Super edge-antimagic total labelings of $mK_{n,n,n}$

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Abstract

An (a,d)-*edge-antimagic total* labeling on (p,q)-graph G is a one-to-one map f from $V(G) \cup E(G)$ onto the integers $1, 2, \dots, p+q$ with the property that the edge-weights, $w(uv) = f(u)+f(v)+f(uv)$ where $uv \in E(G)$, form an arithmetic progression starting from a and having common difference d. Such a labeling is called *super* if the smallest possible labels appear on the vertices. In this paper, we investigate the existence of super (a,d)-edge-antimagic total labeling of disjoint union of multiple copies of complete tripartite graph and disjoint union of stars.

Key Words: *super (a,d)-edge-antimagic total labeling, $mK_{n,n,n}$ and $K_{1,m} \cup 2sK_{1,n}$.*

1 Introduction and Definition

An important subject of research in recent years has been the construction of graphs whose vertices and edges may be labeled by consecutive integers so that the weights of edges have various numerical properties based on these labels (see [5] and [11]).

An (a,d)-edge-antimagic vertex labeling of a (p,q)-graph G with vertex set $V(G)$ is a one-to-one map f from $V(G)$ onto the integers $1, 2, \dots, p$ with the property that the edge-weights form an arithmetic progression $a, a+d, a+2d, \dots, a+(q-1)d$ where edge-weight of an edge uv is the sum of the vertex labels corresponding to the vertices u and v, and $a > 0, d \geq 0$.

Let $f : V(G) \cup E(G) \rightarrow \{1, 2, \dots, p+q\}$ be a total labeling of a (p,q)-graph G with vertex set $V(G)$ and edge set $E(G)$. The edge-weight of an edge uv under the total labeling f is $w(uv) = f(u) + f(uv) + f(v)$.

By an *(a,d)-edge-antimagic total labeling* of a (p,q)-graph G we mean a bijective function $f : V(G) \cup E(G) \to \{1, 2, \ldots, p+q\}$ such that the set of all edge-weights, $\{w(uv) : uv \in E(G)\}$, is $\{a, a+d, a+2d, \ldots, a+(q-1)d\}$, for two integers $a > 0$ and $d \geq 0$. Such a labeling is called *super* if the vertex labels are the integers $1, 2, \ldots, p$ as the smallest possible labels. A graph G is called (a,d)-edge-antimagic total or super (a,d)-edge-antimagic total if there exists an (a,d)-edge-antimagic total or super (a,d)-edge-antimagic total labeling, respectively.

The definition of an (a,d)-edge-antimagic total labeling was introduced by Simanjuntak, Bertault and Miller [8]. The (a,d)-edge-antimagic total labelings and super (a,d)-edge-antimagic total labelings are natural extensions of the notion of edge-magic labeling (see [7], where edge-magic labeling is called *magic valuation*) and super edge-magic labeling introduced by Enomoto, Lladó, Nakamigawa and Ringel in [3].

In this paper we study edge-antimagic properties of disconnected graphs. Some constructions of super $(a,0)$-edge-antimagic total labelings for $nC_k \cup mP_k, K_{1,m} \cup K_{1,n}$ have been described by Ivančo and Lučkaničová in [6] and super (a,d)-edge-antimagic labelings for $P_n \cup P_{n+1}, nP_2 \cup P_n$ and $nP_2 \cup P_{n+2}$ have been shown by Sudarsana, Ismaimuza, Baskoro and Assiyatun in [9].

We will concentrate on the existence of super (a,d)-edge-antimagic total labeling of disjoint union of multiple copies of complete tripartite graph and also disjoint union of stars.

2 Disjoint union of tripartite graph

Let $mK_{n,n,n}$ be a disjoint union of m copies of tripartite graph $K_{n,n,n}$ with a vertex set $V(mK_{n,n,n}) = \{x_i^l : 1 \leq i \leq n,\ 1 \leq l \leq m\} \cup \{y_j^l : 1 \leq j \leq n,\ 1 \leq l \leq m\} \cup \{z_k^l : 1 \leq k \leq n,\ 1 \leq l \leq m\}$ and with edge set $E(mK_{n,n,n}) = \bigcup_{l=1}^{m} \{x_i^l y_j^l, x_i^l z_k^l, y_j^l z_k^l : 1 \leq i \leq n,\ 1 \leq j \leq n,\ 1 \leq k \leq n\}$. Thus $p = |V(mK_{n,n,n})| = 3mn$ and $q = |E(mK_{n,n,n})| = 3mn^2$.

