ASCENTS OF SIZE LESS THAN d IN SAMPLES OF GEOMETRIC RANDOM VARIABLES: MEAN, VARIANCE AND DISTRIBUTION

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ABSTRACT. We consider words $\pi_1\pi_2\pi_3\ldots\pi_n$ of length n, where $\pi_i \in \mathbb{N}$ are independently generated with a geometric probability

$$\mathbb{P}\{\pi = k\} = pq^{k-1} \text{ where } p + q = 1.$$

Let d be a fixed non-negative integer. We say that we have an ascent of size d or more, an ascent of size less than d, a level and a descent if $\pi_{i+1} \geq \pi_i + d$, $\pi_{i+1} < \pi_i + d$, $\pi_{i+1} = \pi_i$, $\pi_i > \pi_{i+1}$ respectively. We determine the mean and variance of the number of ascents of size less than d in a random geometrically distributed word. We also show that the distribution is Gaussian as n tends to infinity.

1. INTRODUCTION

In this paper, we consider words, $\pi_1\pi_2\pi_3\ldots\pi_n$ of length n, where the letters $\pi_i \in \mathbb{N}$ are independently generated with a geometric probability such that

$$\mathbb{P}\{X = k\} = pq^{k-1} \text{ where } p + q = 1.$$

We define an ascent whenever $\pi_i < \pi_{i+1}$, a level whenever $\pi_i = \pi_{i+1}$, and a descent whenever $\pi_i > \pi_{i+1}$. Moreover, if d is a fixed non-negative integer, we say that we have an ascent of size less than d, if $\pi_{i+1} < \pi_i + d$, and an ascent of size d or more if $\pi_{i+1} \geq \pi_i + d$. For an ascent of size less than d to exist, we need d to be greater than 1.

Consider the word 1 1 2 1 3 7 5 4 1 4 2 2 of length 12. It has four ascents: 12, 13, 37, 14, two levels: 11, 22 and five descents: 21, 75, 54, 41, 52. If $d = 3$, there are two ascents of size 3 or more: 37 and 14, and two ascents of size less than 3: 12, 13.

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Ascents of size d or more in samples of random geometric variables have already been done in [1, 2], hence ascents of size less than d are considered. Ascents and descents in geometric random variables have also been studied in [8, 9]. Ascents have also been investigated in partitions and compositions in [4] and [3, 5] respectively.

In Section 2, we find the generating function $G(z,w;r,s,t,u)$ where z counts the size of the word, w the value of the last part π_n, r the descents, s the levels, t the ascents of size less than d and u the ascents of size d or more. To allow for the empty word we use $G(z,w;r,s,t,u) = F(z,w;r,s,t,u) - 1$. As the only variables of interest are the size of the word and the number of ascents of size less than d, we put $w = r = s = u = 1$ and consider $F(z,t) := F(z,1;1,1,t,1)$. The other variables have been studied in [2, 8] and in particular for $d = 1$ in [11]. We extract the mean $\mathbb{E}(n)$ of the number of ascents of size less than d using the standard technique

$$\mathbb{E}(n) = [z^n]\frac{\partial F(z,t)}{\partial t}\Big|_{t=1}.$$

In Section 3, we determine the variance $\mathbb{V}(n)$ using

$$\mathbb{V}(n) = [z^n]\frac{\partial^2 F(z,t)}{\partial t^2}\bigg|_{t=1} + \mathbb{E}(n) - (\mathbb{E}(n))^2.$$

In Section 4, we use the "Meromorphic schema" by Flajolet and Sedgewick in [7] to prove that the distribution of the number of ascents of size less than d converges to a Gaussian distribution as n, the size of the word, tends to infinity.

2. Expected number of ascents of size less than d

2.1. Probability generating function. Let $f_k(z,w;r,s,t,u)$ be the probability generating function where z counts the length of the word k, w the value of the last part, r the descents, s the levels, t the ascents of size less than d and u the ascents of size d or more.

