T_n \odot K_1$ be described as in Figure 5.

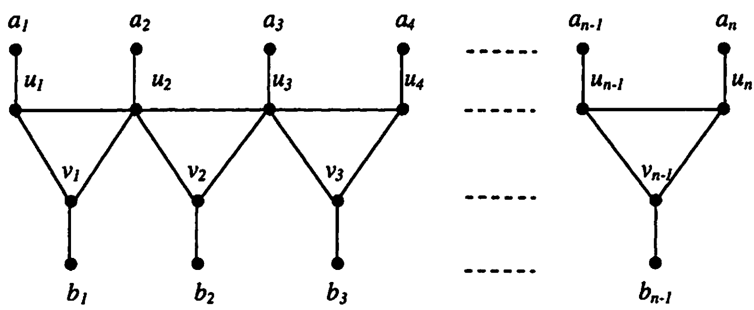


Figure 5.

The graph $T_n \odot K_1$ is a graph of order $2(2n-1)$ and size $5n-4$ .We define the function
$f: V(T_n \odot K_1) \rightarrow \{0,1\}$ as follows :

If n is odd

$f(a_i) = 0$, $1 \leq i \leq n$

$f(u_i) = 1$, $1 \leq i \leq n$

$f(v_i) = \begin{cases} 1 & , \ i = 1,3,5,\ldots, n-2 \\ 0 & , \ i = 2,4,6,\ldots, n-1 \end{cases}$, $f(b_i) = \begin{cases} 1 & , \ i = 1,3,5,\ldots, n-2 \\ 0 & , \ i = 2,4,6,\ldots, n-1 \end{cases}$

The number of vertices labeled with 1 and also the number of vertices labeled with 0 is $2n-1$, the number of edges labeled with 1 is $5(n-1)/2$ and the number of edges labeled with 0 is $(5n-3)/2$.

If n is even

$f(a_i) = 0 \quad , \; 1 \le i \le n$

$f(u_i) = 1 \quad , \; 1 \le i \le n \quad , \quad f(v_i) = \begin{cases} 1 \; , & i = 1,3,5,\ldots, n-1 \\ 0 \; , & i = 2,4,6,\ldots, n-2 \end{cases}$

$f(b_i) = 0 \quad , \; i = n-1$

$$f(b_i) = \begin{cases} 1 \; , & i = 1,3,5,\ldots, n-3 \\ 0 \; , & i = 2,4,6,\ldots, n-2 \end{cases}$$

The number of vertices labeled with 1 and also the number of vertices labeled with 0 is $2n-1$, the number of edges labeled with 1 and also the number of edges labeled with 0 is $(5n-4)/2$.

Theorem 2.9. The graph C_4- snake (some extension of the triangular snake) is product cordial if and only if n is odd.

Proof. Let C_4-snake be described as in Figure 6.

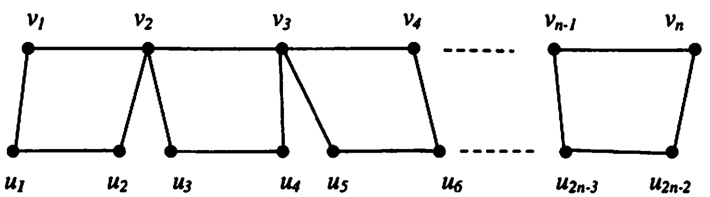


Figure 6.

The graph C_4-snake is a graph of size $4(n-1)$ and order $3n-2$. Let n be odd. We define the function $f\colon V(C_4-\text{snake}) \rightarrow \{0,1\}$ as follows :

$$f(v_i) = \begin{cases} 1 \; , & i = 1,2,3,\ldots, (n+1)/2 \\ 0 \; , & (n+3)/2 \le i \le n \end{cases} \quad , \quad f(u_i) = \begin{cases} 1 \; , & i = 1,2,3,\ldots, n-1 \\ 0 \; , & n \le i \le 2(n-1) \end{cases}$$

The number of vertices labeled with 1 is $(3n-1)/2$, the number of vertices labeled with 0 is $(3n-3)/2$ and the number of edges labeled with 1 is $2(n-1)$, the same for the number of edges labeled with 0.

If n is even, it is also easily seen that the maximum number of edges labeled by 1 is $2n-3$,while the minimum number of edges labeled by 0 is $2n-1$,and the result follows immediately.

