Flexibility of Circular Graphs $C(2n, 2)$ on the Projective Plane*

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Abstract In this paper, we study the flexibility of embeddings of circular graphs $C(2n, 2)$, $n \geq 3$ on the projective plane. The numbers of (nonequivalent) embeddings of $C(2n, 2)$ on the projective plane are obtained, and by describing structures of these embeddings, the numbers of (nonequivalent) weak embeddings and strong embeddings of $C(2n, 2)$ on the projective plane are also obtained.

Keywords Circular graph; embedding; weak embedding; strong embedding; joint tree.

Mathematics Subject Classification 05C10, 05C30

1 Introduction

A *surface* is a compact 2-dimensional manifold without boundary. It can be represented by a polygon of even edges in the plane. Furthermore, it can be also written by words, for example, the plane is written as $O_0 = aa^-$, the projective plane $N_1 = aa$. See [8,13] for more detail. In this way, some topological transformations and operations on surfaces can be represented by words easily. For example, the following relations can be deduced, as shown in, e.g.,[8].

Relation 1: $(AxByCx^-Dy^-) \sim ((ADCB)(xyx^-y^-))$,

Relation 2: $(AxBx) \sim ((AB^-)(xx))$,

Relation 3: $(Axxyzy^-z^-) \sim ((A)(xx)(yy)(zz))$.

In which A, B, C, and D are all linear orders of letters and permitted to be empty. Parentheses are always omitted when the letters in parentheses represent surfaces. $\sim$ means topological equivalence on surfaces.

An *embedding* of a graph G on a surface S is a homeomorphism $h : G \to S$ of G into S such that every component of $S - h(G)$ is a 2-cell. Two embeddings $h : G \to S$ and $g : G \to S$ of G on a surface S are said to be *equivalent* if there is an homeomorphism $f : S \to S$ such that $f \circ h = g$. The connected components of $S - h(G)$ are called *faces* of the embedding. A *weak embedding* of a graph G

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is an embedding of G such that there are no repeated edges (repeated vertices are allowed) on the boundary of each face. A *strong embedding* of a graph G is an embedding of G such that there are no repeated vertices on the boundary of each face.

Given a graph G, how many nonequivalent embeddings of G are there on a given surface? This is an important problem in embedding flexibility, inaugurated by Gross and Furst [3] and some results have been obtained, such as [1-6, 12-14] etc. But as for the enumeration of weak (or strong) embeddings of graphs on surfaces, the results are few [14].

A few years ago, Liu established the *joint tree* model [7] of a graph embedding, by using this model, an embedding of a graph can be represented by a cyclic order of letters with indices, called an *associated surface* [8,13] of the graph. In this way, the problem of enumerating the number of nonequivalent embeddings for a graph on a surface can be transformed into the problem of finding the number of distinct associated surfaces in an equivalent class (up to genus). The joint tree model has been verified useful in the research of this enumeration problem, lots of works have been done with the joint tree model, such as [12-14] etc. The reader is referred to [8,13] for more detail about the joint tree model.

The circular graphs $C(n, m)$ is the graph with vertex set $V = \{v_1, \dots, v_n\}$ and edge set $E = \{(v_i, v_{i+1}), (v_i, v_{i+m}) \mid i = 1, 2, \dots, n,$ subscripts modulo $n\}$. Figure 1 is the circular graph $C(8, 2)$. Ren and Deng [11] obtained the minimum orientable genus and the minimum nonorientable genus of all circular graphs. For example, the minimum orientable genus of $C(2n, 2)$ is 0, and the minimum nonorientable genus of $C(2n, 2)$ is 1. But the number of embeddings on each surface had not been investigated.

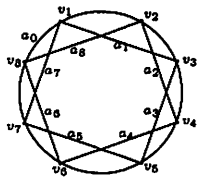

Figure 1 The graph $C(8, 2)$

In this paper, we study the embeddings of circular graphs $C(2n, 2), n \geq 3$ on the projective plane. The numbers of (nonequivalent) embeddings of $C(2n, 2)$ on the projective plane are obtained, and by describing structures of these embeddings, the numbers of (nonequivalent) weak embeddings and strong embeddings of $C(2n, 2)$ on the projective plane are also obtained.

