Pebbling number of Bi-wheel: A diameter three class 0 graph

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Abstract

Given a configuration of pebbles on the vertices of a graph G, a pebbling move consists of taking two pebbles off a vertex v and putting one of them back on a vertex adjacent to v. A graph is called *pebbleable* if for each vertex v there is a sequence of pebbling moves that would place at least one pebble on v. The *pebbling number* of a graph G, is the smallest integer m such that G is pebbleable for every configuration of m pebbles on G. A graph G is *class 0* if the pebbling number of G, is the number of vertices in G. We prove that Bi-wheels, a class of diameter three graphs are class 0.

Key words: Pebbling, Class 0 graph, Diameter three, Bi-wheel.
AMS Subject Classification 05C99.

Introduction

Let $G(V, E)$ be a simple connected graph. A configuration p of pebbles on G is a function $p : V(G) \to \mathbb{N} \cup \{0\}$. The value of $p(v)$ equals the number of pebbles placed at vertex v and let the size $|p|$ be the total number of pebbles in p, that is $|p| = \sum_{v \in V(G)} p(v)$. A pebbling move from a vertex v to a neighbor u takes away two pebbles at v and adds one pebble at u. A pebbling sequence is a sequence of pebbling moves.

Suppose we are given a configuration p and a 'target' vertex v. The configuration is v *solvable* if v has a pebble after some pebbling sequence starting from p.

Definition 1. For a graph G, let $f(G, v)$ be the least k such that every configuration of k pebbles on G is v solvable. A configuration p is *solvable* if every vertex is reachable under p. The *pebbling number* of a graph G denoted by $f(G)$, is the smallest integer m such that for every configuration of m pebbles to the vertices of G, one pebble can be moved to any specified target vertex.

Note that if v is a vertex in a connected graph G, then by placing one pebble each on all vertices in G, except at v, a pebble cannot be moved on to v. Thus we have that $f(G) \geq |V(G)|$, the order of G. Graphs that satisfy $f(G) = |V(G)|$ are known as *class 0* graphs. The goal of this paper is to find a class of diameter 3, class 0 graphs.

T A Clarke et al. [3] characterized diameter two class 0 graphs. They established that a diameter two graph G is class 0, if it has no cut vertex and if G has a cut vertex, then $f(G) = |V(G)|+1$. In [2], Chung proved that $f(Q^n) = 2^n$, where Q^n is an n-cube. In [1], an upper bound for the pebbling number of diameter three graphs was established as $f(G) \leq \frac{3}{2}n + O(1)$.

Definition 2 (Bi-wheels). Bi-wheels B_{2n+2} are graphs having a cycle C_{2n} of length $2n$. Let $\{u_1, v_1, u_2, v_2, \ldots, u_n, v_n, u_1\}$ be the vertex set of C_{2n}. There are, in addition, two distinguished vertices u and v such that u (respectively v) is adjacent to u_i (respectively v_i) for all i.

The Bi-wheel B_8 is simply the 3-cube as illustrated in the figure 1. All bi-wheels are diameter three graphs. We will show that all bi-wheels are class 0 graphs.

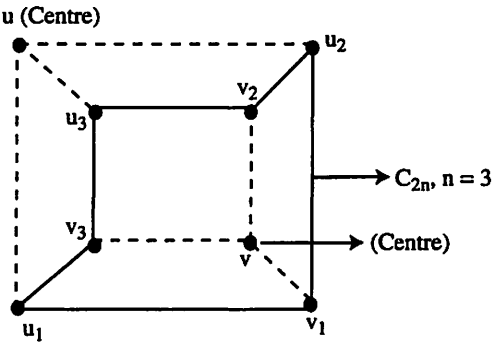

Figure 1: Representation of the 3-cube as a Bi-wheel

Before proving the main result, we have the following lemmas.

Lemma 1. Suppose that S is a set of vertices of G. Let x *adj* y_i, $\forall\, y_i \in S$. Let $p(y_i) \geq 1$, $\forall\, y_i \in S$, where $p(y_i)$ represents the number of pebbles at y_i. Let r denote the number of excess pebbles in S. i.e.,

$$r = \sum_{y_i \in S} p(y_i) - |S|$$

then $\lceil \frac{r}{2} \rceil$ pebbles can be moved to x.

Proof. We prove the result by induction on r.

When $r = 1$ there is one more pebble than the number of vertices in S.

$$\Rightarrow \quad p(y_i) \geq 2, \text{ for at least one } y_i \in S.$$
$$\Rightarrow \quad \text{one pebble can be moved to } x.$$

Assume that the result is true when $n < r$. To prove that the result is true for $n = r$.