Theorem 2.10. The graph $C_n \odot \overline{K_w}$ is product cordial for all $w \ge 1, n \ge 3$.

Proof. Let $C_n \odot \overline{K_w}$ be described as indicated in Figure 7.

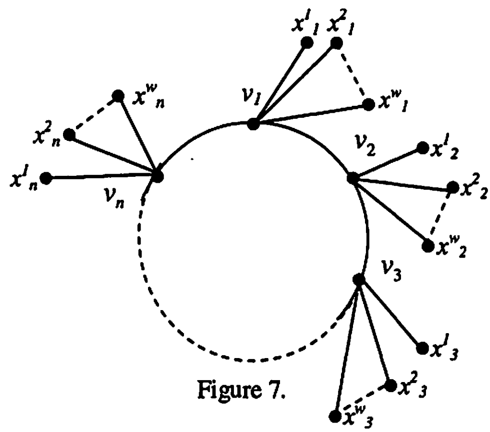


Figure 7.

The graph $C_n \odot \overline{K}_w$ is a graph of size $n + wn$ and order $n + wn$. We define the function $f : V(C_n \odot \overline{K}_w) \rightarrow \{0,1\}$ as follows :

<u>If w is odd</u>

$f(v_i) = 1 \quad , \ 1 \leq i \leq n$

$$f(x^j_i) = \begin{cases} 0 \ , & \text{if } j = 1,3,5,\ldots, w \\ 1 \ , & \text{if } j = 2,4,6,\ldots, w-1 \end{cases} \qquad 1 \leq i \leq n$$

<u>If w is even</u>

$f(v_i) = 1 \quad , \ 1 \leq i \leq n$

$$f(x^j_i) = \begin{cases} 0 \ , & \text{if } j = 1,3,5,\ldots, w-1 \ and \ 1 \leq i \leq n \\ 1 \ , & \text{if } j = 2,4,6,\ldots, w-2 \ and \ 1 \leq i \leq n \end{cases}$$

$$f(x^w_i) = \begin{cases} 1 \ , & i = 1,3,5,\ldots, n-1 \ \text{ if } n \text{ is even} \\ 0 \ , & i = 2,4,6,\ldots, n \ \text{ if } n \text{ is even} \\ 1 \ , & i = 2,4,6,\ldots, n-1 \ \text{ if } n \text{ is odd} \\ 0 \ , & i = 1,3,5,\ldots, n \ \text{ if } n \text{ is odd} \end{cases}$$

As before, it is immediate to check that the graph is product cordial.

Example 2.10 .

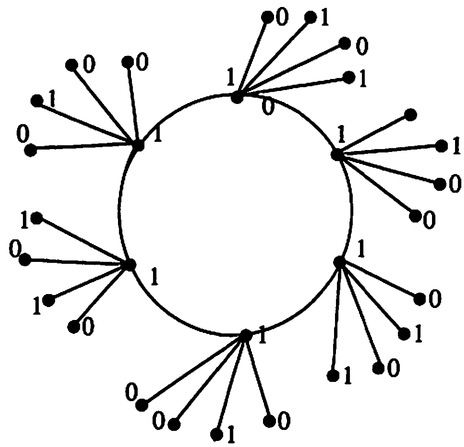


Figure 8.

Definition 2.11.
Let G_1 and G_2 be two graphs . The conjunction ($G_1 \wedge G_2$) of G_1 and G_2 is the graph having vertex set $V(G_1) \times V(G_2)$ and edge set $\{(u_1, v_1)(u_2, v_2) : u_1u_2 \in E(G_1)$ and $v_1v_2 \in E(G_2)\}$ [2] .

Theorem 2.12. The graphs $P_m \wedge P_n$, $n,m \geq 2$ are product cordial .
Proof. Let $P_m \wedge P_n$ be described as indicated in Figure 9.

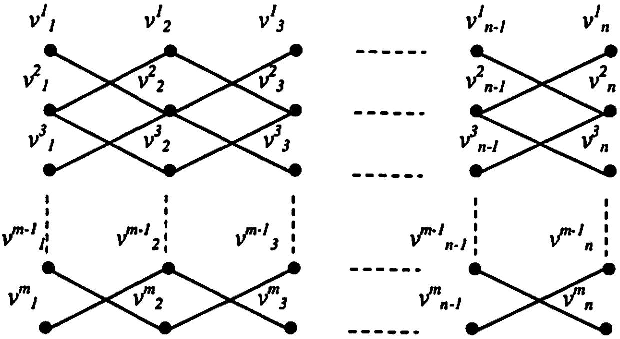


Figure 9.

