The γ-Spectrum of a Graph

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ABSTRACT

Let G be a graph of order n and size m. A γ-labeling of G is a one-to-one function $f : V(G) \to \{0, 1, 2, \ldots, m\}$ that induces a labeling $f' : E(G) \to \{1, 2, \ldots, m\}$ of the edges of G defined by $f'(e) = |f(u) - f(v)|$ for each edge $e = uv$ of G. The value of a γ-labeling f is defined as

$$\text{val}(f) = \sum_{e \in E(G)} f'(e).$$

The γ-spectrum of a graph G is defined as

$$\text{spec}(G) = \{\text{val}(f) : f \text{ is a } \gamma\text{-labeling of } G\}.$$

The γ-spectra of paths, cycles, and complete graphs are determined.

Key Words: γ-labeling, γ-spectrum.

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1 Introduction

For a graph G of order n and size m, a *γ-labeling* of G is defined in [1] as a one-to-one function $f : V(G) \to \{0, 1, 2, \ldots, m\}$ that induces a labeling $f' : E(G) \to \{1, 2, \ldots, m\}$ of the edges of G defined by

$$f'(e) = |f(u) - f(v)| \text{ for each edge } e = uv \text{ of } G.$$

Therefore, a graph G of order n and size m has a γ-labeling if and only if $m \geq n - 1$. In particular, every connected graph has a γ-labeling. If the induced edge-labeling f' of a γ-labeling f is also one-to-one, then f is a *graceful labeling*, one of the most studied graph labelings. An extensive survey of graph labelings as well as their applications has been given by Gallian [4].

In [1] each γ-labeling f of a graph G of order n and size m is assigned a *value* denoted by $\text{val}(f)$ and defined by

$$\text{val}(f) = \sum_{e \in E(G)} f'(e).$$

Since f is a one-to-one function from $V(G)$ to $\{0, 1, 2, \ldots, m\}$, it follows that $f'(e) \geq 1$ for each edge e in G and so

$$\text{val}(f) \geq m. \tag{1}$$

In [1] the *maximum value* and the *minimum value* of a γ-labeling of a graph G are defined, respectively, as

$$\begin{aligned} \text{val}_{\max}(G) &= \max\{\text{val}(f) : f \text{ is a } \gamma\text{-labeling of } G\} \\ \text{val}_{\min}(G) &= \min\{\text{val}(f) : f \text{ is a } \gamma\text{-labeling of } G\}. \end{aligned}$$

A γ-labeling g of G is a *γ-max labeling* if $\text{val}(g) = \text{val}_{\max}(G)$ or a *γ-min labeling* if $\text{val}(g) = \text{val}_{\min}(G)$. These concepts were introduced and studied in [1] and [2]. As an illustration, Figure 1 shows nine γ-labelings $f_1, f_2, \ldots, f_9$ of the path P_5 of order 5, where the vertex labels are shown above each vertex and the induced edge labels are shown below each edge. The value of each γ-labeling is shown in Figure 1 as well. Since $\text{val}(f_1) = 4$ for the γ-labeling f_1 of P_5 shown in Figure 1 and the size of P_5 is 4, it follows by (1) that f_1 is a γ-min labeling of P_5. As we will see later, the γ-labeling f_9 shown in Figure 1 is a γ-max labeling.

For a γ-labeling f of a graph G of size m, the *complementary labeling* $\overline{f} : V(G) \to \{0, 1, 2, \ldots, m\}$ of f is defined in [1] by

$$\overline{f}(v) = m - f(v) \text{ for } v \in V(G).$$

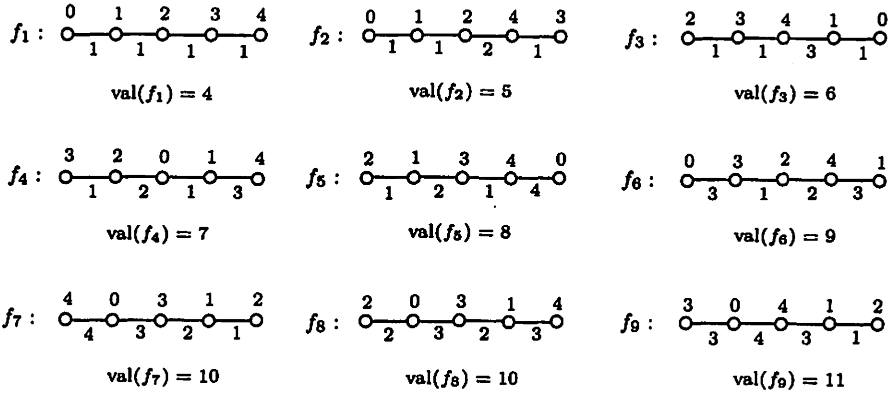


Figure 1: Some γ-labelings of P_5

Not only is $\overline{f}$ a γ-labeling of G as well but $\text{val}(\overline{f}) = \text{val}(f)$. Therefore, a γ-labeling f is a γ-max labeling (γ-min labeling) of G if and only if $\overline{f}$ is a γ-max labeling (γ-min labeling). Figure 2 shows the complementary labelings of the γ-min labeling f_1 and the γ-max labeling f_9 of P_5 as well as the value of each of these two γ-labelings.