To find the generating function, we use the "adding the slice" technique which was originally used by P. Flajolet and H. Prodinger in [6], then by A. Knopfmacher and H. Prodinger in [10] and more recently in [2, 5]. We find a rule for proceeding from a sample with k parts to a sample with $k+1$ parts. We add a new part to the end of a word where the last part has value j, i.e., $\pi_k = j$:

$$\begin{aligned} w^j &\to rzp\sum_{i=1}^{j-1} w^i q^{i-1} + szpw^j q^{j-1} + tzp\sum_{i=j+1}^{j+d-1} w^i q^{i-1} + uzp\sum_{i\ge j+d} w^i q^{i-1} \\ &= \frac{rzpw}{1-wq} + \frac{zp}{q(1-wq)}\left[s - r + (t-s)wq + (u-t)(wq)^d\right](wq)^j. \end{aligned}$$

This implies that

$$f_{k+1}(z,w;r,s,t,u) = \frac{rzpw}{1-wq} f_k(z,1;r,s,t,u)$$
$$+ \frac{zp}{q(1-wq)} \left[s - r + (t-s)wq + (u-t)(wq)^d\right] f_k(z,wq;r,s,t,u). \tag{2.1}$$

This is for a general k, hence we need to sum over all possible values of k. To allow for the empty word we define

$$G(z,w;r,s,t,u) := F(z,w;r,s,t,u) - 1 = \sum_{k\geq 1} f_k(z,w;r,s,t,u).$$

Thus

$$G(z,w;r,s,t,u) = f_1(z,w;r,s,t,u) + \frac{rzpw}{1-wq} G(z,1;r,s,t,u)$$
$$+ \frac{zp}{q(1-wq)} \left[s - r + (t-s)wq + (u-t)(wq)^d\right] G(z,wq;r,s,t,u).$$

The function $f_1(z,w;r,s,t,u)$ represents words that consist of a single letter, where

$$f_1(z,w;r,s,t,u) = \sum_{k\geq 1} zpq^{k-1}w^k = \frac{pzw}{1-wq}.$$

We obtain after the above substitution

$$G(z,w;r,s,t,u) = \frac{rzpw}{1-wq} G(z,1;r,s,t,u) + \frac{pzw}{1-wq}$$
$$+ \frac{zpq^{-1}}{1-wq} \left[s - r + (t-s)wq + (u-t)(wq)^d\right] G(z,wq;r,s,t,u). \tag{2.2}$$

In order to solve this recursion, the following lemma is useful. It is similar to the one found in [5].

Lemma 1. *A recursion of the form $F(z,w) = f(z,w) + g(z,w)F(z,wq)$, has solution*

$$F(z,w) = \sum_{j\geq 1} \left(f(z,wq^{j-1}) \prod_{i=1}^{j-1} g(z,wq^{i-1}) \right), \text{ for } 0 < q < 1.$$

Proof. We perform a few iterations starting with $F(z,w) = f(z,w) + g(z,w)F(z,wq)$:

$$\begin{aligned} F(z,w) &= f(z,w) + g(z,w)[f(z,wq) + g(z,wq)F(z,wq^2)] \\ &= f(z,w) + g(z,w)f(z,wq) + g(z,w)g(z,wq)f(z,wq^2) \\ &\quad + g(z,w)g(z,wq)g(z,wq^2)F(z,wq^3). \end{aligned}$$

If we keep iterating and use the fact that $F(z, wq^j) \to 0$ as $j \to \infty$ we obtain:

$$\begin{aligned} F(z,w) &= f(z,w) + g(z,w)f(z,wq) + g(z,w)g(z,wq)f(z,wq^2) \\ &\quad + g(z,w)g(z,wq)g(z,wq^2)f(z,wq^3) + \cdots \\ &= \sum_{j\geq 1}\left(f(z,wq^{j-1}) \prod_{i=1}^{j-1} g(z,wq^{i-1})\right). \end{aligned}$$

