On Two Bijections from $S_n(321)$ to $S_n(132)$

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Abstract

In [4], Elizalde and Pak gave a bijection $\Theta : S_n(321) \to S_n(132)$ that commutes with the operation of taking inverses and preserves the numbers of fixed points and excedances for every $\sigma \in S_n(321)$. In [1] it was shown that another bijection $\Gamma : S_n(321) \to S_n(132)$ introduced by Robertson in [7] has these same properties, and in [2] a pictorial reformulation of Γ was given that made it clearer why Γ has these properties. Our purpose here is to give a similar pictorial reformulation of Θ, from which it follows that, although the original definitions of Θ and Γ make them appear quite different, these two bijections are in fact related to each other in a very simple way, by using inversion, reversal, and complementation.

1. Introduction

If we write the permutation $\sigma \in S_n$ in one-line notation as $\sigma_1\sigma_2\ldots\sigma_n$, then a triple of entries $\sigma_i\sigma_j\sigma_k$ with $i < j < k$ is called a *321-pattern* if $\sigma_i > \sigma_j > \sigma_k$ and a *132-pattern* if $\sigma_j > \sigma_k > \sigma_i$. We denote by $S_n(321)$ (respectively, $S_n(132)$) the set of all $\sigma \in S_n$ that contain no 321-patterns (respectively, no 132-patterns).

Knuth gave a bijection from $S_n(321)$ to $S_n(132)$ in [5], by giving bijections from each set onto the set D_n of Dyck paths of length $2n$. (Knuth's bijection from $S_n(132)$ to D_n is the same as one given later by Krattenthaler [6], although Krattenthaler's definition of it was very different from Knuth's.) A number of other bijections from $S_n(321)$ to $S_n(132)$ have been given since Knuth's (see [3]). In [4], Elizalde and Pak exhibited a bijection Θ that commutes with the operation of taking inverses and preserves the numbers of fixed points and excedances for every $\sigma \in S_n(321)$. The map Θ was defined by composing Knuth's bijection from $S_n(321)$ to D_n with the inverse of a modification of Krattenthaler's bijection.

On the other hand, Robertson [7] gave a direct bijection $\Gamma : S_n(321) \to S_n(132)$ that iteratively replaces the smallest 132-pattern $\sigma_i\sigma_j\sigma_k$ by $\sigma_j\sigma_k\sigma_i$ ("smallest" meaning smallest in the lexicographic ordering of the triples (i, j, k) for all the 132-patterns present). Robertson conjectured at the 2005 Integers conference that Γ preserves the number of fixed points for every $\sigma \in S_n(321)$, and Bloom and Saracino proved in [1] that it preserves the number of fixed points and the number of excedances, and that it also commutes with the operation of taking inverses. In [2, Theorem 1], Bloom and Saracino gave a non-iterative, pictorial reformulation of Γ that provided a very different view of the map, and made the results on fixed points, excedances, and inverses more transparent.

Our purpose in the present paper is to give a pictorial reformulation of Θ, which, along with Theorem 1 of [2], shows that, despite the disparity in their original definitions, Θ and Γ are in fact related to each other in a very simple way (using inversion, reversal, and complementation). With this simple connection between Θ and Γ in hand, the results for Γ about fixed points, excedances, and inverses follow immediately from the corresponding results for Θ (and vice versa).

While our discovery of the connection between Θ and Γ was motivated by a desire to provide for Θ a pictorial reformulation to parallel that given for Γ in [2], one might also have approached the questions about Γ by trying to prove a connection between Γ and Θ at the outset, motivated by the similarity between the known properties of Θ and the (then) conjectured properties of Γ.

2. Background Definitions and Facts

Consider an $n \times n$ array of squares, and represent the square in the ith row from the top and the jth column from the left by (i, j). Let T be a set of squares in the array, and think of the squares in T as shaded and those not in T as unshaded.

Definition [2]. If $\sigma \in S_n$, we say that T is a *template* for σ if when (inductively) on each row of the array we place a dot in the first unshaded square from the left that has no dot above it, then the dots are in exactly the squares $(i, \sigma(i))$.

The pictorial reformulation of Γ in [2] was obtained by giving a simple way to turn a certain template for $\sigma \in S_n(321)$ into a template for $\Gamma(\sigma) \in S_n(132)$. The template for $\sigma \in S_n(321)$ was obtained by using what were called the *L-corners* of σ.