The minimum possible edge-weight in a super (a,d)-edge-antimagic total labeling of a (p,q) graph is at least $p+4$. On the other hand, the maximum possible edge-weight is at most $3p+q-1$. Then it holds the following inequality

$$a + (q-1)d \leq 3p + q - 1$$

which gives an upper bound on the parameter d

$$d \leq \frac{2p+q-5}{q-1}. \tag{1}$$

If $mK_{n,n,n}$, $m \geq 2$ and $n \geq 1$, is super (a,d)-edge-antimagic total then from (1) it follows that $d < 3$.

Theorem 1 *The graph $mK_{n,n,n}$ has an $(a,1)$-edge-antimagic vertex labeling if and only if $n=1$ and m is odd, $m \geq 3$.*

Proof Assume that $mK_{n,n,n}$ has an $(a,1)$-edge-antimagic vertex labeling $f_1 : V(mK_{n,n,n}) \to \{1,2,\dots,3mn\}$ and $W = \{w(uv) : uv \in E(mK_{n,n,n})\} = \{a, a+1, a+2, \dots, a+(3mn^2-1)\}$ is the set of edge-weights. The sum of the edge-weights in the set W is

$$\sum_{uv\in E(mK_{n,n,n})} w(uv) = 3mn^2\left(a + \frac{3mn^2-1}{2}\right). \qquad (2)$$

In the computation of the edge-weights of $mK_{n,n,n}$, the label of each vertex is used $2n$ times. The sum of all vertex labels used to calculate the edge-weights is equal to

$$2n \sum_{u\in V(mK_{n,n,n})} f_1(u) = 3mn^2(1+3mn). \qquad (3)$$

Combining (2) and (3) gives the following equation

$$\sum_{uv\in E(mK_{n,n,n})} w(uv) = 2n \sum_{u\in V(mK_{n,n,n})} f_1(u)$$

and immediately follows that

$$a = \frac{3mn(2-n)+3}{2}. \qquad (4)$$

The minimum edge-weight a is a positive integer if and only if $n=1$ and m is odd, $m \geq 3$.

The required $\left(\frac{3m+3}{2}, 1\right)$-edge-antimagic vertex labeling f_1 can be defined in the following way.

$$f_1(x_1^l) = \begin{cases} \frac{l+1}{2}, & \text{if } l \text{ is odd} \\ \frac{m+1+l}{2}, & \text{if } l \text{ is even} \end{cases}$$

$$f_1(y_1^l) = \begin{cases} \frac{3m+l}{2}, & \text{if } l \text{ is odd} \\ m+\frac{l}{2}, & \text{if } l \text{ is even} \end{cases}$$

$$f_1(z_1^l) = 3m+1-l, \quad \text{for all } 1 \leq l \leq m.$$

This completes the proof. □

Theorem 2 *For $d \in \{0,2\}$, the graph $mK_{n,n,n}$ is super (a,d)-edge-antimagic total if and only if $n=1$ and m is odd, $m \geq 3$.*

Proof

Case 1. $d = 0$.

Figueroa-Centeno, Ichishima and Muntaner-Batle (see [4], Lemma 1) showed that a (p,q) graph G is super magic (super $(a,0)$-edge-antimagic total) if and only if there exists an $(a-p-q,1)$-edge-antimagic vertex labeling. According to Theorem 1 the graph $mK_{n,n,n}$ has $(\frac{3m+3}{2},1)$-edge-antimagic vertex labeling if and only if $n=1$ and m is odd. With respect to Lemma 1 from [4] and for $p = 3mn, q = 3mn^2$, we have that the graph $mK_{n,n,n}$ has a super $(\frac{15m+3}{2},0)$-edge-antimagic total labeling if and only if $n = 1$ and m is odd.

Case 2. $d = 2$.

Assume that $mK_{n,n,n}$, $m \geq 2, n \geq 1$, has a super (a,d)-edge-antimagic total labeling $f_2 : V(mK_{n,n,n}) \cup E(mK_{n,n,n}) \to \{1,2,\ldots,3mn+3mn^2\}$ and $\{w(uv) = f_2(u)+f_2(uv)+f_2(v) : uv \in E(mK_{n,n,n})\} = \{a, a+d, a+2d, \ldots, a+(3mn^2-1)d\}$ is the set of edge-weights.