□

Applying Lemma 1 to equation (2.2) we get

$$\begin{aligned} G(z,w;r,s,t,u) &= \sum_{j\geq 1}\left(\frac{rzpwq^{j-1}}{1-wq^j}G(z,1;r,s,t,u) + \frac{zpwq^{j-1}}{1-wq^j}\right) \times \\ &\quad \times \prod_{i=1}^{j-1}\left(\left[s-r+(t-s)wq^i+(u-t)(wq^i)^d\right]\frac{zpq^{-1}}{1-wq^i}\right). \end{aligned}$$

The variables of interest are z and t, so we put $w = s = u = v = 1$. For simplicity let $G(z,1;1,1,t,1) = G(z,1;t)$, thus

$$G(z,1;t) = \sum_{j\geq 1}\left(\frac{zpq^{j-1}}{1-q^j}G(z,1;t) + \frac{zpq^{j-1}}{1-q^j}\right)\prod_{i=1}^{j-1}\left((t-1)(q^i-q^{id})\frac{zpq^{-1}}{1-q^i}\right).$$

Solving for $G(z,1;t)$ we have

$$G(z,t) := G(z,1;t) = \frac{\sum_{j\geq 1}\frac{zpq^{j-1}}{1-q^j}\prod_{i=1}^{j-1}\left((t-1)(q^i-q^{id})\frac{zpq^{-1}}{1-q^i}\right)}{1-\sum_{j\geq 1}\frac{zpq^{j-1}}{1-q^j}\prod_{i=1}^{j-1}\left((t-1)(q^i-q^{id})\frac{zpq^{-1}}{1-q^i}\right)}.$$

Thus,

Theorem 1. *The generating function for samples with geometric random variables with ascents less than d is*

$$G(z,t) = \frac{\tau(z,t)}{1-\tau(z,t)},$$

where

$$\tau(z,t) = \sum_{j\geq 1}\frac{zpq^{j-1}}{1-q^j}\prod_{i=1}^{j-1}\left((t-1)(q^i-q^{id})\frac{zpq^{-1}}{1-q^i}\right).$$

□

Thus,

$$F(z,t) = G(z,t) + 1 = \frac{1}{1-\tau(z,t)}.$$

The expected value is $[z^n]\frac{\partial F}{\partial t}\big|_{t=1}$, where

$$\frac{\partial F}{\partial t} = \frac{\tau'(z,t)}{(1-\tau(z,t))^2}.$$

$$\tau'(z,t)\Big|_{t=1} = \frac{zpq}{1-q^2}(q-q^d)\frac{zpq^{-1}}{1-q} = \frac{z^2(q-q^d)}{1+q},$$

and

$$\frac{1}{(1-\tau(z,t))^2}\Big|_{t=1} = \frac{1}{(1-z)^2}.$$

Hence,

$$[z^n]\frac{\partial F}{\partial t}\Big|_{t=1} = [z^n]\frac{z^2(q-q^d)}{(1+q)(1-z)^2} = [z^{n-2}]\frac{(q-q^d)}{(1+q)(1-z)^2} = (n-1)\frac{q-q^d}{1+q}.$$

Thus, finally we obtain

Theorem 2. *The expected number of ascents of size less than d in a word consisting of n geometric random variables is, for* $d > 1$,

$$\mathbb{E}(n) = (n-1)\frac{q-q^d}{1+q}.$$

□

Brennan and Knopfmacher in [2], found the expected number of ascents of size d or more in geometric samples to be

$$\mathbb{E}_{d\ or\ more}(n) = (n-1)\frac{q^d}{1+q}.$$

Putting $d = 1$ gives the expected number of all ascents of geometric samples to be

$$\mathbb{E}_{all}(n) = (n-1)\frac{q}{1+q}.$$

Using the linearity property for the means $\mathbb{E}(X+Y) = \mathbb{E}(X) + \mathbb{E}(Y)$, we can obtain the expected number of ascents of size less than d by a simple subtraction

$$\mathbb{E}_{less\ than\ d}(n) = \mathbb{E}_{all}(n) - \mathbb{E}_{d\ or\ more}(n).$$

This, clearly, matches the result found in Theorem 2. Thus the mean found in Theorem 2 could have been worked out without the generating function found in Theorem 1. However, the linearity property is not true for the variances as the two events are not independent. The generating function found in Theorem 1 is therefore needed for the next section.