Case 1: When $r = 2k$

By induction, when there is an excess of $2k - 1$ pebbles, $\lceil \frac{2k-1}{2} \rceil = k$ pebbles can be moved to x. A fortiori, $k = \frac{2k}{2}$ pebbles can be moved to x if $r = 2k$.

Case 2: When $r = 2k + 1$

The number of extra pebbles in at least one vertex of S, say y_i, must be odd. Now, remove one pebble from y_i and consider the resulting pebbling configuration. The number of extra pebbles now equals $2k$ and by induction k pebbles can be moved to x. We then replace the pebble at y_i. Now y_i has at least two pebbles remaining. Using the two pebbles, one more pebble can be moved to x.

Therefore $k+1 = \lceil \frac{2k+1}{2} \rceil$ pebbles can be moved to x. Hence the lemma. □

Remark 1. We can actually prove that $\frac{r+s}{2}$ pebbles can be moved to x where $s = |\{i : p(y_i) \text{ is even}\}|$.

Remark 2. Lemma 1 is used mainly in computing the number of pebbles which can be transferred to the centre of a star from the end vertices.

Similarly, we can prove the following result.

Lemma 2. Let S and T be disjoint nonempty subsets of $V(G)$. Let p be a pebbling configuration of G with r excess pebbles in S. Assume that every vertex of S is adjacent to atleast one vertex of T. Then $\lceil \frac{r}{2} \rceil$ pebbles can be transferred to T from S by pebbling process.

Theorem 1. $f(B_{2n+2}) = 2n+2, n \geq 3$

Proof. When $n = 3$, we get the 3-cube which is a class 0 graph as already noted. So, hereafter we assume $n \geq 4$. The following figure represents Bi-wheel B_{2n+2}, with $2n+2$ vertices.

Let $G \cong B_{2n+2}$, given in figure 2.

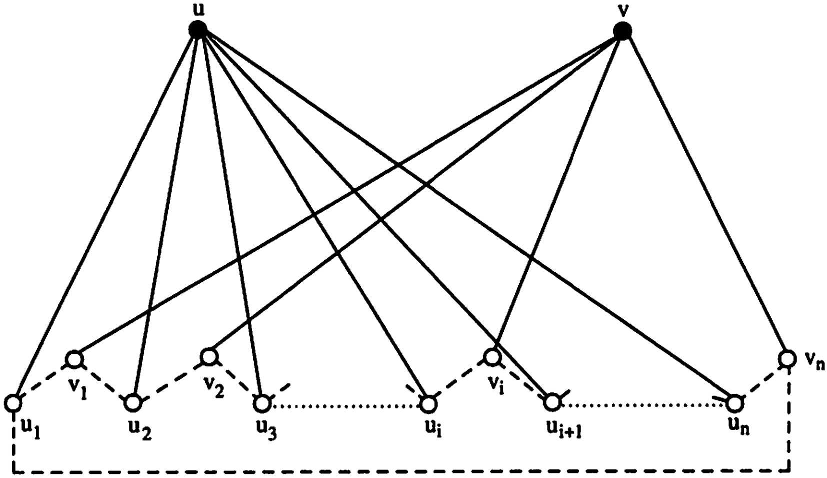

Figure 2: The graph B_{2n+2}

Since $f(G) \geq |V(G)| = 2n+2$, it is enough to prove that $f(G) \leq 2n+2$.

Notation: We follow the notation given below.

1. $p(u_i) = \alpha_i, p(u) = \alpha, p(v_i) = \beta_i$ and $p(v) = \beta$.
2. G_1 denotes the graph induced by $\{u, u_1, u_2, \ldots, u_n\}$
3. G_2 denotes the graph induced by $\{v, v_1, v_2, \ldots, v_n\}$.
4. $p(U)$ is the total number of pebbles in $U \subseteq V(G)$.
5. $\beta^* = \max\{\beta_i : 1 \leq i \leq n\}, T = \{i : \beta_i \geq 2\}$, $t = |T|$, $r = |\{i : \beta_i = 2\}|$, $s = |\{i : \beta_i = 3\}|$.

Note that G_1 and G_2 are isomorphic to $K_{1,n}$, a diameter two graph with a cut vertex, and hence $f(G_1) = f(G_2) = n+2$.