$$\sum_{uv\in E(mK_{n,n,n})} w(uv) = 3mn^2\left(a + \frac{(3mn^2-1)d}{2}\right) \tag{5}$$

is the sum of all the edge-weights. In the computation of the edge-weights of $mK_{n,n,n}$ under the labeling f_2, the label of each vertex is used $2n$ times and the label of each edge is used once. Thus

$$2n\sum_{l=1}^{m}\left(\sum_{i=1}^{n} f_2(x_i^l) + \sum_{j=1}^{n} f_2(y_j^l) + \sum_{k=1}^{n} f_2(z_k^l)\right) +$$
$$\sum_{l=1}^{m}\left(\sum_{i=1}^{n}\sum_{j=1}^{n} f_2(x_i^l y_j^l) + \sum_{i=1}^{n}\sum_{k=1}^{n} f_2(x_i^l z_k^l) + \sum_{j=1}^{n}\sum_{k=1}^{n} f_2(y_j^l z_k^l)\right) =$$
$$9mn^2\left(\frac{mn^2+4mn+1}{2}\right). \tag{6}$$

Since we assume that f_2 is super (a,d)-edge-antimagic total labeling then the sum of edge-weights is equal to the sum of vertex and edge labels. Combining (5) and (6) gives the following equation

$$a = \frac{3mn^2+12mn+3-(3mn^2-1)d}{2}. \tag{7}$$

The minimum possible edge-weight under the labeling f_2 is at least $3mn+4$. So, for $d = 2$ the equation (7) gives the following inequalities

$$\begin{aligned} 3mn+4 &\leq \frac{12mn-3mn^2+5}{2} \\ mn(n-2) &\leq -1. \end{aligned}$$

The last inequality is true if and only if $n = 1$. Then from (7) it follows that $a = \frac{9m+5}{2}$ and is an integer if and only if m is odd.

Bača, Lin, Miller and Simanjuntak (see [1], Theorem 5) have proved that if (p,q)-graph G has an (a,d)-edge-antimagic vertex labeling then G has a super $(a+p+1, d+1)$-edge-antimagic total labeling. Since labeling f_1 from the proof of Theorem 1 is a $(\frac{3m+3}{2}, 1)$-edge-antimagic vertex labeling of $mK_{1,1,1}$ when m is odd, then with respect to Theorem 5 from [1] we have that $mK_{1,1,1}$, for m odd, $m \geq 3$, has a super $(\frac{9m+5}{2}, 2)$-edge-antimagic total labeling. □

Theorem 3 *The graph* $mK_{n,n,n}$ *has a super* $(6mn+2, 1)$*-edge-antimagic total labeling for every* $m \geq 2$ *and* $n \geq 1$.

Proof If $d = 1$ then from (7) it follows that $a = 6mn + 2$. Define the bijective function $f_3 : V(mK_{n,n,n}) \cup E(mK_{n,n,n}) \rightarrow \{1, 2, \ldots, 3mn + 3mn^2\}$ for $m \geq 2$ and $n \geq 1$ in the following way:

$$\begin{aligned} f_3(x_i^l) &= (3i-3)m + l, \text{ for } 1 \leq i \leq n \text{ and } 1 \leq l \leq m, \\ f_3(y_j^l) &= (3j-2)m + l, \text{ for } 1 \leq j \leq n \text{ and } 1 \leq l \leq m, \\ f_3(z_k^l) &= (3k-1)m + l, \text{ for } 1 \leq k \leq n \text{ and } 1 \leq l \leq m. \end{aligned}$$

If $1 \leq l \leq m$ then

$$f_3(x_i^l y_j^l) = 3mn(n+1-2j+2i) + 3m \sum_{t=0}^{j-i-1} (1+2t) + 1 - l - 3m(i-1),$$

for $1 \leq i \leq n$ and $i \leq j \leq n$,

$$f_3(x_i^l y_j^l) = 3mn(n+2-2i+2j) + 6m \sum_{t=0}^{i-j-1} t + 1 - l - 3m(j-1),$$

for $1 \leq j \leq n-1$ and $j+1 \leq i \leq n$,

$$f_3(y_j^l z_k^l) = 3mn(n+1-2k+2j) + 3m \sum_{t=0}^{k-j-1} (1+2t) + 1 - l - m(3j-2),$$

for $1 \leq j \leq n$ and $j \leq k \leq n$,

$$f_3(y_j^l z_k^l) = 3mn(n+2-2j+2k) + 6m \sum_{t=0}^{j-k-1} t + 1 - l - m(3k-2),$$

for $1 \leq k \leq n-1$ and $k+1 \leq j \leq n$,

$$f_3(z_k^l x_i^l) = 3mn(n+3-2i+2k) + 3m \sum_{t=0}^{i-k-2} (1+2t) + 1 - l - m(3k-1),$$

for $1 \leq k \leq n-1$ and $k+1 \leq i \leq n$,

$$f_3(z_k^l x_i^l) = 3mn(n-2k+2i) + 6m \sum_{t=0}^{k-i} t + 1 - l - m(3i-4),$$

for $1 \le i \le n$ and $i \le k \le n$.