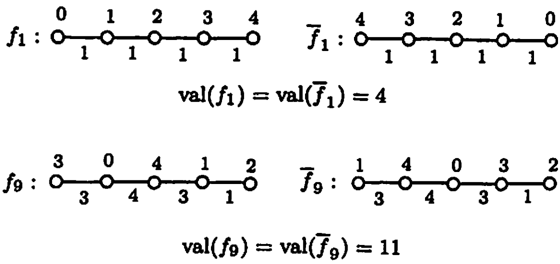


Figure 2: Complementary labelings of γ-labelings

The γ-*spectrum* of a graph G is defined in [1] as

$$\text{spec}(G) = \{\text{val}(f) : f \text{ is a } \gamma\text{-labeling of } G\}.$$

Thus, $\{4,5,6,7,8,9,10,11\} \subseteq \text{spec}(P_5)$. (In fact, $\{4,5,6,7,8,9,10,11\} = \text{spec}(P_5)$.) Observe that $\text{val}_{\min}(G), \text{val}_{\max}(G) \in \text{spec}(G)$ for every graph G. For integers a and b with $a \leq b$, let

$$[a,b] = \{a, a+1, \ldots, b\}$$

be the set of integers between a and b. Thus for every graph G,

$$\text{spec}(G) \subseteq [\text{val}_{\min}(G),\ \text{val}_{\max}(G)].$$

The spectrum of a star $K_{1,t}$, where $t \geq 2$, was determined in [1], which we state next.

Theorem 1.1 ([1]) *For each integer $t \geq 2$,*

$$\operatorname{spec}(K_{1,t}) = \left\{ \binom{t+1-k}{2} + \binom{k+1}{2} : 0 \leq k \leq t \right\}.$$

In this work, we determine the γ-spectra of some well-known classes of graphs, namely paths, cycles, and complete graphs. We refer to the book [3] for graph theory notation and terminology not described in this paper.

2 The γ-spectrum of a path

For each integer $n \geq 2$, let $P_n : v_1, v_2, \ldots, v_n$ be the path of order n. The maximum and minimum values of a γ-labeling of P_n were determined in [1].

Theorem 2.1 ([1]) *For any path P_n of order $n \geq 2$,*

$$\operatorname{val}_{\min}(P_n) = n - 1 \quad \text{and} \quad \operatorname{val}_{\max}(P_n) = \left\lfloor \frac{n^2-2}{2} \right\rfloor.$$

The γ-labeling $f_{\min}$ of P_n defined by $f_{\min}(v_i) = i - 1$ $(1 \leq i \leq n)$ has $\operatorname{val}(f_{\min}) = n - 1$ and so $f_{\min}$ is a γ-min labeling of P_n. In fact, $f_{\min}$ and its complementary labeling $\overline{f}_{\min}$ are the only γ-min labelings of P_n for each integer $n \geq 2$. On the other hand, this is not the case for the γ-max labelings of P_n. A γ-max labeling of P_n was given in [1] for each integer n as follows: For an odd integer $n = 2k + 1$, a γ-max labeling f_o of P_n is defined by

$$f_o(v_i) = \begin{cases} k + \frac{i+1}{2} & \text{if } i \text{ is odd and } i < n \\ k & \text{if } i = n \\ \frac{i-2}{2} & \text{if } i \text{ is even.} \end{cases}$$

For an even integer $n = 2k$, a γ-max labeling f_e of P_n is defined by

$$f_e(v_i) = \begin{cases} k + \frac{i-1}{2} & \text{if } i \text{ is odd} \\ \frac{i-2}{2} & \text{if } i \text{ is even.} \end{cases}$$

There are other γ-max labelings for P_n. For example, for an odd integer $n = 2k + 1$, define a γ-labeling g_o of P_n by

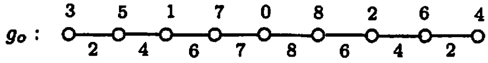
$$\begin{aligned} g_o(v_{k+1-i}) &= \begin{cases} n-1-i & \text{if } i \text{ is odd and } 1 \leq i \leq k \\ i-1 & \text{if } i \text{ is even and } 2 \leq i \leq k \end{cases} \\ g_o(v_{k+1}) &= 0 \\ g_o(v_{k+1+i}) &= \begin{cases} n-i & \text{if } i \text{ is odd} \\ i & \text{if } i \text{ is even.} \end{cases} \end{aligned}$$

Then g_o is a γ-max labeling of P_n for each odd integer $n \geq 3$. For an even integer $n = 2k$, define a γ-labeling g_e of P_n by

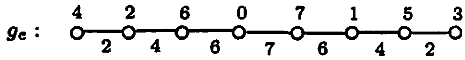
$$g_e(v_{k+1-i}) = \begin{cases} i-1 & \text{if } i \text{ is odd and } 1 \leq i \leq k \\ n-i & \text{if } i \text{ is even and } 2 \leq i \leq k \end{cases}$$

$$g_e(v_{k+i}) = \begin{cases} n-i & \text{if } i \text{ is odd and } 1 \leq i \leq k \\ i-1 & \text{if } i \text{ is even and } 2 \leq i \leq k. \end{cases}$$

Then g_e is a γ-max labeling of P_n for each even integer $n \geq 2$. Figure 3 shows the γ-max labelings g_o and g_e for P_9 and P_8, respectively.



val(g_o) = 39



val(g_e) = 31

Figure 3: The γ-max labelings g_o and g_e for P_9 and P_8, respectively

In order to determine the γ-spectrum of the path P_n of order $n \geq 2$, we first establish some additional definitions and notation. For a γ-labeling f of P_n and each integer $j \in \{2, 3, \ldots, n-1\}$, a *$j$-right arrangement* $R_j(f)$ of f is defined as a γ-labeling of P_n for which

$$R_j(f)(v_\ell) = \begin{cases} f(v_\ell) & \text{if } 1 \leq \ell \leq j-1 \\ f(v_{\ell+1}) & \text{if } j \leq \ell \leq n-1 \\ f(v_j) & \text{if } \ell = n. \end{cases}$$

That is, if f is a γ-labeling of P_n: v_1, v_2, $\ldots$, v_n such that the labels are assigned by f to the vertices of P_n are in the order

$$(f(v_1), f(v_2), \ldots, f(v_{j-1}), f(v_j), f(v_{j+1}) \ldots, f(v_n)),$$

then $R_j(f)$ is the γ-labeling of P_n for which the labels are assigned by $R_j(f)$ to the vertices of P_n are in the order

$$(f(v_1), f(v_2), \ldots, f(v_{j-1}), f(v_{j+1}), \ldots, f(v_n), f(v_j)).$$

Analogously, a *j-left arrangement* $L_j(f)$ of f is defined as

$$L_j(f)(v_\ell) = \begin{cases} f(v_j) & \text{if } \ell = 1 \\ f(v_{\ell-1}) & \text{if } 2 \leq \ell \leq j \\ f(v_\ell) & \text{if } j+1 \leq \ell \leq n. \end{cases}$$

We are now prepared to present the main result of this section.