3. Variance of the number of ascents of size less than d in samples of n geometric random variables

Lemma 2. *The variance* $\mathbb{V}(n)$ *is*

$$\mathbb{V}(n) = [z^n]\frac{\partial^2 F(z,t)}{\partial t^2}\bigg|_{t=1} + \mathbb{E}(n) - (\mathbb{E}(n))^2 .$$

We need the following:

$$\frac{\partial^2 F(z,t)}{\partial t^2}\bigg|_{t=1} = \frac{\tau''(z,t)(1-\tau(z,t)) + 2(\tau'(z,t))^2}{(1-\tau(z,t))^3}\bigg|_{t=1},$$

where

$$(1-\tau(z,t))\Big|_{t=1} = 1-z \ ; \qquad \tau'(z,t)\Big|_{t=1} = \frac{z^2(q-q^d)}{1+q},$$

and

$$\tau''(z,t)\Big|_{t=1} = 2\frac{zpq^2}{1-q^3}\left[(q-q^d)\frac{zpq^{-1}}{1-q}(q^2-q^{2d})\frac{zpq^{-1}}{1-q^2}\right]$$
$$= \frac{2z^3(q-q^d)(q^2-q^{2d})p}{(1-q^3)(1+q)}.$$

Thus

$$\frac{\partial^2 F(z,t)}{\partial t^2}\bigg|_{t=1} = \frac{2z^3(q-q^d)(q^2-q^{2d})p}{(1-q^3)(1+q)(1-z)^2} + \frac{2z^4(q-q^d)^2}{(1+q)^2(1-z)^3}.$$

Hence we get

$$[z^n]\frac{\partial^2 F(z,t)}{\partial t^2}\bigg|_{t=1} = [z^{n-3}]\frac{2(q-q^d)(q^2-q^{2d})p}{(1-q^3)(1+q)(1-z)^2} + [z^{n-4}]\frac{2(q-q^d)^2}{(1+q)^2(1-z)^3}$$
$$= \frac{2(q-q^d)(q^2-q^{2d})p}{(1-q^3)(1+q)}(n-2) + \frac{(q-q^d)^2}{(1+q)^2}(n-3)(n-2).$$

Putting it all together we obtain

$$\begin{aligned}\mathbb{V}(z,n) &= \frac{2(q-q^d)(q^2-q^{2d})p}{(1-q^3)(1+q)}(n-2) + \frac{(q-q^d)^2}{(1+q)^2}(n-3)(n-2)\\ &+ (n-1)\frac{q-q^d}{(1+q)} - (n-1)^2\frac{(q-q^d)^2}{(1+q)^2}\\ &= \frac{2(q-q^d)(q^2-q^{2d})p}{(1-q^3)(1+q)}(n-2) + (n-1)\frac{q-q^d}{(1+q)} + \frac{(q-q^d)^2}{(1+q)^2}(5-3n).\end{aligned}$$

Thus

Theorem 3. *The variance of the number of ascents of size less than d in samples of n geometric random variables is, for $d > 1$,*

$$\mathbb{V}(n) = (q - q^d)\left(\left[\frac{2(q^2 - q^{2d})p}{(1-q^3)(1+q)} - \frac{3(q-q^d)}{(1+q)^2} + \frac{1}{1+q}\right] n \right.$$
$$\left. - \frac{4(q^2 - q^{2d})p}{(1-q^3)(1+q)} + \frac{5(q-q^d)}{(1+q)^2} - \frac{1}{1+q}\right).$$