For the undefined terminologies see [8,10].

2 The number of embeddings of $C(2n, 2)$ on the projective plane

Let S be a surface. If $x, y \in S$ are in the form as $S = AxByCx^{\varepsilon_1}Dy^{\varepsilon_2}$ where $\varepsilon_i (i = 1, 2)$ is a binary index, it can be +(always omit) or −, then they are said to be *interlaced*; otherwise, *parallel*. Suppose $A = a_1a_2 \cdots a_t, t \geq 1$ is a word, then $A^{-} = a_t^{-} \cdots a_2^{-} a_1^{-}$ is called the *inverse* of A.

Lemma 2.1 [8] *An orientable surface S is a surface of orientable genus 0 if and only if there is no form as $AxByCx^{-}Dy^{-}$ in it.*

Lemma 2.2 [13] *Let S be a nonorientable surface, if there is a form as $AxByCx^{-}Dy^{-}$ in S, then the genus of S will be not less than 3; if there is a form as $AxByCx^{-}Dy$ or $AxByCyDx$ in S, then the genus of S will be not less than 2.*

According to the joint tree model, we can choose the tree $v_1 v_2 \cdots v_{2n}$ as a spanning tree of $C(2n, 2)$, then label cotree edges $v_1 v_{2n}$ by a_0, $v_i v_{i+2}$ by a_i, for $1 \leq i \leq 2n$, subscripts modulo $2n$. The spanning tree we choose and the cotree edge a_0 form a circuit in $C(2n, 2)$, denote by C_{2n} . In the embeddings of $C(2n, 2)$, the circuit C_{2n} can be classified into two cases: contractible or noncontractible.

2.1 C_{2n} is contractible

When C_{2n} is contractible, from the joint tree model, the associated surface is divided by a_0 and a_0^{-} into 2 segments: A and B, as shown in Figure 2. And the associated surface of $C(2n, 2)$ is $a_0 A a_0^{-} B$.

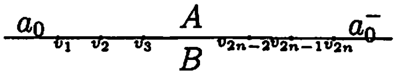

Figure 2 The joint tree of $C(2n, 2)$ on the projective plane when C_n is contractible

Lemma 2.3 *When C_{2n} is contractible, the number of embeddings of $C(2n, 2), n \geq 3$ on the projective plane is*

$$N_{1.1}(C(2n, 2)) = \begin{cases} 40 & \textit{when } n = 3; \\ 12n & \textit{when } n \geq 4. \end{cases}$$

Proof According to Lemma 2.2, $a_0 A a_0^{-} B \sim N_1$ if and only if one of the following two cases holds:

Case 1 $A \sim N_1$ and $B \sim O_0$; **Case 2** $A \sim O_0$ and $B \sim N_1$.

According to the symmetry, the numbers of embeddings of the two cases are the same. So we only discuss Case 1.

Claim There are at most n edges in B and at most three twisted edges in A.

According the graph $C(2n, 2)$, the edge a_i is interlaced with the two edges incident with the vertex v_{i+1}, $1 \leq i \leq 2n$, subscripts modulo $2n$, if they are all in B or A. So there are at most n edges in B, otherwise two edges in B will be interlaced, the genus of B will be at least one, from Lemma 2.1. For the same reason, if there are more than three twisted edges in A, then at least one pair of them will be parallel, so the nonorientable genus of it will be more than one, from Lemma 2.2. The claim holds.

In the following, we classify the embeddings according to the number of twisted edges in A.

Subcase 1 There is one twisted edge in A.

From Lemma 2.2, all the edges are parallel with others in A, so there are at most n edges in A. And from the claim above, there are n edges in both A and B, and no edge interlace with others. So the associated surfaces in this subcase are of forms:

$$a_0 a_{2n} a_2 a_2^{\varepsilon_2} a_4 a_4^{\varepsilon_4} \cdots a_{2n-2} a_{2n-2}^{\varepsilon_{2n-2}} a_{2n}^{\varepsilon_{2n}} a_0^- a_{2n-1} a_{2n-3} a_{2n-3}^- \cdots a_3 a_3^- a_1 a_1^- a_{2n-1}^-,$$

in which one of $\varepsilon_2, \varepsilon_4 \ldots \varepsilon_{2n}$ is $+$ (always omitted), the others are $-$.