Theorem 2.2 *For each integer $n \geq 2$,*

$$\operatorname{spec}(P_n) = [\operatorname{val}_{\min}(P_n),\ \operatorname{val}_{\max}(P_n)] = \left[n-1,\quad \left\lfloor \frac{n^2-2}{2} \right\rfloor\right].$$

Proof. We show, for each integer

$$s \in [\operatorname{val}_{\min}(P_n),\ \operatorname{val}_{\max}(P_n)],$$

that there exists a γ-labeling of P_n whose value is s. We consider two cases, according to whether n is even or n is odd.

Case 1. n **is even.** Define the sets Γ_k^n for $0 \leq k \leq \frac{n-2}{2}$ by

$$\Gamma_k^n = \begin{cases} [0,\ n-1] & \text{if } k = 0 \\ [0,\ n-1] - \{1, 2, \ldots, k, n-1-k, \ldots, n-2\} & \text{if } 1 \leq k \leq \frac{n-2}{2}. \end{cases}$$

Then

$$|\Gamma_k^n| = n - 2k$$

for $0 \leq k \leq \frac{n-2}{2}$. Suppose that

$$\Gamma_k^n = \{a_1, a_2, \ldots, a_{n-2k}\}$$

such that $a_1 < a_2 < \cdots < a_{n-2k}$. Let

$$\begin{aligned} \Delta_{2k}^n &= 2k(n-k-1) \text{ for } 0 \leq k \leq \tfrac{n-2}{2} \\ \Delta_n^n &= \Delta_{n-2}^n + 1. \end{aligned}$$

We now define a γ-labeling f^k, where $0 \leq k \leq \frac{n-2}{2}$, of P_n by

$$\begin{aligned} f^k(v_{k+1-i}) &= \begin{cases} n-i-1 & \text{if } i \text{ is odd and } 1 \leq i \leq k \\ i & \text{if } i \text{ is even and } 2 \leq i \leq k \end{cases} \\ f^k(v_{k+i}) &= a_i \quad \text{if } 1 \leq i \leq n-2k \\ f^k(v_{n-k+i}) &= \begin{cases} i & \text{if } i \text{ is odd and } 1 \leq i \leq k \\ n-i-1 & \text{if } i \text{ is even and } 2 \leq i \leq k. \end{cases} \end{aligned}$$

Observe that

$$f^k(v) \in \begin{cases} [0,\ n-1] - \Gamma_k^n & \text{if } v \in V(P_n) - \{v_{k+1}, v_{k+2}, \ldots, v_{n-k}\} \\ \Gamma_k^n & \text{if } v \in \{v_{k+1}, v_{k+2}, \ldots, v_{n-k}\}. \end{cases}$$

Furthermore,

$$\operatorname{val}(f^0) = n - 1$$

and for $1 \le k \le \frac{n-2}{2}$

$$\operatorname{val}(f^k) = n - 1 + 2(n-2) + \cdots + 2(n-2k) = n - 1 + \Delta^n_{2k}.$$

For each integer $s \in \left[n-1, \ \frac{n^2-2}{2}\right]$, we construct a γ-labeling whose value is s by the following procedure:

1. find $k \in \left[0, \frac{n-2}{2}\right]$ such that

$$s - n + 1 \in \left[\Delta^n_{2k}, \quad \Delta^n_{2(k+1)} - 1\right];$$

2. find a smallest nonnegative integer t such that $t = s - n + 1$ if $k = 0$ and

$$t \equiv s - n + 1 \pmod{\Delta^n_{2k}} \ \text{ otherwise.}$$

3. if $t = 0$, then f^k is the solution; else

4. if k is even, then

4.1. if $1 \le t \le n - 2k - 2$, then find $\ell \in [k+2, \ n-k-1]$ such that

$$f^k(v_\ell) = t + k;$$

the solution is $L_\ell(f^k)$; else

4.2. if $n - 2k - 1 \le t \le 2n - 4k - 5$, then

4.2.1 set $\tilde{f}^k = L_{n-k-1}(f^k)$;

4.2.2 find $\ell \in [k+2, \ n-k-1]$ such that

$$\tilde{f}^k(v_\ell) = 2n - 3k - t - 4;$$

the solution is $L_\ell(\tilde{f}^k)$;

5. if k is odd, then proceed with **4.** and replace L by R.

Observe that if k is even, then $f^k(v_1) = k$. Thus, for t and ℓ as described in **4.1.**, we have

$$\operatorname{val}(L_\ell(f^k)) = \operatorname{val}(f^k) + (t + k - k) = s.$$

In the condition **4.2.**,

$$\operatorname{val}(\tilde{f}^k) = \operatorname{val}(f^k) + (n - k - 2 - k)$$

and

$$\mathrm{val}(L_\ell(\tilde{f}^k)) = \mathrm{val}(\tilde{f}^k) + [n-k-2-(2n-3k-t-4)] = s.$$

Case 2. n **is odd.** Define the sets Γ_k^n for $0 \le k \le \frac{n-3}{2}$ by

$$\Gamma_k^n = \begin{cases} [0,\ n-1] & \text{if } k=0 \\ [0,\ n-1]-\{1,2\} & \text{if } k=1 \\ [0,\ n-1]-\{1,\ldots,k,n-k-2,\ldots,n-3\} & \text{if } k \text{ is and even} \\ & 2 \le k \le \frac{n-3}{2} \\ [0,\ n-1]-\{1,\ldots,k+1,n-k-1,\ldots,n-3\} & \text{if } k \text{ is odd and} \\ & 3 \le k \le \frac{n-3}{2}. \end{cases}$$