Let $A^l = (a^l_{ij})$ be a system of square matrices for all $l = 1, 2, \ldots, m$, where $a^l_{ij} = f_3(x^l_i) + f_3(y^l_j)$ for $1 \le i \le n, 1 \le j \le n$ and $\alpha = 3mn + 2l, \beta = 6mn + 2l$.

$$A^l = \begin{bmatrix} m+2l & 4m+2l & 7m+2l & \ldots & \alpha-5m & \alpha-2m \\ 4m+2l & 7m+2l & 10m+2l & \ldots & \alpha-2m & \alpha+m \\ 7m+2l & 10m+2l & 13m+2l & \ldots & \alpha+m & \alpha+4m \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha-5m & \alpha-2m & \alpha+m & \ldots & \beta-11m & \beta-8m \\ \alpha-2m & \alpha+m & \alpha+4m & \ldots & \beta-8m & \beta-5m \end{bmatrix}$$

Let $B^l = (b^l_{jk})$ be a system of square matrices for all $l = 1, 2, \ldots, m$, where $b^l_{jk} = f_3(y^l_j) + f_3(z^l_k)$ for $1 \le j \le n, 1 \le k \le n$ and $\alpha = 3mn + 2l, \beta = 6mn + 2l$.

$$B^l = \begin{bmatrix} 3m+2l & 6m+2l & 9m+2l & \ldots & \alpha-3m & \alpha \\ 6m+2l & 9m+2l & 12m+2l & \ldots & \alpha & \alpha+3m \\ 9m+2l & 12m+2l & 15m+2l & \ldots & \alpha+3m & \alpha+6m \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha-3m & \alpha & \alpha+3m & \ldots & \beta-9m & \beta-6m \\ \alpha & \alpha+3m & \alpha+6m & \ldots & \beta-6m & \beta-3m \end{bmatrix}$$

Let $C^l = (c^l_{ki})$ be a system of square matrices for all $l = 1, 2, \ldots, m$, where $c^l_{ki} = f_3(z^l_k) + f_3(x^l_i)$ for $1 \le k \le n, 1 \le i \le n$ and $\alpha = 3mn + 2l, \beta = 6mn + 2l$.

$$C^l = \begin{bmatrix} 2m+2l & 5m+2l & 8m+2l & \ldots & \alpha-4m & \alpha-m \\ 5m+2l & 8m+2l & 11m+2l & \ldots & \alpha-m & \alpha+2m \\ 8m+2l & 11m+2l & 14m+2l & \ldots & \alpha+2m & \alpha+5m \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha-4m & \alpha-m & \alpha+2m & \ldots & \beta-10m & \beta-7m \\ \alpha-m & \alpha+2m & \alpha+5m & \ldots & \beta-7m & \beta-4m \end{bmatrix}$$

The systems of square matrices A^l, B^l and C^l, for $l = 1, 2, \ldots, m$, describe the edge-weights of $mK_{n,n,n}$ under vertex labeling. The labels of edges of $mK_{n,n,n}$ described by labeling f_3 can be exhibited by the systems of square matrices $H^l = (h^l_{ij}), P^l = (p^l_{jk})$ and $R^l = (r^l_{ki})$ for $l = 1, 2, \ldots, m$, where

$$\begin{aligned} h^l_{ij} &= f_3(x^l_i y^l_j), \text{ for } 1 \le i \le n \text{ and } 1 \le j \le n \\ p^l_{jk} &= f_3(y^l_j z^l_k), \text{ for } 1 \le j \le n \text{ and } 1 \le k \le n \\ r^l_{ki} &= f_3(z^l_k x^l_i), \text{ for } 1 \le k \le n \text{ and } 1 \le i \le n, \text{ respectively.} \end{aligned}$$

For $\gamma = 3mn^2 + 1 - l$, $\xi = 3mn$ and $\delta = \xi + 1 - l$, the systems of square matrices are as follows.