The graph $P_m \wedge P_n$ is a graph of size $2(m-1)(n-1)$ and order mn . We define the function $f: V(P_m \wedge P_n) \rightarrow \{0,1\}$ as follows :

$$f(v^i_j) = \begin{cases} 0 , & i = 1,3,5,\ldots, m-1 \text{ (} m \text{ is even), or } m \text{ (} m \text{ is odd)}, \\ 1 , & i = 2,4,6,\ldots, m \text{ (} m \text{ is even), or } m-1 \text{ (} m \text{ is odd)}, \end{cases}$$
$$j = 1,3,5,\ldots, n-1 \text{ (} n \text{ is even), or } n \text{ (} n \text{ is odd)}$$

$$f(v^i_j) = \begin{cases} 1 , & i = 1,3,5,\ldots, m-1 \text{ (} m \text{ is even), or } m \text{ (} m \text{ is odd)}, \\ 0 , & i = 2,4,6,\ldots, m \text{ (} m \text{ is even), or } m-1 \text{ (} m \text{ is odd)}, \end{cases}$$
$$j = 2,4,6,\ldots, n \text{ (} n \text{ is even), or } n-1 \text{ (} n \text{ is odd)}$$

Theorem 2.13. The graphs $P_n \wedge S_m$, such that $S_m = \overline{K_m} + K_1$, $n, m \geq 2$ are product cordial .

Proof. Let $P_n \wedge S_m$ be described as indicated in Figure 10.

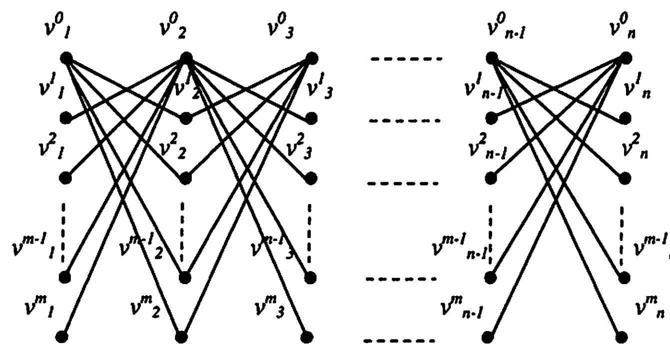


Figure 10.

The graph $P_n \wedge S_m$ is a graph of size $2(n-1)\,m$ and order $n\,(m+1)$. We define the following function $f: V(P_n \wedge S_m) \rightarrow \{0,1\}$ as follows :

<u>**If *n* is even, *m* is even or odd**</u>

$$f(v^0_j) = \begin{cases} 0 , & j = 2,4,6,\ldots, n \\ 1 , & j = 1,3,5,\ldots, n-1 \end{cases}$$

$$f(v^i_j) = \begin{cases} 0 , & i = 1,2,3,\ldots, m \text{ and } j = 1,3,5,\ldots, n \\ 1 , & i = 1,2,3,\ldots, m \text{ and } j = 2,4,6,\ldots, n \end{cases}$$

If n is odd,m is even or odd

$$f(v^0_j)=\begin{cases} 0 & , j=2,4,6,\ldots,n-1 \\ 1 & , j=1,3,5,\ldots,n \end{cases}$$

$$f(v^i_j)=\begin{cases} 0 & , i=1,2,3,\ldots,m \text{ and } j=1,3,5,\ldots,n-2 \\ 1 & , i=1,2,3,\ldots,m \text{ and } j=2,4,6,\ldots,n-1 \end{cases}$$

$$f(v^i_n)=\begin{cases} 0 & , i=1,2,3,\ldots,(m+2)/2 \text{ , } m \text{ is even ,or } (m+1)/2 \text{ ,} m \text{ is odd} \\ 1 & , (m+4)/2 \le i \le m \text{ , } m \text{ is even , or } (m+3)/2 \le i \le m \text{ , } m \text{ is odd} \end{cases}$$

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