Thus

$$|\Gamma_0^n| = n \text{ and } |\Gamma_k^n| = n-2k$$

for $1 \le k \le \frac{n-3}{2}$. We now define the γ-labeling f^0 of P_n by

$$f^0(v_i) = i-1 \text{ for } 1 \le i \le n$$

and the γ-labeling g^k $(0 \le k \le \frac{n-3}{2})$ of P_n as follows:

$$\begin{aligned} g^k(v_{k+1}) &= n-2 \\ g^k(v_{k+1+i}) &= a_i \quad (1 \le i \le n-2k-1) \end{aligned}$$

where

$$\Gamma_k^n - \{n-2\} = \{a_1, a_2, \ldots, a_{n-2k-1}\}$$

with $a_1 < a_2 < \cdots < a_{n-2k-1}$ and

$$\begin{aligned} g^k(v_{k+1-i}) &= g^k(v_{n-k+i}) - 1 \\ &= \begin{cases} i & \text{if } i \text{ is odd and } 1 \le i \le k \\ n-i-2 & \text{if } i \text{ is even and } 2 \le i \le k. \end{cases} \end{aligned}$$

Furthermore, define

$$\Delta_{2k}^n = 3n-8+2k(n-k-4) \quad \text{for } 0 \le k \le \tfrac{n-5}{2}$$

and

$$\Delta_{n-3}^n = \Delta_{n-5}^n + 1.$$

Then $\mathrm{val}(f^0) = n-1$, $\mathrm{val}(g^0) = 2n-3$, and for $0 \le k \le \frac{n-5}{2}$,

$$\begin{aligned}\mathrm{val}(g^{k+1}) &= \mathrm{val}(g^0) + 2(n-3) + 2(n-5) + \cdots + 2(n-2k-3) \\ &= n - 1 + \Delta^n_{2k}.\end{aligned}$$

For each integer $s \in \left[n-1, \ \frac{n^2-3}{2}\right]$, we construct a γ-labeling of P_n whose value is s by the following procedure:

1. if $s-n+1 \in [0, \ 3n-9]$, then

1.1. if $s-n+1 = 0$, then the solution is f^0;

1.2. if $s-n+1 \in [1, \ n-2]$, then the solution is $L_{s-n+2}(f^0)$;

1.3. if $s-n+1 \in [n-1, \ 2n-5]$, then solution is $L_{3n-s-3}(g^0)$;

1.4. if $s-n+1 \in [2n-4, \ 3n-9]$, then

1.4.1. set $\tilde{g} = L_3(g^0)$;

1.4.2. find $\ell \in [4, \ n-1]$ such that

$$\tilde{g}(v_\ell) = s - 3n + 7;$$

the solution is $L_\ell(\tilde{g})$; else

2. find $k \in [0, \ \frac{n-5}{2}]$ such that $s-n+1 \in \left[\Delta^n_{2k}, \ \Delta^n_{2(k+1)} - 1\right]$;

3. find a smallest nonnegative t such that

$$s - n + 1 \equiv t \ (\mathrm{mod} \ \Delta^n_{2k});$$

[Note that $\Delta^n_0 > 0$ if $k = 0$.]

4. if $t = 0$, then g^{k+1} is the solution; else

5. if k is even, then

5.1. if $1 \le t \le n-2k-5$, then the solution is $R_{k+t+3}(g^{k+1})$; else

5.2. if $n-2k-4 \le t \le 2n-4k-11$, then

5.2.1 set $\tilde{g}^k = R_{n-k-2}(g^{k+1})$;

5.2.2 find $\ell \in [k+4, \ n-k-3]$, such that

$$\tilde{g}^k(v_\ell) = 2n - 3k - t - 8;$$

the solution is $R_\ell(\tilde{g}^k)$; else

6. if k is odd, then

6.1. if $1 \le t \le n-2k-5$, then the solution is $L_{n-k-t-1}(g^{k+1})$; else

6.2. if $n-2k-4 \le t \le 2n-4k-11$, then

6.2.1 set $\tilde{g}^k = L_{k+4}(g^{k+1})$;

6.2.2 find $\ell \in [k+5,\ n-k-2]$ such that $\tilde{g}^k(v_\ell) = 3k-n+t+7$; the solution is $L_\ell(\tilde{g}^k)$.

It can be verified that

$$\text{val}(L_{s-n+2}(f^0)) = \text{val}(f^0) + s - n + 1 = s$$

and

$$\text{val}(L_{3n-s-3}(g^0)) = \text{val}(g^0) + n - 2 - (3n-s-5) = s.$$

Also,

$$\text{val}(\tilde{g}) = \text{val}(g^0) + n - 2 - 1 = 3n - 6$$

and

$$\text{val}(L_\ell(\tilde{g})) = \text{val}(\tilde{g}) + (s - 3n + 7) - 1 = s$$

for ℓ as described in **1.4.2.**

Notice that in **5.**,

$$\text{val}(R_{k+t+3}(g^{k+1})) = \text{val}(g^{k+1}) + t + k + 2 - (k+2) = s$$

and

$$\begin{aligned}
\text{val}(R_\ell(\tilde{g}^k)) &= \text{val}(\tilde{g}^k) + n - k - 3 - (2n - 3k - t - 8) \\
&= \big(\text{val}(g^{k+1}) + n - k - 3 - (k+2)\big) + (2k + t - n + 5) \\
&= s.
\end{aligned}$$

These equalities still hold for $\text{val}(L_\ell(\tilde{g}^k))$, where k and t are described in **6.2.** Finally, we observe that

$$\text{val}(L_{n-k-t-2}(g^{k+1})) = \text{val}(g^{k+1}) + n - k - 3 - (n-k-t-3) = s,$$

for t as described in **6.1.** ∎

We now illustrate the proof of Theorem 2.2. The table below shows all variables in the proof of Theorem 2.2 that we use to find $\text{spec}(P_8)$.