□

Note. The variance of the number of ascents of size d or more samples of n geometric random variables in [2] was

$$\mathbb{V}(n) = \left[\frac{2q^{3d}p}{(1-q^3)(1+q)} - \frac{3q^{2d}}{(1+q)^2} + \frac{q^d}{1+q}\right] n$$
$$- \frac{4q^{3d}p}{(1-q^3)(1+q)} + \frac{5q^{2d}}{(1+q)^2} - \frac{q^d}{1+q}.$$

4. Limiting distribution of number of ascents of size less than d

In this section, we prove that the limiting distribution of the number of ascents of size less than d is Gaussian. For this we use Proposition IX.9 from Flajolet and Sedgewick [7], which we state below: We introduce the notation

$$\mathfrak{v}(f) = \frac{f''(1)}{f(1)} + \frac{f'(1)}{f(1)} - \left(\frac{f'(1)}{f(1)}\right)^2.$$

Proposition 1. *"Meromorphic schema" Let $F(z,u)$ be a bivariate function that is bivariate analytic at $(z,u) = (0,0)$ and has nonnegative coefficients there. Assume that $F(z,1)$ is meromorphic in $z \le r$ with only a simple pole at $z = \rho$ for some positive $\rho < r$. In [2], this was done for ascents of size d or more.*
Assume also the following conditions.

i) *Meromorphic perturbation: there exists $\epsilon > 0$ and $r > \rho$ such that in the domain $\mathrm{D} = \{|z| \le r\} \times \{|u-1| < \epsilon\}$, the function $F(z,u)$ admits the representation*

$$F(z,u) = \frac{B(z,u)}{C(z,u)},$$

where $B(z,u)$, $C(z,u)$ are analytic for $(z,u) \in D$ with $B(\rho,1) \neq 0$. (Thus ρ is a simple zero of $C(z,1)$).

ii) *Nondegeneracy: one has $\partial_z C(\rho,1) \cdot \partial_u C(\rho,1) \neq 0$, ensuring the existence of a nonconstant $\rho(u)$ analytic at $u = 1$, such that $C(\rho(u),u) = 0$ and $\rho(1) = \rho$.*

iii) *Variability: one has*

$$\mathfrak{v}\left(\frac{\rho}{\rho(u)}\right) \neq 0.$$

Then, the random variable with probability generating function

$$p_n(u) = \frac{[z^n]F(z,u)}{[z^n]F(z,1)}$$

converges in distribution to a Gaussian variable with a speed of convergence that is $O(n^{-1/2})$. The mean and the variance of X_n are asymptotically linear in n.

In addition, we have the following results from [7].
We introduce the notation

$$c_{i,j} := \frac{\partial^{i+j}}{\partial z^i \partial u^j} C(z,u)\bigg|_{(\rho,1)}, \tag{4.1}$$

then if $\rho(u)$ denotes the analytic solution of the implicit equation $C(\rho(u),u) = 0$,

$$\rho(u) = \rho - \frac{c_{0,1}}{c_{1,0}}(u-1) - \frac{c_{1,0}^2 c_{0,2} - 2c_{1,0}c_{1,1}c_{0,1} + c_{2,0}c_{0,1}^2}{2c_{1,0}^3}(u-1)^2 + O((u-1)^3). \tag{4.2}$$

Condition (ii) corresponds to

$$c_{0,1}c_{1,0} \neq 0. \tag{4.3}$$

The variability condition (iii) corresponds to

$$\rho c_{1,0}^2 c_{0,2} - \rho c_{1,0}c_{1,1}c_{0,1} + \rho c_{2,0}c_{0,1}^2 + c_{0,1}^2 c_{1,0} + c_{0,1}c_{1,0}^2\rho \neq 0. \tag{4.4}$$

For our specific problem

$$C(z,t) = 1 - \sum_{j\geq 1} \frac{zpq^{j-1}}{1-q^j} \prod_{i=1}^{j-1} \left((t-1)(q^i - q^{id})\frac{zpq^{-1}}{1-q^i}\right).$$