or

$$a_0 a_{2n-1} a_1 a_1^{\varepsilon_1} a_3 a_3^{\varepsilon_3} \cdots a_{2n-3} a_{2n-3}^{\varepsilon_{2n-3}} a_{2n-1}^{\varepsilon_{2n-1}} a_0^- a_{2n} a_{2n-2} a_{2n-2}^- \cdots a_4 a_4^- a_2 a_2^- a_{2n}^-,$$

in which one of $\varepsilon_1, \varepsilon_3 \ldots \varepsilon_{2n-1}$ is $+$, the others are $-$.

So there are $2n$ embeddings in this subcase.

Subcase 2 There are two twisted edges in A.

The two twisted edges in A must be incident with the same vertex, otherwise the two twisted edges will be parallel or one twisted edge will be interlaced with an untwisted edge in A, or two untwisted edges will be interlaced in B, then the genus will be more than one, from Lemma 2.2.

So A has the from as $A_1 a_i a_{i+2} A_2 a_i A_3 a_{i+2}$ in which $A_2 = \emptyset$, otherwise one twisted edge will be interlaced with an untwisted edge. By Relation 2,

$$A_1 a_i a_{i+2} a_i A_3 a_{i+2} \sim A_1 a_{i+2}^- A_3 a_{i+2} a_i a_i,$$

$$A_1 a_{i+2}^- A_3 a_{i+2} a_i a_i \sim N_1 \Leftrightarrow A_1 \sim O_0, A_3 \sim O_0.$$

From the claim, $| A_1 | + | A_3 | \leq 2(n-2)$, and $| B | \leq 2n$, and $| A_1 | + | A_3 | + | B | = 2(2n-2)$, hence $| B | = 2n, | A_1 | + | A_3 | = 2(n-2)$.

For $| B | = 2n$ and $B \sim O_0$, the form of B is in one of the two subcases:

Subcase 2.1 $B = a_{2n-1} a_{2n-3} a_{2n-3}^- \cdots a_3 a_3^- a_1 a_1^- a_{2n-1}^-,$

Subcase 2.2 $B = a_{2n} a_{2n-2} a_{2n-2}^- \cdots a_4 a_4^- a_2 a_2^- a_{2n}^-.$

According to the symmetry, the numbers of embeddings in the two subcases are the same. So we only discuss subcase 2.1. As for $A = A_1 a_i a_{i+2} a_i A_3 a_{i+2}$, in which i is even, $2 \leq i \leq 2n$, subscripts modulo $2n$, and $A_1 \sim O_0, A_3 \sim O_0$, we can get that A has the form as:

$$A = \underline{a_2 a_{2n} a_2} a_4 a_4^- \cdots a_{2n-2} a_{2n-2}^- \underline{a_{2n}}$$

or

$$A = a_{2n} \underline{a_2 a_4 a_2 a_4} a_6 a_6^- \cdots a_{2n-2} a_{2n-2}^- a_{2n}^-$$

or

$$A = a_{2n} a_2 a_2^- \underline{a_4 a_6 a_4 a_6} a_8 a_8^- \cdots a_{2n-2} a_{2n-2}^- a_{2n}^-$$

......

or

$$A = \underline{a_{2n}} a_2 a_2^- \cdots a_{2n-4} a_{2n-4}^- \underline{a_{2n-2} a_{2n} a_{2n-2}}.$$

And the number of embeddings in subcase 2 is $2n$.

Subcase 3 There are three twisted edges in A.

The two of the three twisted edges must be incident with the same vertex, otherwise one of the two twisted edges will be parallel with the third. The two twisted edges incident with the same vertex are a_{i-1} and a_{i+1}, and the other twisted edge must be a_i, $1 \le i \le 2n$, subscripts modulo $2n$ (there are two exceptions when $n = 3$, for $C(6,2)$, the three twisted edges can also be a_2, a_4, a_6 or a_1, a_3, a_5), otherwise this twisted edge will be parallel with at least one of these two twisted edges incident with the same vertex, then the genus will be more than one.