Definition [2]. If $\sigma \in S_n(321)$ and $\sigma_i\sigma_j$ is a 21-pattern in σ (i.e., if $i < j$ and $\sigma_i > \sigma_j$), then we call σ_i a *2-element* and σ_j a *1-element.*

It is clear that if $\sigma \in S_n(321)$ then no entry in σ can be both a 2-element and a 1-element. It is also easy to see that the 2-elements in σ are in increasing order from left to right, and so are the 1-elements.

Fact 1 ([2], Lemma 2). Suppose $\sigma \in S_n(321)$ and there are 21-patterns in σ. Then

(a) If σ_i is the largest 2-element in σ and σ_j is the largest 1-element, then $\sigma_i\sigma_j$ is a 21-pattern in σ.

(b) Let $\sigma_s\sigma_t$ be a 21-pattern in σ and suppose there exist 21-patterns $\sigma_x\sigma_y$ in σ such that $\sigma_x < \sigma_s$ and $\sigma_y < \sigma_t$. Let σ_i and σ_j be, respectively, the largest 2-element and the largest 1-element occurring in these patterns $\sigma_x\sigma_y$. Then $\sigma_i\sigma_j$ is a 21-pattern in σ.

Definition [2]. Suppose $\sigma \in S_n(321)$ and there are 21-patterns in σ. The *first L-corner* of σ is the pair (i, σ_j), where σ_i and σ_j are, respectively, the largest 2-element and the largest 1-element in σ. If (s, σ_t) is the kth L-corner of σ and there are 21-patterns $\sigma_x\sigma_y$ in σ such that $\sigma_x < \sigma_s$ and $\sigma_y < \sigma_t$ then the *(k+1)th L-corner* of σ is the pair (i, σ_j), where σ_i and σ_j are, respectively, the largest 2-element and the largest 1-element occurring in these patterms $\sigma_x\sigma_y$.

If the L-corners of σ are $(p_1, v_1), \ldots, (p_t, v_t)$ ("p" for "position", "v" for "value"), then for each (p_i, v_i) let T_i be the set of squares

$$T_i = \{(p_i, j) : j \le v_i\} \cup \{(j, v_i) : j \le p_i\}.$$

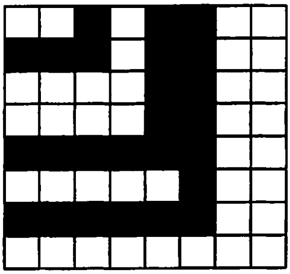
The squares in T_i form a reversed L having its corner at (p_i, v_i) and extending horizontally to the left-hand border of the array and vertically to its top border. Let T_σ be the union of the T_i's, so that T_σ is the union of a set of nested reversed L's. (Note that the smallest reversed L may degenerate to a line segment.)

Fact 2 ([2], Lemma 3). For every $\sigma \in S_n(321)$, the set T_σ is a template for σ.

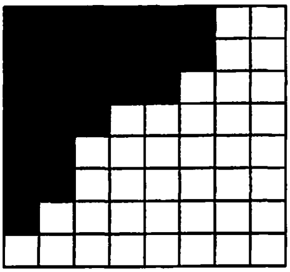
If the L-corners of σ are $(p_1, v_1), \ldots, (p_t, v_t)$ with $p_t < \ldots < p_1$ and $v_t < \ldots < v_1$ then let $\widehat{T_\sigma}$ be the set of squares obtained from T_σ by replacing each T_i by an inverted L having its corner at (i, i) and vertical and horizontal legs consisting of p_i and v_i squares, respectively (including the corner in each leg).

Fact 3 ([2], Theorem 1). For every $\sigma \in S_n(321)$, $\widehat{T_\sigma}$ is a template for $\Gamma(\sigma)$.

Example. If $\sigma \in S_8(321)$ is 14237586 then the L-corners of σ are (7,6), (5,5), and (2,3), so T_σ is



and $\widehat{T_\sigma}$ is



so $\Gamma(\sigma)$ is 78643521.

Since we want to obtain a similar pictorial reformulation of Θ, we recall the original definition from [4].