$H^l =$

$$\begin{bmatrix} \gamma+\xi & \gamma-\xi+3m & \gamma-3\xi+12m & \dots & \delta+12m & \delta+3m \\ \gamma & \gamma+\xi-3m & \gamma-\xi & \dots & \delta+24m & \delta+9m \\ \gamma-2\xi+6m & \gamma-3m & \gamma+\xi-6m & \dots & \delta+42m & \delta+21m \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \delta+18m & \delta+33m & \delta+54m & \dots & \gamma+6m & \gamma-\xi+6m \\ \delta+6m & \delta+15m & \delta+30m & \dots & \gamma-\xi+6m & \gamma+3m \end{bmatrix},$$

$P^l =$

$$\begin{bmatrix} \gamma+\xi-m & \gamma-\xi+2m & \gamma-3\xi+11m & \dots & \delta+11m & \delta+2m \\ \gamma-m & \gamma+\xi-4m & \gamma-\xi-m & \dots & \delta+23m & \delta+8m \\ \gamma-2\xi+5m & \gamma-4m & \gamma+\xi-7m & \dots & \delta+41m & \delta+20m \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \delta+17m & \delta+32m & \delta+53m & \dots & \gamma+5m & \gamma-2\xi+8m \\ \delta+5m & \delta+14m & \delta+29m & \dots & \gamma-\xi+5m & \gamma+2m \end{bmatrix},$$

$R^l =$

$$\begin{bmatrix} \gamma+m & \gamma+\xi-2m & \gamma-\xi+m & \dots & \delta+25m & \delta+10m \\ \gamma-2\xi+7m & \gamma-2m & \gamma+\xi-5m & \dots & \delta+43m & \delta+22m \\ \gamma-4\xi+19m & \gamma-2\xi+4m & \gamma-5m & \dots & \delta+67m & \delta+40m \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \delta+7m & \delta+16m & \delta+31m & \dots & \gamma-\xi+7m & \gamma+4m \\ \delta+m & \delta+4m & \delta+13m & \dots & \gamma-3\xi+13m & \gamma-\xi+4m \end{bmatrix}.$$

All edge-weights of $mK_{n,n,n}$ under the total labeling f_3 can be presented as the systems of square matrices: $S^l = A^l + H^l, T^l = B^l + P^l$ and $U^l = C^l + R^l$ for $l = 1, 2, \dots, m$. It is not difficult to verify by a routine procedure that the systems of square matrices S^l, T^l and U^l, for $l = 1, 2, \dots, m$, are formed from consecutive integers $6mn+2, 6mn+3, 6mn+4, \dots, 3mn^2+6mn, 3mn^2+6mn+1$. This implies that the total labeling f_3 is super $(6mn+2, 1)$-edge-antimagic total for every $m \geq 2$ and $n \geq 1$. □

3 Disjoint union of stars

In this section we will study super edge-antimagicness of a disjoint union of $K_{1,m}$ and $2sK_{1,n}$, denoted by $K_{1,m} \cup 2sK_{1,n}$. The disjoint union of $K_{1,m}$ and $2sK_{1,n}$ is the disconnected graph with vertex set $V(K_{1,m} \cup 2sK_{1,n}) = \{x_{1,j} : 0 \leq j \leq m\} \cup \{x_{i,k} : 2 \leq i \leq 2s+1, 0 \leq k \leq n\}$ and edge set $E(K_{1,m} \cup 2sK_{1,n}) = \{x_{1,0}x_{1,j} : 1 \leq j \leq m\} \cup \{x_{i,0}x_{i,k} : 2 \leq i \leq 2s+1, 1 \leq k \leq n\}$. Thus $p = |V(K_{1,m} \cup 2sK_{1,n})| = m + 2s(n+1) + 1$ and $q = |E(K_{1,m} \cup 2sK_{1,n})| = m + 2sn$.

If the graph $K_{1,m} \cup 2sK_{1,n}$ is super (a,d)-edge-antimagic total then from (1) we have that

$$d \leq 3 + \frac{4s}{m+2sn-1}. \tag{8}$$

By applying the equation (8) for values of m, n and s we obtain the following.

(i) For graph $K_{1,m} \cup 2sK_{1,n}, m = n = 1, s \geq 1$, there is no super (a,d)-edge-antimagic total labeling with $d > 5$.

(ii) For graph $K_{1,m} \cup 2sK_{1,n}$, $n + m = 3$, $s \geq 1$, there is no super (a,d)-edge-antimagic total labeling with $d > 4$.