The three twisted edge in A must be interlaced with each other, and no other edges interlaced with any of the three twisted edges, so A has the form as $A_1 a_{i-1} a_i a_{i+1} a_{i-1} a_i a_{i+1} A_2$,

$$A_1 a_{i-1} a_i a_{i+1} a_{i-1} a_i a_{i+1} A_2 \sim N_1 \Leftrightarrow A_1 A_2 \sim O_0.$$

And from the claim, $\mid A_1 \mid + \mid A_2 \mid \le 2(n-2)$, and $\mid B \mid \le 2(n-1)$, and $\mid A_1 \mid + \mid A_2 \mid + \mid B \mid = 2(2n-3)$, hence $\mid B \mid = 2(n-1), \mid A_1 \mid + \mid A_3 \mid = 2(n-2)$.

For $\mid B \mid = 2(n-1)$, $B \sim O_0$, and $A = A_1 a_{i-1} a_i a_{i+1} a_{i-1} a_i a_{i+1} A_2$, the form of B is in one of the two subcases:

Subcase 3.1 $B = a_{2n-1} a_{2n-3} a_{2n-3}^{-} \cdots a_{2i+1} a_{2i+1}^{-} a_{2i-3} a_{2i-3}^{-} \cdots a_3 a_3^{-} a_1 a_1^{-} a_{2n-1}^{-}$, $1 \le i \le n$, subscripts modulo $2n$,

Subcase 3.2 $B = a_{2n} a_{2n-2} a_{2n-2}^{-} \cdots a_{2i+2} a_{2i+2}^{-} a_{2i-2} a_{2i-2}^{-} \cdots a_4 a_4^{-} a_2 a_2^{-} a_{2n}^{-}$, $1 \le i \le n$, subscripts modulo $2n$.

According to the symmetry, the numbers of embeddings in the two subcases are the same. So we only discuss subcase 3.1. Similar with the discussion in subcase 2, we can get that, when B is given, the from of A follows, the associated surfaces in this subcase are of forms:

$$a_0 \underline{a_1 a_2 a_{2n} a_1 a_2} a_4 a_4^{-} \cdots a_{2n-2} a_{2n-2}^{-} \underline{a_{2n}} a_0^{-} a_{2n-1} a_{2n-3} a_{2n-3}^{-} \cdots a_5 a_5^{-} a_3 a_3^{-} a_{2n-1}^{-},$$

$$a_0 a_{2n} \underline{a_2 a_3 a_4 a_2 a_3 a_4} a_6 a_6^{-} \cdots a_{2n-2} a_{2n-2}^{-} a_{2n}^{-} a_0^{-} a_{2n-1} a_{2n-3} a_{2n-3}^{-} \cdots a_5 a_5^{-} a_1 a_1^{-} a_{2n-1}^{-},$$

$$a_0 a_{2n} a_2 a_2^{-} \underline{a_4 a_5 a_6 a_4 a_5 a_6} a_8 a_8^{-} \cdots a_{2n-2} a_{2n-2}^{-} a_{2n}^{-} a_0^{-} a_{2n-1} a_{2n-3} a_{2n-3}^{-} \cdots a_1 a_1^{-} a_{2n-1}^{-},$$

......

$$a_0 \underline{a_{2n-1} a_{2n}} a_2 a_2^{-} a_4 a_4^{-} \cdots a_{2n-4} a_{2n-4}^{-} \underline{a_{2n-2} a_{2n-1} a_{2n} a_{2n-2}} a_0^{-} a_{2n-3} a_{2n-3}^{-} \cdots a_1 a_1^{-}.$$

For $1 \le i \le n$, there are n embeddings in subcase 3.1. So there are $2n$ embeddings in subcase 3, when $n \ge 4$. When $n = 3$, we need to add $2n$ to the two exceptions we mentioned above which are $a_0 a_1 a_5 a_3 a_1 a_5 a_3 a_0^{-} a_6 a_4 a_4^{-} a_2 a_2^{-} a_6^{-}$ and $a_0 a_2 a_6 a_4 a_2 a_6 a_4 a_0^{-} a_5 a_3 a_3^{-} a_1 a_1^{-} a_5^{-}$, so the number is $2n + 2 = 8$.