$s = \text{val}(f)$ of $P_8 \in$	$\left[n-1, \frac{n^2-2}{2}\right]$	$= [7, 31]$										
$k = 0, s-n+1 \in$	$[\Delta_0^n, \Delta_{2(1)}^n - 1]$	$= [0, 11]$										
s	$s-n+1$	t	*labeling*	$:v_1$	v_2	v_3	v_4	v_5	v_6	v_7	v_8	val(f)
7	0	0	f^0	: 0	1	2	3	4	5	6	7	7
8	1	1	$L_2(f^0)$	: 1	0	2	3	4	5	6	7	8
9	2	2	$L_3(f^0)$	: 2	0	1	3	4	5	6	7	9
10	3	3	$L_4(f^0)$	: 3	0	1	2	4	5	6	7	10
11	4	4	$L_5(f^0)$	: 4	0	1	2	3	5	6	7	11
12	5	5	$L_6(f^0)$	: 5	0	1	2	3	4	6	7	12
13	6	6	$L_7(f^0)$	: 6	0	1	2	3	4	5	7	13
			$f^0 = L_7(f^0)$	: 6	0	1	2	3	4	5	7	
14	7	7	$L_7(f^0)$	: 5	6	0	1	2	3	4	7	14
15	8	8	$L_6(f^0)$	: 4	6	0	1	2	3	5	7	15
16	9	9	$L_5(f^0)$	: 3	6	0	1	2	4	5	7	16
17	10	10	$L_4(f^0)$	: 2	6	0	1	3	4	5	7	17
18	11	11	$L_3(f^0)$	: 1	6	0	2	3	4	5	7	18
$k = 1, s-n+1 \in$	$[\Delta_{2(1)}^n, \Delta_{2(2)}^n - 1]$	$= [12, 19]$										
s	$s-n+1$	t	*labeling*	$:v_1$	v_2	v_3	v_4	v_5	v_6	v_7	v_8	val(f)
19	12	0	f^1	: 6	0	2	3	4	5	7	1	19
20	13	1	$R_3(f^1)$	: 6	0	3	4	5	7	1	2	20
21	14	2	$R_4(f^1)$	: 6	0	2	4	5	7	1	3	21
22	15	3	$R_5(f^1)$	: 6	0	2	3	5	7	1	4	22
23	16	4	$R_6(f^1)$	: 6	0	2	3	4	7	1	5	23
			$f^1 = R_6(f^1)$	: 6	0	2	3	4	7	1	5	
24	17	5	$R_5(f^1)$	: 6	0	2	3	7	1	5	4	24
25	18	6	$R_4(f^1)$	: 6	0	2	4	7	1	5	3	25
26	19	7	$R_3(f^1)$	: 6	0	3	4	7	1	5	2	26
$k = 2, s-n+1 \in$	$[\Delta_{2(2)}^n, \Delta_{2(3)}^n - 1]$	$= [20, 23]$										
s	$s-n+1$	t	*labeling*	$:v_1$	v_2	v_3	v_4	v_5	v_6	v_7	v_8	val(f)
27	20	0	f^2	: 2	6	0	3	4	7	1	5	27
28	21	1	$L_4(f^2)$	: 3	2	6	0	4	7	1	5	28
29	22	2	$L_5(f^2)$	: 4	2	6	0	3	7	1	5	29
			$f^2 = L_5(f^2)$	: 4	2	6	0	3	7	1	5	
30	23	3	$L_5(f^2)$	: 3	4	2	6	0	7	1	5	30
$k = 3, s-n+1 \in$	$[\Delta_3^n, \Delta_n^n - 1]$	$= [24, 24]$										
s	$s-n+1$	t	*labeling*	$:v_1$	v_2	v_3	v_4	v_5	v_6	v_7	v_8	val(f)
31	24	0	f^3	: 4	2	6	0	7	1	5	3	31

The table below shows all variables that we use to find spec(P_9).