In Section 1, we prove that the pebbling number of an arbitrary vertex in the outer cycle (say u_1) to be $2n+2$. In Section II, we prove that the pebbling number of a distinguished vertex (say u) to be $2n+2$.

Section 1. We prove that $f(G, u_1) = 2n + 2,\ n \geq 4$

It is enough to prove that we can pebble u_1 from any configuration of $2n+2$ pebbles in G. If $\alpha_1 \geq 1, \alpha \geq 2, \beta_1 \geq 2$ or $\beta_n \geq 2$, u_1 can be pebbled. We may assume none of these conditions hold.

1.1. We assume that $\beta_1 = 1$ and show that u_1 can be pebbled.

If $p(G_2) \geq f(G_2)+1 = n+3$, one more pebble can be moved to v_1 from any configuration of $(n + 2)$ pebbles in G_2. The resulting configuration will have at least two pebbles at v_1 and hence u_1 can be pebbled. If $p(G_1) = n+2$, u_1 can be pebbled. Therefore we may assume $p(G_1) \leq n+1$ and $p(G_2) \leq n+2$. Since $p(G_1)+p(G_2) = 2n+2$, there are two possibilities.

1.1.1. We assume $p(G_1) = (n + 1) = p(G_2)$.

If there is some gap in G_1, [i.e., some vacant $u_i, i \geq 1$], u_i can be pebbled. Therefore we may assume there is no gap in G_1. The only non pebbling situation is

$$\alpha = 0, \alpha_1 = 0, \alpha_i = 3 \text{ for some } i, \alpha_j = 1,\ \forall\, j \neq 1, i.$$

If there are two gaps in G_2, v_1 lies in a $K_{1,n-2}$. With a total number of $(n + 1)$ pebbles, two pebbles can be placed at v_1. Now u_1 can be pebbled. Therefore there is at most one gap in G_2. Therefore, there is no gap in one of the sections $[v_1, u_i]$ or $[u_i, v_n]$ (figure 3). Therefore one pebble can always be moved to u_1.

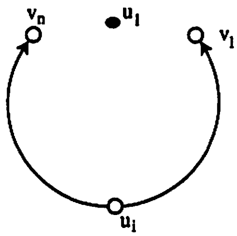

Figure 3: Diagram showing paths through which a pebble can be moved to u_1.

Next, we consider the other possibility.

1.1.2. We assume $p(G_1) = n, p(G_2) = n + 2$

If there are two gaps in G_1, u_1 can be pebbled. We may assume at most one gap in G_1, (say at u_i). As before, we may assume there is no gap in G_2. So, similar to the previous case, the only non pebbling situation is $\beta = 0, \beta_i = 3$ for some i.

Similarly, the case $\beta_n = 1$ may be reduced to the case $\beta_1 = 1$ by relabeling the vertices and we can prove that u_1 can be pebbled.

As before, one of the sections $[v_1, v_i]$ or $[v_i, v_n]$ contains no gap. We can now move a pebble to u_1.

1.2. If $\alpha = 1$, we prove that u_1 can be pebbled.

From 1.1, we may assume $\beta_1 = \beta_n = 0$. First we note that, if $\alpha_i = \beta_i = 1$ for some i, u_1 can be pebbled. For, if $\alpha_j \geq 2$ for some j, u_1 can be pebbled. Therefore we may assume $\alpha_j \leq 1 \; \forall \; j$.

Therefore $p(G_1) \leq n$, and $p(G_2) \geq n+2$. Further v_i lies in a $K_{1,n-2}$ in G_2. Therefore one more pebble can be moved to v_i. In the resulting configuration $\beta_i = 2, \alpha_i = 1$, and $\alpha = 1$. Therefore u_1 can be pebbled.

Clearly, if $\beta^* \geq 4, u_1$ can be pebbled. Assume $\beta^* \leq 3$. Hence (α_i, β_i) is one of the type $(0,0), (1,0), (0,1), (0,2)$ or $(0,3)$.

Also $(\alpha_1, \beta_1)=(0,0)$. According to our notation, $r=|\{i \,:\, (\alpha_i, \beta_i)=(0,2)\}|$ and $s = |\{i \,:\, (\alpha_i, \beta_i) = (0,3)\}|$.