Summarizing above, when $n \ge 4$, there are $6n$ embeddings in Case 1; when $n = 3$, the number is $6n + 2$. So the number of embeddings of $C(2n, 2)$ in Case 1 and Case 2 is $12n$, when $n \ge 4$; and $12n + 4$, when $n = 3$. The theorem is obtained. □

2.2 C_{2n} is noncontractible

Similar to the proof in Lemma 2.3, we discuss the number of twisted edges in the embeddings of $C(2n, 2)$ on the projective plane, when C_{2n} is noncontractible. We list all embeddings in this condition instead of the proof and counting, because it is similar to Lemma 2.3 and it is routine.

When C_{2n} is noncontractible, the associated surfaces of $C(2n,2)(n \geq 4)$ on the projective plane are:

$$a_0a_{2n-1}A_{1,2n-3}a_{2n-1}^{-}a_0a_{2n}A_{2n-2,2}a_{2n}^{-}, \tag{2.1}$$

$$a_0a_{2n-1}A_{1,2n-3}a_{2n-1}^{-}a_{2n}a_0A_{2n-2,2}a_{2n}, \tag{2.2}$$

$$a_0A_{1,2n-3}a_{2n-1}a_0a_{2n}A_{2n-2,2}a_{2n}^{-}a_{2n-1}, \tag{2.3}$$

$$a_0a_{2n-1}A_{1,2n-3}a_{2n-1}^{-}a_{2n-2}a_{2n}a_0a_{2n-2}A_{2n-4,2}a_{2n}, \tag{2.4}$$

$$a_0a_{2n-1}a_1a_{2n}A_{2,2n-2}a_{2n}^{-}a_0a_{2n-1}A_{2n-3,3}a_1, \tag{2.5}$$

$$a_0a_{2n-1}a_1A_{2,2n-2}a_{2n}a_0a_{2n-1}A_{2n-3,3}a_1a_{2n}, \tag{2.6}$$

$$a_0a_{2n-1}A_{1,2t+1}a_{2t+2}A_{2t+4,2n-2}a_{2n}a_0a_{2n-1}A_{2n-3,2t+3}a_{2t+2}A_{2t,2}a_{2n}, \quad (0 \leq t \leq n-3), \tag{2.7}$$

$$a_0a_{2n-1}A_{1,2t+1}a_{2t+3}A_{2t+4,2n-2}a_{2n}a_0a_{2n-1}A_{2n-3,2t+5}a_{2t+3}A_{2t+2,2}a_{2n}, \quad (0 \leq t \leq n-4), \tag{2.8}$$

$$a_0a_{2n-1}A_{1,2n-5}a_{2n-3}A_{2n-2,2n-2}a_{2n}a_0a_{2n-1}a_{2n-3}A_{2n-4,2}a_{2n}, \tag{2.9}$$

$$a_0a_{2n-1}A_{1,2n-3}a_{2n-2}a_{2n}a_0a_{2n-1}a_{2n-2}A_{2n-4,2}a_{2n}, \tag{2.10}$$

$$a_0a_{2n-1}a_1a_2A_{4,2n-2}a_{2n}a_0a_{2n-1}A_{2n-3,3}a_1a_2a_{2n}, \tag{2.11}$$

$$a_0a_{2n-1}A_{1,2t+1}a_{2t+3}a_{2t+4}A_{2t+6,2n-2}a_{2n}a_0a_{2n-1}A_{2n-3,2t+5}a_{2t+3}a_{2t+4}A_{2t+2,2}a_{2n}, \quad (0 \leq t \leq n-4), \tag{2.12}$$

$$a_0a_{2n-1}A_{1,2n-5}a_{2n-3}a_{2n-2}a_{2n}a_0a_{2n-1}a_{2n-3}a_{2n-2}A_{2n-4,2}a_{2n}, \tag{2.13}$$

$$a_0a_{2n-1}A_{1,2t+1}a_{2t+3}a_{2t+2}A_{2t+4,2n-2}a_{2n}a_0a_{2n-1}A_{2n-3,2t+5}a_{2t+3}a_{2t+2}A_{2t,2}a_{2n}, \quad (0 \leq t \leq n-4), \tag{2.14}$$