For $\sigma \in S_n(321)$, we first use the Robinson-Schensted-Knuth correspondence (see, for example, [8]) to obtain the insertion tableau and recording tableau for σ. Each tableau will consist of the elements of $\{1, 2, \ldots, n\}$ arranged in two rows (unless σ is the identity element of S_n, in which case each will have one row). The insertion tableau is formed in stages, by using the entries of σ from left to right. Assuming that $\sigma_1, \ldots, \sigma_{i-1}$ have been placed in the insertion tableau, we place σ_i as follows. If σ_i is larger than all the elements that are in the first row, then we place σ_i at the right end of the first row. If, on the other hand, σ_j is the leftmost element in the first row that is larger than σ_i, we replace σ_j by σ_i and place σ_j at the right end of the second row. (We then say that σ_i has "bumped" σ_j to the second row.) The recording tableau for σ has the same shape as the insertion tableau and is obtained by placing each i in the position that became occupied for the first time when σ_i was placed in the insertion tableau.

Example. If σ is 14237586, the insertion and recording tableaux are

$$\begin{matrix} 1 & 2 & 3 & 5 & 6 \\ 4 & 7 & 8 & & \end{matrix}$$

and

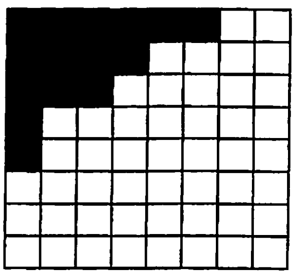
$$\begin{matrix} 1 & 2 & 4 & 5 & 7 \\ 3 & 6 & 8 & & \end{matrix} .$$

To continue with the definition of $\Theta(\sigma)$, we next form a Dyck path of length $2n$ by getting the first n steps from the insertion tableau and the last n steps from the recording tableau. For the first half, for each $i \in \{1, \ldots, n\}$ we take an upstep (u) if i is in the first row of the insertion tableau and a downstep (d) if i is in the second row. For the second half, we form a sequence of u's and d's in the same way from the recording tableau, and then write this sequence in reverse order and interchange u's and d's. We append the result to the first half to complete the Dyck path of length $2n$.

Example. If σ is 14237586, the first half of the Dyck path is *uuuduudd* and the second half is *ududdudd*.

To obtain $\Theta(\sigma)$, we start with an $n \times n$ array of squares. We draw a path consisting of edges of the squares, by starting at the point in the lower left-hand corner of the array and (reading the Dyck path from left to right) going up one edge for each u and right one edge for each d.

Example. If σ is 14237586 then we obtain the path that is the border between the shaded and unshaded regions in



Finally, $\Theta(\sigma)$ is defined to be the element of $S_n(132)$ whose diagram is the portion of the array that is to the left of the path. As always for elements of $S_n(132)$, the diagram is a template for the permutation.

It will be helpful in Section 3 to note that the second half of the path drawn in the last step of the definition of $\Theta(\sigma)$ can be obtained from the recording tableau for σ by starting at the point in the upper right-hand corner of the array and, for each $i \in \{1, \ldots, n\}$, going left one edge if i is in the first line of the tableau and down one edge if i is in the second line.

3. Θ and its Relationship to Γ

Notation. Fix n. For any $i \in \{1, \ldots, n\}$, let $\bar{i} = n + 1 - i$.

Lemma 1. Let $\sigma \in S_n(321)$ and suppose the second rows of the insertion and recording tableaux for σ are

$$a_1\ a_2\ \ldots\ a_n \text{ and } b_1\ b_2\ \ldots\ b_n$$

respectively. Then the template for $\Theta(\sigma)$ produced by the definition of $\Theta(\sigma)$ consists of k inverted L's, $L_1, \ldots, L_k$, such that the corner of L_i is the square (i, i) and the vertical and horizontal legs of L_i consist of $\overline{a_i}$ and $\overline{b_i}$ squares, respectively (including the corner in each leg).

Proof. Considering the construction of the path of edges of squares in the definition of $\Theta(\sigma)$, we see that the horizontal edges in the first half of the path occur in the leftmost k columns of the array, and that for $1 \le i \le k$ the ith horizontal edge is $a_i - i \le n - i$ edge-lengths above the bottom border of the array. Likewise, viewing the second half of the path as originating at the upper right-hand corner of the array, the vertical edges in the second half occur in the top k rows of the array, and for $1 \le i \le k$ the ith vertical edge from the top is $b_i - i \le n - i$ edge-lengths to the left of the right border of the array. Therefore the template consisting of those squares to the left of the path is comprised of k inverted L's, $L_1, \ldots, L_k$, such that the corner of L_i is the square (i, i).