Therefore, we have

$$\begin{aligned} (2n+2) &\leq (n-1-r-s)+1+2r+3s+\beta \\ &= n+r+2s+\beta \\ \Rightarrow \quad n+2 &\leq r+2s+\beta \qquad (1) \end{aligned}$$

If $r+s+\beta \geq 4$, we can have four pebbles at v and u_1 can be pebbled. Therefore we may assume $r+s+\beta \leq 3$ which implies $s \leq 3$. Therefore using (1), $n+2 \leq 6$. Since we have assumed $n \geq 4, n = 4$. Then (1) implies $6 \leq 3+s$. But if $n = 4$, $s \geq 3$, there must be i such that $\beta_i = \beta_{i+1} = 3$, which implies 2 pebbles can be moved to α_i. Thereafter one more pebble can be moved to α and u_1 can be pebbled.

From 1.1 and 1.2, we may assume $\alpha = \alpha_1 = \beta_1 = \beta_n = 0$

1.3. If $\alpha_i = 3$ for some i, we prove that u_1 can be pebbled.

If $\alpha_j = 2$ for some $j \neq i, u_1$ can be pebbled. Thus we may assume $\alpha_j \leq 1, \; \forall \; j \neq i$. Therefore $p(G_1) \leq n+1$ and $p(G_2) \geq n+1$.

First suppose that $i = 2$. Then, two pebbles can be moved to v_1, one pebble from u_2 and another from $K_{1,n-1}$ containing $n+1$ pebbles in which v_1 lies. Hence u_1 can be pebbled. Therefore we may assume $\alpha_i = 3$ for some $i \neq 1, 2, n$.

Also if $\beta_j \geq 2, (j \neq i-1, i+1)$ and $\alpha_j \geq 1$ or $\alpha_{j+1} \geq 1, u_1$ can be pebbled. *Therefore for every j for which $\beta_j \geq 2$, there is a gap in G_1.*

Hence G_1 must contain at least t gaps. (2)

Again, we may assume $\beta_{i-1} = 0$. This is because, if $\beta_{i-1} \geq 2$, u_1 can be pebbled. If $\beta_{i-1} = 1$, then v_{i-1} lies in a $K_{1,n-2}$ containing at least $(n+1)$ pebbles, which is one more than the pebbles needed for this graph. Therefore we can move one more pebble to v_{i-1} and u_1 can be pebbled. Therefore we may assume $\beta_{i-1} = 0$. Similarly, we may assume $\beta_i = 0$.

Hence, we assume $\alpha = \alpha_1 = \beta_1 = \beta_{i-1} = \beta_i = \beta_n = 0$. Therefore, there are at least $(n+1)$ pebbles in $K_{1,n-4}$ in G_2.

If, either $t+\beta \geq 4$ or $\beta^* \geq 4, u_1$ can be pebbled. So, we assume $\beta^* \leq 3$ and $t+\beta \leq 3$. Now,

$$\begin{aligned} p(G_2) &\leq (n-4-t+3t+\beta) = n-4+2t+\beta \qquad \text{and} \\ p(G_1) &= 2n+2-p(G_2) \\ &\geq n+6-2t-\beta \\ &\geq n+3-t \text{ since } t+\beta \leq 3. \end{aligned}$$

Now (2) implies that u_1 lies in a $K_{1,n-t}$ which contains at least $n-t+3$ pebbles. Hence u_1 can be pebbled.

1.4. We assume $\alpha_i = 2$ for some i, $\alpha_j \leq 1$, $j \neq i$ and show that u_1 can be pebbled.

Hence $p(G_1) \leq n$ and $p(G_2) \geq n+2$. If $\beta \geq 4$ or $\beta^* \geq 4$, u_1 can be pebbled. If $r+s+\beta \geq 4$, then 4 pebbles can be placed at v and u_1 can be pebbled. Thus we may assume $r+s+\beta \leq 3, \beta \leq 3$ and $\beta^* \leq 3$. Thus we have,

$$\begin{aligned} n+2 \leq p(G_2) &\leq n-2-r-s+2r+3s+\beta \\ &= n-2+r+2s+\beta. \end{aligned}$$

Hence $4 \leq r+2s+\beta$. We have assumed $r+s+\beta \leq 3$. These two inequalities imply $s \geq 1$.

Suppose $\beta_j = 3$. Further, if $r+(s-1)+\beta \geq 2$, we can place two pebbles on v (without affecting v_j). Then 4 pebbles can be placed at v_j and u_1 can be pebbled. Thus, we may assume $r+s+\beta \leq 2$. This together with the inequality $r+2s+\beta \geq 4$ implies $s = 2$ and $r = \beta = 0$.