(iii) For graph $K_{1,m} \cup 2sK_{1,n}, n \geq 2, m \geq 2$ and $s \geq 1$, there is no super (a,d)-edge-antimagic total labeling with $d > 3$.

If $m = n = 1$ then the graph $K_{1,m} \cup 2sK_{1,n}$ is a disjoint union of $2s+1$ copies of P_2, denoted by $(2s+1)P_2$. We have proved in [2] that for every $s \geq 1$ and $d \in \{0,1,2,3,4,5\}$ the graph $(2s+1)P_2$ has a super (a,d)-edge-antimagic total labeling.

Theorem 4 *The graph $K_{1,m} \cup 2sK_{1,n}$, $m \geq 1, n \geq 1$ and $s \geq 1$, has an $(3s+3,1)$-edge-antimagic vertex labeling.*

Proof Let us distinguish two cases.

Case 1. $m \geq n$.

Define the vertex labeling $f_4 : V(K_{1,m} \cup 2sK_{1,n}) \to \{1, 2, \ldots, m + 2s(n+1) + 1\}$ in the following way:

$$f_4(x_{i,0}) = \begin{cases} s+i, & \text{if } 1 \leq i \leq s+1 \\ i-s-1, & \text{if } s+2 \leq i \leq 2s+1. \end{cases}$$

$$f_4(x_{1,j}) = \begin{cases} (2s+1)j+1, & \text{if } 1 \leq j \leq n+1 \\ 2s(n+1)+j+1, & \text{if } n+2 \leq j \leq m. \end{cases}$$

$$f_4(x_{i,k}) = (2s+1)k + i, \text{ for } 2 \leq i \leq 2s+1 \text{ and } 1 \leq k \leq n.$$

Clearly, the values of f_4 are $1, 2, \ldots, m + 2s(n+1) + 1$. The edge-weights of $K_{1,m} \cup 2sK_{1,n}$ under the labeling f_4 constitute the sets

$$\begin{aligned}
W^1_{f_4} &= \{w^1_{f_4}(x_{1,0}x_{1,j}) = f_4(x_{1,0}) + f_4(x_{1,j}) : \ 1 \leq j \leq n+1\} \\
&= \{(2s+1)j + s + 2 : \ 1 \leq j \leq n+1\}, \\
W^2_{f_4} &= \{w^2_{f_4}(x_{1,0}x_{1,j}) = f_4(x_{1,0}) + f_4(x_{1,j}) : \ n+2 \leq j \leq m\} \\
&= \{(2n+3)s + j + 2 : \ n+2 \leq j \leq m\}, \\
W^3_{f_4} &= \{w^3_{f_4}(x_{i,0}x_{i,k}) = f_4(x_{i,0}) + f_4(x_{i,k}) : \ 2 \leq i \leq s+1 \text{ and } 1 \leq k \leq n\} \\
&= \{(2s+1)k + 2i + s : \ 2 \leq i \leq s+1 \text{ and } 1 \leq k \leq n\} \text{ and} \\
W^4_{f_4} &= \{w^4_{f_4}(x_{i,0}x_{i,k}) = f_4(x_{i,0}) + f_4(x_{i,k}) : \ s+2 \leq i \leq 2s+1 \text{ and } 1 \leq k \leq n\} \\
&= \{(2s+1)k + 2i - s - 1 : \ s+2 \leq i \leq 2s+1 \text{ and } 1 \leq k \leq n\}.
\end{aligned}$$

It is not difficult to check that the set $\bigcup_{r=1}^{4} W_{f_4}^r = \{3s+3, 3s+4, \ldots, (3+2n)s+m+2\}$.

Case 2. $m < n$.

For $m \geq 1, n \geq 1$ and $s \geq 1$ define the bijection $f_5 : V(K_{1,m} \cup 2sK_{1,n}) \to \{1, 2, \ldots, m + 2s(n+1) + 1\}$ as follows:

$$\begin{aligned} f_5(x_{i,0}) &= f_4(x_{i,0}), \\ f_5(x_{1,j}) &= (2s+1)j + 1, \text{ for } 1 \leq j \leq m, \end{aligned}$$

$$f_5(x_{i,k}) = \begin{cases} (2s+1)k + i, & \text{if } 2 \leq i \leq 2s+1 \text{ and } 1 \leq k \leq m \\ 2sk + m + i, & \text{if } 2 \leq i \leq 2s+1 \text{ and } m+1 \leq k \leq n. \end{cases}$$

Then for the edge-weights of $K_{1,m} \cup 2sK_{1,n}$ we have:

$W_{f_5}^1 = \{w_{f_5}^1(x_{1,0}x_{1,j}) = f_5(x_{1,0}) + f_5(x_{1,j}) : 1 \leq j \leq m\} = \{(2s+1)j + s + 2 : 1 \leq j \leq m\}$,

$W_{f_5}^2 = \{w_{f_5}^2(x_{i,0}x_{i,k}) = f_5(x_{i,0}) + f_5(x_{i,k}) : 2 \leq i \leq s+1 \text{ and } 1 \leq k \leq m\} = \{(2s+1)k + 2i + s : 2 \leq i \leq s+1 \text{ and } 1 \leq k \leq m\}$,

$W_{f_5}^3 = \{w_{f_5}^3(x_{i,0}x_{i,k}) = f_5(x_{i,0}) + f_5(x_{i,k}) : s+2 \leq i \leq 2s+1 \text{ and } 1 \leq k \leq m\} = \{(2s+1)k + 2i - s - 1 : s+2 \leq i \leq 2s+1 \text{ and } 1 \leq k \leq m\}$,

$W_{f_5}^4 = \{w_{f_5}^4(x_{i,0}x_{i,k}) = f_5(x_{i,0}) + f_5(x_{i,k}) : 2 \leq i \leq s+1 \text{ and } m+1 \leq k \leq n\} = \{(2k+1)s + 2i + m : 2 \leq i \leq s+1 \text{ and } m+1 \leq k \leq n\}$,

$W_{f_5}^5 = \{w_{f_5}^5(x_{i,0}x_{i,k}) = f_5(x_{i,0}) + f_5(x_{i,k}) : s+2 \leq i \leq 2s+1 \text{ and } m+1 \leq k \leq n\} = \{(2k-1)s + 2i + m - 1 : s+2 \leq i \leq 2s+1 \text{ and } m+1 \leq k \leq n\}$,

and $\bigcup_{r=1}^{5} W_{f_5}^r = \{3s+3, 3s+4, \ldots, (3+2n)s+m+2\}$ consists of consecutive integers. This implies that f_4 and f_5 are $(3s+3, 1)$-edge-antimagic vertex labelings. □

Theorem 5 *For* $m \geq 1, n \geq 1$ *and* $s \geq 1$ *the graph* $K_{1,m} \cup 2sK_{1,n}$ *has a super* $((4n+5)s+2m+4, 0)$*-edge-antimagic total labeling and a super* $((2n+5)s+m+5, 2)$*-edge-antimagic total labeling.*

Proof From Theorem 4 we have that for $m \geq 1, n \geq 1$ and $s \geq 1$ the graph $K_{1,m} \cup 2sK_{1,n}$ has $(3s+3, 1)$-edge-antimagic vertex labeling.

According to Lemma 1 from [4] for $p = m + 2s(n+1) + 1$ and $q = m + 2sn$ there is a super $((4n+5)s + 2m + 4, 0)$-edge-antimagic total labeling.

In the same way, in the light of Theorem 5 proved in [1] (see proof of Theorem 2), we have that for $p = m + 2s(n+1) + 1$ the graph $K_{1,m} \cup 2sK_{1,n}$, $m \geq 1, n \geq 1$ and $s \geq 1$, has a super $((2n+5)s + m + 5, 2)$-edge-antimagic total labeling. □

Sugeng et al. in [10] proved the following lemma.

Lemma 1 *Let* $\mathfrak{U}$ *be a sequence* $\mathfrak{U} = \{c, c+1, c+2, \ldots c+k\}$, k *even. Then there exists a permutation* $\Pi(\mathfrak{U})$ *of the elements of* $\mathfrak{U}$ *such that* $\mathfrak{U} + \Pi(\mathfrak{U}) = \{2c + \frac{k}{2}, 2c + \frac{k}{2} + 1, 2c + \frac{k}{2} + 2, \ldots, 2c + \frac{3k}{2} - 1, 2c + \frac{3k}{2}\}$.

This lemma will be useful for proving the following theorem.