Now consider the vertical leg of L_i. This leg consists of those squares in the lower $n - i + 1$ rows of the array that are above the ith horizontal edge in the first half of the path. Since this edge is $a_i - i$ edge-lengths above the bottom border of the array, the number of squares in the vertical leg of L_i (including the corner) is

$$(n - i + 1) - (a_i - i) = \overline{a_i}.$$

Viewing the second half of the path as originating at the upper right-hand corner of the array, we see in the same way that the horizontal leg of L_i consists of $\overline{b_i}$ squares. □

We next need an easy way to read the second rows of the tableaux for σ directly from σ.

Notation. If σ is the permutation $\sigma_1\sigma_2\ldots\sigma_n$, then $r\sigma$ denotes the permutation $\sigma_n\sigma_{n-1}\ldots\sigma_1$ and $c\sigma$ denotes the permutation $\overline{\sigma_1}\,\overline{\sigma_2}\ldots\overline{\sigma_n}$. We denote the inverse of σ by $i\sigma$.

Definition. If $\sigma \in S_n(321)$ then the *rcL-corners* of σ are $(v_1, p_1), \ldots, (v_t, p_t)$, where $(\overline{p_1}, \overline{v_1}), \ldots, (\overline{p_t}, \overline{v_t})$ are the *L*-corners of $rc\sigma$.

In this definition, each v_i is a 2-element and each p_i is the position of a 1-element.

Example. If σ is 14237586 then $rc\sigma$ is 31426758, the *L*-corners of which are $(6,5), (3,2)$, and $(1,1)$, so the *rcL*-corners of σ are $(4,3), (7,6)$, and $(8,8)$.

Lemma 2. If $\sigma \in S_n(321)$ and the *rcL*-corners of σ are $(v_1, p_1), \ldots, (v_t, p_t)$ in increasing order, then

(a) The smallest 2-element in σ is v_1, the smallest 1-element is σ_{p_1}, and $v_1\sigma_{p_1}$ is a 21-pattern

(b) If $1 \leq j \leq t$ and S is the set of all 21-patterns $\sigma_x\sigma_y$ in σ such that $\sigma_x > v_j$ and $\sigma_y > \sigma_{p_j}$, then v_{j+1} is the smallest 2-element occurring in the members of S and $\sigma_{p_{j+1}}$ is the smallest 1-element, and $v_{j+1}\sigma_{p_{j+1}}$ is a 21-pattern. There are no 21-patterns $\sigma_x\sigma_y$ in σ such that $\sigma_x > v_t$ and $\sigma_y > \sigma_{p_t}$.

Proof. This follows from Fact 1 and the definition of the *rcL*-corners. □

Lemma 3. If $\sigma \in S_n(321)$ and the *rcL*-corners of σ are $(v_1, p_1), \ldots, (v_t, p_t)$ in increasing order, then the second rows of the insertion and recording tableaux for σ are

$$v_1\ v_2\ \ldots\ v_t \quad \text{and} \quad p_1\ p_2\ \ldots\ p_t,$$

respectively.

Proof. We must show that σ_{p_i} "bumps" v_i for $1 \leq i \leq t$, and that these are the only "bumps" that occur in the construction of the insertion and recording tableaux. Note that if σ_s bumps σ_t then σ_t is a 2-element and σ_s is a 1-element.

Claim: Fix $1 \leq j \leq t$. Then σ_{p_i} bumps v_i for $1 \leq i \leq j$. If $\sigma_k \leq v_j$ and σ_k gets bumped during the construction of the tableaux for σ, then $\sigma_k \in \{v_1, \ldots, v_j\}$. If $\sigma_\ell \leq \sigma_{p_j}$ and σ_ℓ bumps some σ_z during the construction of the tableaux, then $\sigma_\ell \in \{\sigma_{p_1}, \ldots, \sigma_{p_j}\}$.

If we can establish this claim then Lemma 3 will be proved, for there cannot exist $\sigma_x > v_t$ and $\sigma_y > \sigma_{p_t}$ such that σ_y bumps σ_x. (Any such σ_x and σ_y would yield a 21-pattern $\sigma_x\sigma_y$ that would contradict the last sentence of Lemma 2(b).)

We will prove the claim by induction on j.