Hence there exists $j \neq i$ such that $\beta_j = 3$. If now $\alpha_j \geq 0$, we can place 2 pebbles at α_j. Two pebbles can then be moved to u (considering the pebbles at u_i and u_j). Thus u_1 can be pebbled. Thus $\alpha_j = 0$. Similarly, if $\beta_k = 3$ with $k \neq j, i$ then $\alpha_k = 0$. If $\beta_i = 3, \alpha_{i+1} = 0$. If $j = i+1$, then a similar argument gives $\alpha_{i+2} = 0$. *Thus in any case, there exist distinct j and $k \neq 1$ such that $\alpha_j = \alpha_k = 0$.* We have $p(G_2) \leq n+2, p(G_1) \geq n$. Now, u_1 can be pebbled since u_1 lies in a $K_{1,n-2}$ on which n pebbles are placed.

1.5. We assume $\alpha_i \leq 1$, $\forall\, i$ and show that u_1 can be pebbled.

Let $c = |\{i : \alpha_i = 0, i \neq 1\}|$ and $d = |\{j : \beta_j = 0, j \neq 1, n\}|$. Then $p(G_1) = n - c - 1, p(G_2) = n + 3 + c$.

Let $s = \{v_j : \beta_j \neq 0\}, |s| = n - d - 2$. Let $T = V(G_1)$. $p(s) = n + 3 - c - \beta$. Therefore $|p(s)| - |s| = 5 + c + d - \beta$. (We note that if $\beta \geq 4$, u_1 can be pebbled.). Therefore using lemma 1, we can transfer $\lceil \frac{5+c+d-\beta}{2} \rceil$ pebbles from S to v. Taking into account the β pebbles already present at v, the total number of pebbles at v is atleast $\lceil \frac{5+c+d-\beta}{2} \rceil + \beta = \lceil \frac{5+c+d+\beta}{2} \rceil$.

If $c + d + \beta \geq 2$, there will be 4 pebbles of v and u_1 can be pebbled. So, we have to consider only the case $c + d + \beta \leq 1$.

1.5.1. We suppose $c = d = \beta = 0$.

In the case $p(G_1) = n - 1, p(G_2) = n + 3$. The $n + 3$ pebble on G_2 are placed on the $n - 2$ vertices $v_2, v_3, \ldots, v_{n-1}$ without any gap. Thus, there are $(n+3) - (n-2) = 5$ excess pebble in $p(G_2)$ of which 3 can be transfered to G_1 using lemma 2. But $p(G_1)$ becomes $n + 2$ allowing u_1 to be pebbled. All the cases involving $\beta = 0$ can be proved similarly.

1.5.2. We suppose $c = d = 0, \beta = 1$.

In this, we will be able to transfer only two pebbles to G_1. But, it can be done in such a way that one pebble each is transfered to two different u_i's with $p(u_i) \geq 1$ allowing u_1 to be pebbled.

Section 2. We prove that $f(G, u) = 2n + 2, n \geq 4$.

Suppose p is a pebbling configuration with $|p| = 2n + 2$. We then prove that u can be pebbled.

Each of the following five conditions defines a situation where u can be pebbled.

$$(1)\ \alpha \geq 1, \quad (2)\ \alpha_i \geq 2 \text{ for some } i, \quad (3)\ \beta \geq 8,$$
$$(4)\ (\alpha_i, \beta_i) = (1, 2) \text{ for some } i, \quad (5)\ \beta^* \geq 4$$

We assume none of those conditions hold.

2.1. We prove that if $\alpha_i = 1, \forall\ i, u$ can be pebbled.

If $\beta^* \geq 2$, u can be pebbled. If $\beta^* \leq 1$, $p(G_1) \leq n$ and $p(G_2) \geq n + 2$ implies $\beta \geq 2$. If $\beta_i = 1$ for some i, u can be pebbled. Otherwise $\beta \geq n + 2 \geq 6$, u can be pebbled. We note that a similar proof holds if $\alpha_i = 1$ for all but one i.

2.2. We prove that if $\alpha_i = 0, \forall\ i, u$ can be pebbled.

For, $\alpha_i = 0\ \forall\ i \Rightarrow p(G_1) = 0$ which implies $p(G_2) = 2n + 2$. We consider the four possibilities which arise.

2.2.1. We suppose $\beta^* = 3$.

If $(t-1)+\beta \geq 2$ we can place two pebbles at v. We can then move one more pebble to the vertex v_i which has 3 pebbles. Now $\beta^* = 4$ and u can be pebbled.

Suppose $t+\beta \leq 2$, then,

$$2n+2 \leq (n-t)+3t+\beta = n+2t+\beta$$

That is, $n+2 \leq 2t+\beta \leq 4$, which is not possible as $n \geq 4$.