We have $\rho(1) = \rho = 1$ when $z = 1$.
According to (4.1)

$$c_{0,1} = \frac{\partial}{\partial t}C(z,t)\bigg|_{(1,1)} = -\frac{q-q^d}{1+q}.$$

$$c_{1,0} = \frac{\partial}{\partial z}C(z,t)\bigg|_{(1,1)} = -1.$$

$$c_{1,1} = \frac{\partial^2}{\partial z \partial t}C(z,t)\bigg|_{(1,1)} = \frac{-2(q-q^d)}{1+q}.$$

$$c_{0,2} = \frac{\partial^2}{\partial t^2} C(z,t)\bigg|_{(1,1)} = \frac{-2p(q-q^d)(q^2-q^{2d})}{(1-q^3)(1+q)}.$$

$$c_{2,0} = \frac{\partial^2}{\partial z^2} C(z,t)\bigg|_{(1,1)} = 0.$$

Condition (ii) gives $c_{1,0}c_{0,1} = \frac{q-q^d}{1+q} \neq 0$, which is true for all $q > 0$ for $d > 1$. This is fine since according to our definition, $d = 1$ does not yield an ascent.
Using (4.2),

$$\rho(t) = 1 + \frac{q^d - q}{1+q}(t-1) + \left(\frac{p(q^d-q)(q^2-q^{2d})}{(1+q)(1-q^3)} + \frac{2(q^d-q)^2}{(1+q)^2} \right)(t-1)^2 + O\left((t-1)^3\right). \tag{4.5}$$

The left hand side of the variability condition (4.4) is

$$\frac{1}{(1+q)^2(1-q^3)}\Big[-q - 2q^3 + q^4 + 2q^5 + q^d - q^{d+1} + 2q^{d+2} - q^{d+3} - q^{d+4} + q^{2d} + 2q^{2d+1} - 3q^{2d+3} - 2q^{3d} + 2q^{3d+2}\Big]$$

which for $0 < q \leq 1$ and $d \geq 0$ is never zero. Thus

Theorem 4. *The distribution of the number of ascents of size less than d in samples of n geometric random variables converges to a Gaussian distribution with a speed of convergence of $O(n^{-1/2})$, where the mean μ_n and the variance σ_n^2 are as given in Theorems 2 and 3, where $d > 1$.*

Remark. In Flajolet and Sedgewick [7] it is also shown that under the conditions of the proposition, the mean μ_n and variance σ_n^2 are of the form

$$\mu_n = \mathfrak{m}\left(\frac{\rho(1)}{\rho(u)}\right) n + O(1), \qquad \sigma_n^2 = \mathfrak{v}\left(\frac{\rho(1)}{\rho(u)}\right) n + O(1),$$

where

$$\mathfrak{m}(f) = \frac{f'(1)}{f(1)} \quad \text{and} \quad \mathfrak{v}(f) = \frac{f''(1)}{f(1)} + \frac{f'(1)}{f(1)} - \left(\frac{f'(1)}{f(1)}\right)^2.$$

This gives

$$\mu_n = \left(\frac{1}{\rho(t)}\right)'\bigg|_{t=1} n + O(1) = (n-1)\frac{q-q^d}{1+q} + O(1).$$

which is in agreement with our exact result in Theorem 2.

In the case of the variance

$$\mathfrak{v}\left(\frac{\rho(1)}{\rho(t)}\right)\bigg|_{t=1} n = \left[\left(\frac{1}{\rho(t)}\right)''\bigg|_{t=1} + \left(\frac{1}{\rho(t)}\right)'\bigg|_{t=1} - \left(\left(\frac{1}{\rho(t)}\right)'\bigg|_{t=1}\right)^2\right] n$$
$$= \left[\frac{2q^{3d}p}{(1-q^3)(1+q)} - \frac{3q^{2d}}{(1+q)^2} + \frac{q^d}{1+q}\right] n,$$

which corresponds to the main term of the exact result found in Theorem 3.

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