$s = \text{val}(f)$ of $P_9 \in [n-1, \frac{n^2-3}{2}] = [8, 39]$													
$s-n+1 \in [0, 3n-9] = [0, 18]$													
s	$s-n+1$		*labeling*	$:v_1$	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	val(f)
8	0		f^0	:0	1	2	3	4	5	6	7	8	8
9	1		$L_2(f^0)$	:1	0	2	3	4	5	6	7	8	9
10	2		$L_3(f^0)$	:2	0	1	3	4	5	6	7	8	10
11	3		$L_4(f^0)$	:3	0	1	2	4	5	6	7	8	11
12	4		$L_5(f^0)$	:4	0	1	2	3	5	6	7	8	12
13	5		$L_6(f^0)$	:5	0	1	2	3	4	6	7	8	13
14	6		$L_7(f^0)$	:6	0	1	2	3	4	5	7	8	14
15	7		$L_8(f^0)$	:7	0	1	2	3	4	5	6	8	15
			g^0	:7	0	1	2	3	4	5	6	8	
16	8		$L_8(g^0)$	:6	7	0	1	2	3	4	5	8	16
17	9		$L_7(g^0)$	:5	7	0	1	2	3	4	6	8	17
18	10		$L_6(g^0)$	:4	7	0	1	2	3	5	6	8	18
19	11		$L_5(g^0)$	:3	7	0	1	2	4	5	6	8	19
20	12		$L_4(g^0)$	:2	7	0	1	3	4	5	6	8	20
21	13		$L_3(g^0)$	:1	7	0	2	3	4	5	6	8	21
			$\bar{g} = L_3(g^0)$	:1	7	0	2	3	4	5	6	8	
22	14		$L_4(\bar{g})$	:2	1	7	0	3	4	5	6	8	22
23	15		$L_5(\bar{g})$	:3	1	7	0	2	4	5	6	8	23
24	16		$L_6(\bar{g})$	:4	1	7	0	2	3	5	6	8	24
25	17		$L_7(\bar{g})$	:5	1	7	0	2	3	4	6	8	25
26	18		$L_8(\bar{g})$	:6	1	7	0	2	3	4	5	8	26
$k = 0, s-n+1 \in [\Delta_0^n, \Delta_{2(1)}^n - 1] = [19, 26]$													
s	$s-n+1$		*labeling*	$:v_1$	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	val(f)
27	19	0	g^1	:1	7	0	3	4	5	6	8	2	27
28	20	1	$R_4(g^1)$	:1	7	0	4	5	6	8	2	3	28
29	21	2	$R_5(g^1)$	:1	7	0	3	5	6	8	2	4	29
30	22	3	$R_6(g^1)$	:1	7	0	3	4	6	8	2	5	30
31	23	4	$R_7(g^1)$	:1	7	0	3	4	5	8	2	6	31
			$g^0 = R_7(g^1)$	:1	7	0	3	4	5	8	2	6	
32	24	5	$R_6(g^0)$	:1	7	0	3	4	8	2	6	5	32
33	25	6	$R_5(g^0)$	:1	7	0	3	5	8	2	6	4	33
34	26	7	$R_4(g^0)$	:1	7	0	4	5	8	2	6	3	34
$k = 1, s-n+1 \in [\Delta_{2(1)}^n, \Delta_{2(2)}^n - 1] = [27, 30]$													
s	$s-n+1$		*labeling*	$:v_1$	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	val(f)
35	27	0	g^2	:5	1	7	0	3	4	8	2	6	35
36	28	1	$L_6(g^2)$	:4	5	1	7	0	3	8	2	6	36
37	29	2	$L_5(g^2)$	:3	5	1	7	0	4	8	2	6	37
			$g^1 = L_5(g^2)$	:3	5	1	7	0	4	8	2	6	
38	30	3	$L_6(g^1)$	:4	3	5	1	7	0	8	2	6	38
$k = 2, s-n+1 \in [\Delta_{2(2)}^n, \Delta_{2(3)}^n - 1] = [\Delta_{2(2)}^n, \Delta_{n-3}^n - 1] = [31, 31]$													
s	$s-n+1$		*labeling*	$:v_1$	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	val(f)
39	31	0	g^3	:3	5	1	7	0	8	2	6	4	39

If $n = 8$ and $s = 30$, then $k = 2$ (since $23 \in [20, 23]$) and $t = 3$. Applying **4.1.** and **4.2.**, we obtain the γ-labelings f^2 and $\tilde{f}^2$ of P_8 as shown in Figure 4. Then the solution is $L_5(\tilde{f}^2)$, which is also shown in Figure 4, and $\text{val}(L_5(\tilde{f}^2)) = 30$.

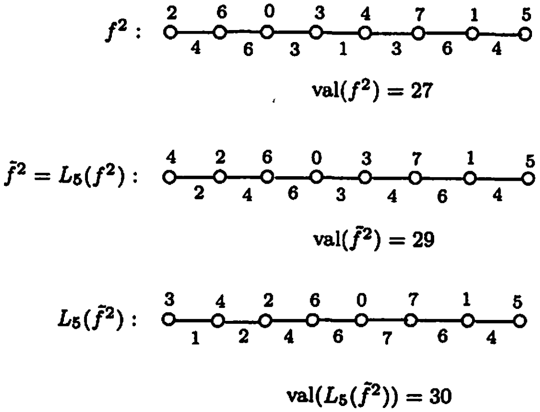


Figure 4: The γ-labelings f^2, $\tilde{f}^2$, and $L_5(\tilde{f}^2)$ of P_8

If $n = 9$ and $s = 32$, then $k = 0$ (since $24 \in [19, 26]$) and $t = 5$. Applying **5.1.** and **5.2.**, we obtain the γ-labelings g^1 and $\tilde{g}^0$ of P_9 as shown in Figure 5. Then the solution is $R_6(\tilde{g}^0)$, which is also shown in Figure 5, and $\text{val}(R_6(\tilde{g}^0)) = 32$.

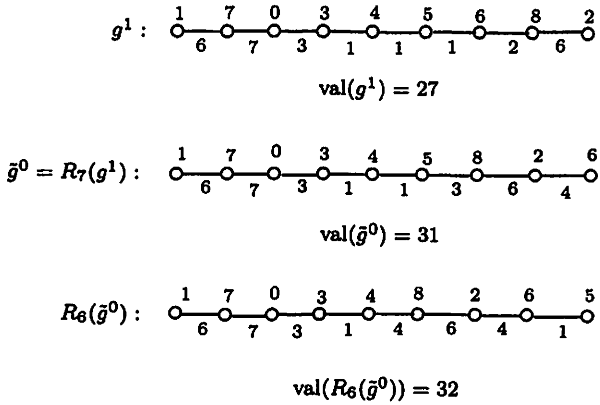


Figure 5: The γ-labelings g^1, $\tilde{g}^0$, and $R_6(\tilde{g}^0)$ of P_9

3 The γ-spectrum of a cycle

For an integer $n \geq 3$, let C_n: $v_1, v_2, \ldots, v_n, v_1$ be the cycle of order n. The maximum and minimum values of a γ-labeling of C_n were determined in [1].

Theorem 3.1 ([1]) *For every integer $n \geq 3$,*

$$\begin{aligned} \text{val}_{\min}(C_n) &= 2(n-1) \\ \text{val}_{\max}(C_n) &= \begin{cases} \frac{(n-1)(n+3)}{2} & \text{if } n \text{ is odd} \\ \frac{n(n+2)}{2} & \text{if } n \text{ is even.} \end{cases} \end{aligned}$$

In order to determine the γ-spectrum of C_n for each integer $n \geq 3$, we first establish two lemmas. The proof of the first lemma is straightforward and is therefore omitted.