For $j = 1$, note that no $\sigma_k < v_1$ can ever get bumped, because no such σ_k is a 2-element, by Lemma 2(a). Similarly, no $\sigma_\ell < \sigma_{p_1}$ can bump any σ_z because no such σ_ℓ is a 1-element. Since v_1 is placed in the first row of the insertion tableau before σ_{p_1} is, and no $\sigma_\ell < \sigma_{p_1}$ can bump v_1, it follows from the fact that the elements of the first row of the tableau are in increasing order from left to right that σ_{p_1} bumps v_1, because v_1 is to the left of all the other 2-elements in the first row.

Now assume that the claim holds for $1 \leq j < t$. Note that if $\sigma_x > v_j$ and $\sigma_y > \sigma_{p_j}$ and σ_y bumps σ_x, then since $\sigma_x\sigma_y$ is a 21-pattern, we have $\sigma_x \geq v_{j+1}$ and $\sigma_y \geq \sigma_{p_{j+1}}$ by Lemma 2(b). It remains to show that $\sigma_{p_{j+1}}$ bumps v_{j+1}. But by the induction hypothesis and what we have already said, no $\sigma_y < \sigma_{p_{j+1}}$ bumps v_{j+1}, and no $\sigma_x < v_{j+1}$ is bumped by $\sigma_{p_{j+1}}$. □

Theorem 1. If $\sigma \in S_n(321)$ and the *rcL*-corners of σ are $(v_1, p_1), \ldots, (v_t, p_t)$ in increasing order then we get a template for $\Theta(\sigma)$ by taking inverted L's, $L_i, \ldots, L_t$ such that the corner of L_i is the square (i, i) and the vertical and horizontal legs of L_i consist of $\overline{v_i}$ and $\overline{p_i}$ squares, respectively.

Proof. This follows from Lemmas 1 and 3. □

To give an analog of Fact 3 for Θ, we need an appropriate pictorial representation for σ. We will use a modification of the notion of a template.

Definition. If T is a subset of an $n \times n$ array and

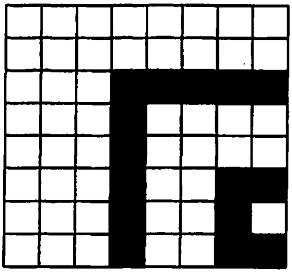
$$\overline{T} = \{(\overline{i}, \overline{j}) : (i, j) \in T\}$$

then we say that T is an *rc-template* for $\sigma \in S_n$ if $\overline{T}$ is a template for $rc\sigma$.

So T is an rc-template for σ if when we start with the bottom row and (inductively) on each row we place a dot in the first unshaded square from the right that has no dot below it, then the dots are in exactly the squares $(i, \sigma(i).)$

It is immediate that if $\sigma \in S_n(321)$ and $T_{rc\sigma}$ is the template for $rc\sigma$ specified in Fact 2 then $\overline{T_{rc\sigma}}$ is an rc-template for σ. If the rcL-corners of σ are $(v_1, p_1), \ldots, (v_t, p_t)$, then $\overline{T_{rc\sigma}}$ is the union of t inverted L's having their corners at the squares (p_i, v_i) and extending to the right-hand and bottom borders of the array.

Example. If σ is 14237586 then $\overline{T_{rc\sigma}}$ is



Theorem 2. If $\sigma \in S_n(321)$ then we obtain a template for $\Theta(\sigma)$ by taking $\overline{T_{rc\sigma}}$, sliding the largest inverted L so that its vertex becomes the square $(1,1)$, then sliding the second largest so that its vertex becomes the square $(2,2)$, and so on, and then finally flipping all the inverted L's across the diagonal from upper left to lower right.

Proof. This follows from Theorem 1, since in $\overline{T_{rc\sigma}}$ the inverted L corresponding to (v_i, p_i) has horizontal and vertical legs consisting of $\overline{v_i}$ and $\overline{p_i}$ squares, respectively. □

Theorem 3. For every $\sigma \in S_n(321)$ we have $\Theta(\sigma) = \Gamma(irc\sigma)$.

Proof. If σ has rcL-corners $(v_1, p_1), \ldots, (v_t, p_t)$ then $T_{rc\sigma}$ is the union of reversed L's, $L_1, \ldots, L_t$, where L_i has its corner at $(\overline{p_i}, \overline{v_i})$ and extends to the left-hand and top borders of the array. $T_{irc\sigma}$ is obtained from $T_{rc\sigma}$ by replacing each corner $(\overline{p_i}, \overline{v_i})$ by $(\overline{v_i}, \overline{p_i})$. The result now follows from Theorem 1 and Fact 3. □

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