2.2.2. We suppose $\beta^* = 2$.

Again, if $(r-1)+\beta \geq 4$, the previous argument shows some β_i can be made at least 4. So assume $(r-1)+\beta \leq 3$. That is, $r+\beta \leq 4$. Then

$$\begin{aligned} p(G_2) &\leq (n-r)+2r+\beta \\ &= n+r+\beta \\ &\leq n+4 \end{aligned}$$

That is,

$$2n+2 \leq n+4 \quad \Rightarrow \quad n \leq 2.$$

But by assumption $n \geq 4$.

2.2.3. We suppose $\beta^* = 1$.

We have

$$\begin{aligned} &2n+2 \leq p(G_2) \leq n+\beta \\ \Rightarrow \quad &\beta \geq n+2 \geq 6 \end{aligned}$$

Again, u can be pebbled.

2.2.4. We suppose $\beta^* = 0$.

Here $\beta = 2n+2 \geq 10$ and u can be pebbled.

2.3. If $\alpha_i = 1$ and $\alpha_j = 0$ for some i, j, u can be pebbled.

Divide $\{u_1, u_2, \ldots, u_n\}$ into blocks R_i and S_i, $1 \leq i \leq n$ such that

$$\begin{aligned} p(x) = 1 \quad &\text{if } x \in R_i, 1 \leq i \leq k \\ p(x) = 0 \quad &\text{if } x \in S_i, 1 \leq i \leq k \end{aligned}$$

We may assume $\alpha_1 \in R_1$ and $\alpha_n \in S_k$.

Let $r_i = p(R_i)$ and $r = \sum r_i$.

(We may assume $r \leq n-2$ from remark following 2.1).

There are at least $(r+k)$ v_i's adjacent to at least one u_i with $p(u_i) \neq 0$. If any such β_i has value at least 2, we are done.

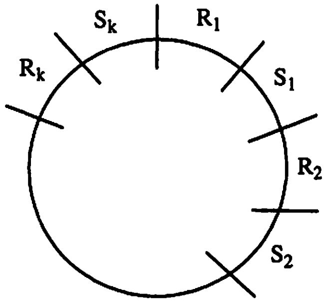

Figure 4: Diagram showing blocks into which the vertices in $\{u_1, u_2, \ldots, u_n\}$ are divided.

If pebbling u is not possible, $\beta_i \leq 1$ for all such v_i.

Therefore for a non pebbling situation, at most $r+k$ pebbles are at these locations. Therefore there are at least $(2n+2)-(r+k)-r = 2n+2-2r-k$ pebbles unused. They are to be placed in $n-(r+k)+1$ locations in G_2. Therefore there are $n-r+1$ extra pebbles.

(i) $\beta_i = 1$ for at least one such i

Suppose $\beta = 1$. One more pebble can then be moved to v as $(n-r) \geq 1$. Now $\beta = 2$ and u can be pebbled.

We also note that the pebbles involved in moving one more pebble to v must come from outside the v_i's adjacent to u_i's since we are considering locations in G_2 other than the $r+k$ positions.

Next suppose β=0: There are $(n-r+1)$ extra pebbles. As $n-r \geq 2$, $n-r+1 \geq 3$, two pebbles can be moved to v and u can be pebbled. Movement of pebbles is similar to that of $\beta = 1$.

(ii) All such $\beta_i = 0$

Then the total number of extra pebbles $= 2n+2-r$.

Using lemma 1, the total number of pebbles which can be placed on v is at least

$$\left\lceil \frac{2n+2-r-\beta}{2} \right\rceil + \beta \geq \left\lceil \frac{2n+2-r}{2} \right\rceil \geq \left\lceil \frac{2n+2-(n-1)}{2} \right\rceil = \left\lceil \frac{n+3}{2} \right\rceil \geq 4$$

as $n \geq 4$.

Hence, at least 4 pebbles can be placed on v.

Since at least one $\alpha_i = 1$, this implies u can be pebbled. □

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References

[1] Boris Bukh, "Maximum pebbling number of graphs of diameter three". *Preprint* Oct. 12, 2003.

[2] F R K Chung, "Pebbling in Hypercubes", SIAM. *J. Disc. Math.* 2 (1989), 467–472.

[3] T A Clarke, R A Hochberg and G H Hulbert, "Pebbling in diameter two graphs and products of paths". *J. Graph Theory* Vol. 25, Number 2, (1997) 119–128.