Lemma 3.2 *Let f be a γ-labeling of a graph G. If $P_k : u_1, u_2, \cdots, u_k$ is a path of order k in G such that $f(u_i) < f(u_{i+1})$ for $1 \leq i \leq k-1$, then*

$$\text{val}(f) = [f(u_k) - f(u_1)] + \sum_{e \in E(G)-E(P)} f'(e).$$

Lemma 3.3 *For every γ-labeling f of C_n, where $n \geq 3$, the value* val(f) *of f is even.*

Proof. Let f be a γ-labeling of C_n. If 0 is in the sequence of the images of f, we may assume, without loss of generality, that $f(v_1) = 0$; otherwise $f(v_1) = 1$. Let

$$\begin{aligned} S_1(f) &= (f(v_{i_1}), f(v_{i_1+1}), \ldots, f(v_{i_2})) \\ S_2(f) &= (f(v_{i_2}), f(v_{i_2+1}), \ldots, f(v_{i_3})) \\ &\vdots \\ S_{k-1}(f) &= (f(v_{i_{k-1}}), f(v_{i_{k-1}+1}), \ldots, f(v_{i_k})) \\ S_k(f) &= (f(v_{i_k}), f(v_{i_k+1}), \ldots, f(v_{i_{k+1}})) \end{aligned}$$

be k maximal monotone sequences of the vertices of C_n, where $2 \leq k \leq n$ and

$$1 = i_{k+1} = i_1 < i_2 < \cdots < i_k.$$

Since $f(v_1) = 0$ or $f(v_1) = 1$ if 0 is not an image of f, it follows that $S_1(f)$ is an increasing sequence; $S_2(f)$ is decreasing and so on. (Notice that k is always even.) By Lemma 3.2,

$$\begin{aligned}\text{val}(f) &= [f(v_{i_2}) - f(v_{i_1})] + [f(v_{i_2}) - f(v_{i_3})] + [f(v_{i_4}) - f(v_{i_3})] + \cdots \\ &\quad + [f(v_{i_k}) - f(v_{i_{k-1}})] + [f(v_{i_k}) - f(v_{i_1})] \\ &= 2[f(v_{i_2}) + f(v_{i_4}) + \cdots + f(v_{i_k})] \\ &\quad -2[f(v_{i_1}) + f(v_{i_3}) + \cdots + f(v_{i_{k-1}})],\end{aligned}$$

which is even, as desired. ∎

For even integers a and b with $a < b$, let

$$E[a,b] = \{a, a+2, a+4, \cdots, b\}$$

be the set of all even integers i for which $a \le i \le b$. We are now prepared to present the main result of this section.

Theorem 3.4 *For each integer $n \ge 3$,*

$$\text{spec}(C_n) = E[\text{val}_{\min}(C_n),\ \text{val}_{\max}(C_n)].$$

Proof. By Lemma 3.3,

$$\text{spec}(C_n) \subseteq E[\text{val}_{\min}(C_n),\ \text{val}_{\max}(C_n)].$$

Next, we show that for each even integer

$$s \in E[\text{val}_{\min}(C_n),\ \text{val}_{\max}(C_n)],$$

there exists a γ-labeling of C_n whose value is s. Certainly, the statement is true if $s = \text{val}_{\min}(C_n)$ or $s = \text{val}_{\max}(C_n)$.

For $s = \text{val}_{\min}(C_n) + 2$, let $C_n : u_1, u_2, \ldots, u_n, u_1$ and define a γ-labeling h of C_n by

$$h(u_i) = \begin{cases} 0 & \text{if } i = 1 \\ i & \text{if } 2 \le i \le n. \end{cases}$$

Then

$$\text{val}(h) = [h(u_n) - h(u_1)] + [h(u_n) - h(u_1)] = 2n = \text{val}_{\min}(C_n) + 2.$$

We now assume that

$$s \in E[\text{val}_{\min}(C_n) + 4,\ \text{val}_{\max}(C_n) - 2].$$

There exists an integer k with $0 \le k \le \lceil \frac{n-4}{2} \rceil$ such that

$$s \in [\Delta_k^n + 2,\ \Delta_{k+1}^n],$$

where

$$\Delta_k^n = 2[n + k(n-k-1)].$$

Label the vertices of C_n as

$$C_n : x_1, z_1, z_2, \ldots, z_{n-2k-4}, y_1, x_2, y_2, x_3, y_3, \ldots, x_{k+1}, y_{k+1}, x_{k+2}, y_{k+2}, x_1.$$

Define a γ-labeling f of C_n by

$$\begin{cases} f(x_i) = i-1 & \text{if } 1 \le i \le k+2 \\ f(y_i) = n-i+1 & \text{if } 1 \le i \le k+1 \\ f(y_i) = (k+2) + (\frac{s}{2} - n) - k(n-k-1) - 1 & \text{if } i = k+2 \\ f(z_i) \in [2,\ n-1] - W & \text{if } 1 \le i \le n-2k-4 \end{cases}$$

where

$$W = \{f(x_1), f(y_1), f(x_2), f(y_2), \ldots, f(x_{k+2}), f(y_{k+2})\}$$

and

$$f(z_1) < f(z_2) < \cdots < f(z_{n-2k-4}).$$

Then

$$\begin{aligned} \text{val}(f) &= \sum_{i=1}^{k+1}(f(y_i) - f(x_i)) + \sum_{i=1}^{k+1}(f(x_{i+1}) - f(y_i)) \\ &\quad + f(y_{k+2}) - f(x_{k+2}) + f(y_{k+2}) - f(x_1) \\ &= [n + n - 2 + n - 4 + \cdots + n - 2k] \\ &\quad + [n - 1 + n - 3 + \cdots + n - (2k+1)] \\ &\quad + 2\left[(k+2) + \left(\frac{s}{2} - n\right) - k(n-k-1) - 1\right] - (k+1) - 0 \\ &= (2k+2)n - \frac{(2k+1)(2k+2)}{2} \\ &\quad + 2\left[(k+2) + \left(\frac{s}{2} - n\right) - k(n-k-1) - 1\right] - (k+1) = s, \end{aligned}$$

as desired. ∎

We now illustrate the γ-labeling f of C_n described in the proof of Theorem 3.4. If $n = 8$ and $s = 32 \in \text{spec}(C_8) = E[18,\ 38]$, then $k = 1$. Let

$$C_8 : x_1, z_1, z_2, y_1, x_2, y_2, x_3, y_3, x_1.$$

The γ-labeling f of C_8 is shown in Figure 6 with $\text{val}(f) = 32$.