$$a_0a_{2n-1}A_{1,2n-5}a_{2n-3}a_{2n-4}A_{2n-2,2n-2}a_{2n}a_0a_{2n-1}a_{2n-3}a_{2n-4}A_{2n-6,2}a_{2n}, \tag{2.15}$$

and their inverses, where

$$A_{s,k} = \begin{cases} \prod_{i=0}^{l} a_{s+2i}a_{s+2i}^{-} & \text{when } k = s+2l,\ l \geq 0,\ s \geq 1; \\ \prod_{i=0}^{l} a_{s-2i}a_{s-2i}^{-} & \text{when } k = s-2l,\ l \geq 0,\ s \geq 1; \\ \emptyset & \text{otherwise.} \end{cases}$$

When $n = 3$, the graph is $C(6,2)$. The associated surfaces of $C(6,2)$ on the projective plane when C_{2n} is noncontractible, are (2.1) – (2.7), (2.9) – (2.11), (2.13), (2.15) with their inverses and the two listed below and their inverses:

$$a_0a_1a_1^{-}a_4a_2a_5a_0a_4a_6a_3a_3^{-}a_6^{-}a_2a_5, \tag{2.16}$$

$$a_0a_1a_5a_2a_2^{-}a_5^{-}a_3a_6a_0a_4a_4^{-}a_1a_3a_6. \tag{2.17}$$

So when C_{2n} is noncontractible, $C(6,2)$ has $14 \times 2 = 28$ embeddings on the projective plane, $C(2n,2)(n \geq 4)$ has $4n \times 2 = 8n$ embeddings on the projective plane. The lemma 2.4 follows.

Lemma 2.4 *When C_{2n} is noncontractible, the number of embeddings of $C(2n,2)$ on the projective plane is*

$$N_{1.2}(C(2n,2)) = \begin{cases} 28 & \text{when } n = 3; \\ 8n & \text{when } n \geq 4. \end{cases}$$

The summation of $N_{1.1}(C(2n,2))$ and $N_{1.2}(C(2n,2))$ is the number of embeddings of $C(2n,2)$ on the projective plane. From Lemma 2.3 and 2.4, the theorem follows.

Theorem 2.1 *The number of embeddings of $C(2n,2), n \geq 3$ on the projective plane is*

$$N_1(C(2n,2)) = \begin{cases} 68 & \text{when } n = 3; \\ 20n & \text{when } n \geq 4. \end{cases}$$

3 The numbers of weak embeddings and strong embeddings of $C(2n,2)$ on the projective plane

From [9], we get that $C(2n,2)$ have strong embeddings on the projective plane, because $C(2n,2)$ contain triangles. An embedding is a strong embedding, then it is also a weak embedding. So $C(2n,2)$ also have weak embeddings on the projective plane. We have all associated surfaces of $C(2n,2)$ on the projective plane. And according to the face traversal procedure in [10] and the joint tree model, the faces of an embedding can be got easily. By checking the faces of all embeddings of $C(2n,2)$ on the projective plane, one can obtain strong and weak embeddings among them.

Theorem 3.1 *The number of strong embeddings of $C(2n,2), n \geq 3$ on the projective plane is*

$$SN_1(C(2n,2)) \begin{cases} 20 & \text{when } n = 3; \\ 4n & \text{when } n \geq 4. \end{cases}$$

Proof In the proof of Lemma 2.3, we get all associated surfaces of $C(2n,2)$ on the projective plane, when C_n is contractible. From the face traversal procedure and joint tree model, we get that there is no strong embeddings in subcase 1, because in each embedding the two vertices incident with the twisted edge are repeated in one face. Figure 3 gives two examples in which the faces with repeated vertices can be traced by the dash line, in Figure 3(a), a_2 is the twisted edge, vertices v_2 and v_4 are repeated in a face; in Figure 3(b), a_{2n} is the twisted edge, vertices v_2 and v_{2n} are repeated in a face.

In subcase 2, there is no strong embedding in this subcase, because in each embedding the vertex incident with the two twisted edges is repeated in one face. Figure 4 gives two examples, vertices v_2 and v_4 are repeated in Figure 4(a) and 4(b), respectively.