Theorem 6 *If m is odd then the graph $K_{1,m} \cup 2sK_{1,n}$ for $m \geq 1, n \geq 1$ and $s \geq 1$ has a super $(s(3n+5)+\frac{3m+9}{2}, 1)$-edge-antimagic total labeling.*

Proof. Consider the vertex labelings f_4 and f_5 of the graph $K_{1,m} \cup 2sK_{1,n}$ from Theorem 4 which are $(3s+3, 1)$-edge-antimagic vertex labeling. The set of edge-weights gives the sequence $\mathfrak{U} = \{c, c+1, c+2, \ldots, c+k\}$ for $c = 3s+3$ an $k = 2ns+m-1$. The value k is even for m odd. According to Lemma 1 [10] there exists a permutation $\Pi(\mathfrak{U})$ of the elements of $\mathfrak{U}$, such that $\mathfrak{U} + [\Pi(\mathfrak{U}) - c + p + 1] = \{s(3n+5)+\frac{3m+9}{2}, s(3n+5)+\frac{3m+11}{2}, s(3n+5)+\frac{3m+13}{2}, \ldots, s(5n+5)+\frac{5m+7}{2}\}$. If $[\Pi(\mathfrak{U}) - c + p + 1]$ is an edge labeling of $K_{1,m} \cup 2sK_{1,n}$ for m odd, $m \geq 1, n \geq 1, s \geq 1$, then $\mathfrak{U} + [\Pi(\mathfrak{U}) - c + p + 1]$ determines the set of edge-weights of the graph $K_{1,m} \cup 2sK_{1,n}$ and the resulting total labeling is super $(s(3n+5)+\frac{3m+9}{2}, 1)$-edge-antimagic total. □

For m even we have not found any super $(a, 1)$-edge-antimagic total labeling. Therefore we propose the following open problem.

Open Problem 1 *For m even, $m \geq 2, n \geq 1$ and $s \geq 1$, determine if there is a super $(a, 1)$-edge-antimagic total labeling of $K_{1,m} \cup 2sK_{1,n}$.*

Theorem 7 *For $s \geq 1$ the graph $K_{1,2} \cup 2sK_{1,1}$ has a super $(5s+7, 4)$-edge-antimagic total labeling.*

Proof For $s \geq 3$ we consider the following function $f_6 : V(K_{1,2} \cup 2sK_{1,1}) \to \{1, 2, \ldots, 4s+3\}$, where

$$f_6(x_{i,0}) = \begin{cases} 3s+3, & \text{if } i = 1 \\ s+2i-2, & \text{if } 2 \leq i \leq s+3. \\ 2s+i+1, & \text{if } s+4 \leq i \leq 2s+1. \end{cases}$$

$$f_6(x_{1,j}) = \begin{cases} s+3, & \text{if } j = 1 \\ 4s+3, & \text{if } j = 2. \end{cases}$$

$$f_6(x_{i,1}) = \begin{cases} i-1, & \text{if } 2 \leq i \leq s+2 \\ 2i-s-1, & \text{if } s+3 \leq i \leq 2s+1. \end{cases}$$

In the case $s = 1$, label $f_7(x_{1,0}) = 6, f_7(x_{1,1}) = 4, f_7(x_{1,2}) = 7, f_7(x_{2,0}) = 3, f_7(x_{2,1}) = 1, f_7(x_{3,0}) = 5$ and $f_7(x_{3,1}) = 2$.

If $s = 2$ then label $f_8(x_{1,0}) = 9$, $f_8(x_{1,1}) = 5$, $f_8(x_{1,2}) = 11$, $f_8(x_{2,0}) = 4$, $f_8(x_{2,1}) = 1$, $f_8(x_{3,0}) = 6$, $f_8(x_{3,1}) = 2$, $f_8(x_{4,0}) = 8$, $f_8(x_{4,1}) = 3$, $f_8(x_{5,0}) = 10$ and $f_8(x_{5,1}) = 7$.

It is a matter of routine checking to see that the vertex labelings f_6, f_7 and f_8 are $(s+3, 3)$-edge-antimagic vertex. According to Theorem 5 proved in [1] for $p = 4s+3, s \geq 1$, there is a super $(5s+7, 4)$-edge-antimagic total labeling of $K_{1,m} \cup 2sK_{1,n}$. □

Open Problem 2 *For $s \geq 1$ determine if there is a super $(a, 4)$-edge-antimagic total labeling of $K_{1,1} \cup 2sK_{1,2}$.*

In the case when $d = 3$, $m \geq 2$, $n \geq 2$ and $s \geq 1$, we do not have any answer for super edge-antimagicness of $K_{1,m} \cup 2sK_{1,n}$. Therefore we propose the following open problem.

Open Problem 3 *For the graph $K_{1,m} \cup 2sK_{1,n}$, $m \geq 2$, $n \geq 2$ and $s \geq 1$, determine if there is a super $(a, 3)$-edge-antimagic total labeling.*

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