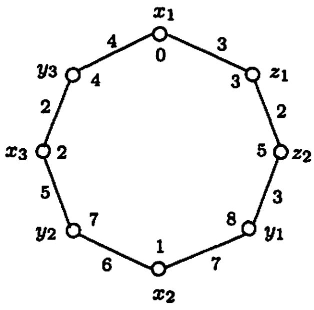


Figure 6: The γ-labeling f of C_8 with $\text{val}(f) = 32$

4 The γ-spectrum of a complete graph

For each integer $n \geq 2$, let K_n be a complete graph of order n with

$$V(K_n) = \{v_1, v_2, \ldots, v_n\}.$$

The maximum and minimum values of a γ-labeling of K_n were determined in [1].

Theorem 4.1 ([1]) *For every integer $n \geq 2$,*

$$\begin{aligned} \text{val}_{\min}(K_n) &= \binom{n+1}{3} \\ \text{val}_{\max}(K_n) &= \begin{cases} \frac{n(3n^3-5n^2+6n-4)}{24} & \text{if } n \text{ is even} \\ \frac{(n^2-1)(3n^2-5n+6)}{24} & \text{if } n \text{ is odd.} \end{cases} \end{aligned}$$

The minimum value $\text{val}_{\min}(K_n)$ of a γ-labeling of K_n is attained by the γ-labeling f_ℓ defined by

$$f_\ell(v_i) = \ell + i - 1,$$

where $\ell \in \left[0, \frac{n(n-3)}{2}\right]$, while the maximum value $\text{val}_{\max}(K_n)$ of a γ-labeling of K_n is attained by a γ-labeling of K_n that assigns the labels in the set

$$\left\{0, 1, \ldots, \left\lceil \frac{n}{2} \right\rceil - 1, \binom{n}{2} - \left\lfloor \frac{n}{2} \right\rfloor + 1, \binom{n}{2} - \left\lfloor \frac{n}{2} \right\rfloor + 2, \ldots, \binom{n}{2}\right\}$$

to the vertices of K_n (see [1]).

In order to determine the γ-spectrum of a complete graph K_n for $n \geq 2$, we need an additional definition. If f is a γ-labeling of K_n such that

$f(v_i) = a_i$ for $1 \le i \le n$ and

$$0 \le a_1 < \cdots < a_n \le \binom{n}{2}, \tag{2}$$

then

$$\mathrm{val}(f) = \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} (n - 2i + 1)(a_{n-i+1} - a_i).$$

Setting

$$\alpha_i = a_{n-\lfloor \frac{n}{2} \rfloor + i} - a_{\lfloor \frac{n}{2} \rfloor} - i + 1 \text{ for } i = 1, 2, \ldots, \lfloor \tfrac{n}{2} \rfloor,$$

we obtain an increasing sequence $\{\alpha_i\}$ such that

(i) $\alpha_i \ge \alpha_{i-1} + 2$ for $i = 2, 3, \ldots, \lfloor \frac{n}{2} \rfloor$,

(ii) $\alpha_{\lfloor \frac{n}{2} \rfloor} \le \binom{n}{2}$, and

(iii) $\alpha_1 \ge 2$ if n is odd, while $\alpha_1 \ge 1$ if n is even.

On the other hand, if $\{\alpha_i\}$ is an increasing sequence with properties (i)-(iii), then there exist n integers $a_1, a_2, \ldots, a_n$ satisfying (2) such that $\alpha_i = a_{n-i+1} - a_i$ for $i = 1, 2, \ldots, \lfloor \frac{n}{2} \rfloor$. Therefore, an increasing sequence $\{\alpha_i\}$ is called a *γ-sequence* of K_n if $\{\alpha_i\}$ satisfies properties (i)-(iii). The γ-spectrum of the complete graph K_n is then given in terms of γ-sequences of K_n.

Theorem 4.2 *For each integer $n \ge 2$,*

if n is odd, then

$$\mathrm{spec}(K_n) = \left\{ \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} 2i\alpha_i \; : \; \{\alpha_i\} \text{ is a } \gamma\text{-sequence of } K_n \right\},$$

if n is even, then

$$\mathrm{spec}(K_n) = \left\{ \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} (2i-1)\alpha_i \; : \; \{\alpha_i\} \text{ is a } \gamma\text{-sequence of } K_n \right\}.$$

The following is an immediate consequence of Theorem 4.2.

Corollary 4.3 *If $n \ge 3$ is odd, then the value of every γ-labeling of K_n is even.*

As an illustration of Theorem 4.2, we see that

$$\begin{aligned} \text{spec}(K_4) &= \{3\alpha_2 + \alpha_1 : 6 \geq \alpha_2 \geq \alpha_1 + 2 \geq 3\} \\ &= \{10, 13, 14, 16, 17, 18, 19, 20, 21, 22\}; \end{aligned}$$

$$\begin{aligned} \text{spec}(K_5) &= \{4\alpha_2 + 2\alpha_1 : 10 \geq \alpha_2 \geq \alpha_1 + 2 \geq 4\} \\ &= E[20, 56] - \{22\}. \end{aligned}$$

Notice that there are integers in $[\text{val}_{\min}(K_n), \text{val}_{\max}(K_n)]$ for which there is no γ-labeling with that value. For example, if $s \in [\text{val}_{\min}(K_n) + 1, \text{val}_{\min}(K_n) + n - 2]$, then there is no γ-labeling of K_n whose value is s.

5 Acknowledgments

We are grateful to the referee whose valuable suggestions resulted in an improved paper.

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