Finding all the faces of embeddings in subcase 3, we get that all the embeddings in this subcase are strong embeddings. So when C_{2n} is contractible, there are $4n$ embeddings are strong embeddings when $n \geq 4$; and $8 \times 2 = 16$ strong embeddings when $n = 3$.

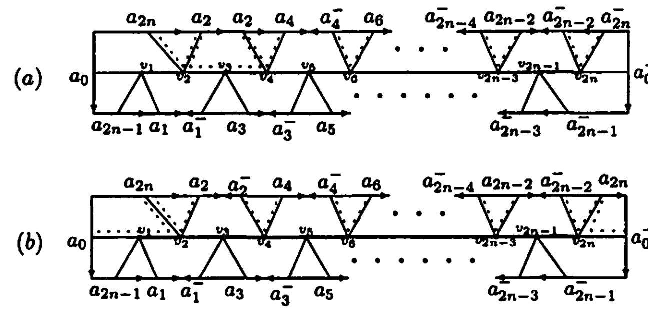

Figure 3 Two embeddings of $C(2n,2)$ on the projective plane in subcase 1

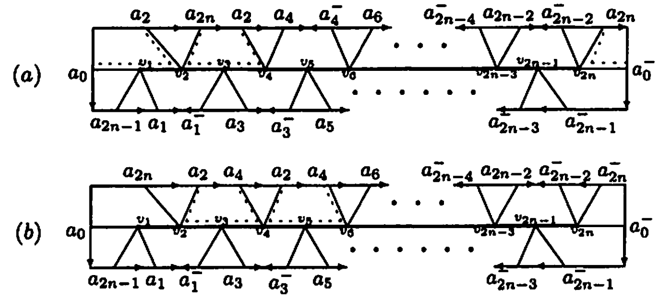

Figure 4 Two embeddings of $C(2n,2)$ on the projective plane in subcase 2

When C_{2n} is noncontractible, by checking all the embeddings in this condition, we obtain that there is no strong embedding when $n \geq 4$, when $n = 3$, there are four strong embeddings, they are (2.16), (2.17) and their inverses.

Summarizing the above, there is $4n$ strong embeddings of $C(2n,2)$ on the projective plane, when $n \geq 4$; and $16 + 4 = 20$ strong embeddings when $n = 3$. The theorem follows. □

Theorem 3.2 *The number of weak embeddings of* $C(2n,2), n \geq 3$ *on the projective plane is*

$$WN_1(C(2n,2)) = \begin{cases} 44 & \text{when } n = 3; \\ 12n & \text{when } n \geq 4. \end{cases}$$

Proof With a similar argument to the proof of Theorem 3.1, We check each embedding of $C(2n,2)$ on the projective plane. When C_{2n} is contractible, there is no weak embeddings in subcase 1, because in each embedding the twisted edge is repeated in one face. See Figure 3 as an example.

In subcase 2, the vertex incident with the two twisted edges is the only one vertex repeated in a face in each embedding. So no edge repeated in one face for all the embeddings in this subcase. From Lemma 2.3, the number of weak embeddings in subcase 2 is $2n$.

From the proof of theorem 3.1, all the embeddings in subcase 3 are strong embeddings, so they are also weak embeddings. The number of weak embeddings in subcase 3 is $2n$, when $n \geq 4$; and 8, when $n = 3$, from Lemma 2.3.

Above all and from the proof of theorem 2.1, when C_{2n} is contractible, there are $8n$ weak embeddings of $C(2n, 2)$ on the projective plane when $n \geq 4$, and 28 weak embeddings, when $n = 3$.

When C_{2n} is noncontractible, by checking all the embeddings on by one, we find that (2.2), (2.3), (2.6) − (2.10) with their inverses are weak embeddings and when $n = 3$, (2.16)(2.17) with their inverses are strong embeddings, so they are also weak embeddings. Hence, when C_{2n} is noncontractible, there are $2n \times 2 = 4n$ weak embeddings of $C(2n, 2)$ on the projective plane when $n \geq 4$, and $4 \times 3 + 2 \times 2 = 16$ weak embeddings, when $n = 3$.

Summarizing the above, the theorem is obtained. □

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