Several q-series identities related to Jackson's

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Abstract

In this paper, using the q-exponential operator technique to two identities due to Jackson, we obtain some q-series identities involving ${}_3\phi_2$.

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1 Notation and Introduction

Throughout this paper, let $0 < |q| < 1$. We adopt the notation and terminology in [2]. The q-shifted factorial is defined by

$$(a;q)_0 = 1, \quad (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k) = \frac{(a;q)_\infty}{(aq^n;q)_\infty}, \tag{1}$$

$$(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k). \tag{2}$$

For brevity, we employ the notation

$$(a_1,\ a_2,\ \ldots,\ a_m;q)_n = (a_1;q)_n(a_2;q)_n\ldots(a_m;q)_n.$$

The q-binomial coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}. \tag{3}$$

Jackson [7] established the following interesting identity:

$$\begin{aligned} &\sum_{k=0}^{2n}(-1)^k \begin{bmatrix} 2n \\ k \end{bmatrix}(a,\ b;q)_k(a,\ b;q)_{2n-k}q^{k(2n-k+1)/2} \\ = \ &(a,\ b,\ q^{n+1},\ abq^n;q)_n. \end{aligned} \tag{4}$$

Jackson obtained (4) by a curious transformation of his q-analogue of Dougall's theorem [6]. In fact, Jackson had already stated (4) without proof in [5]. Recently, Guo [3] gave an elementary proof of (4) and obtained the following identity by the same argument.

$$\begin{aligned} &\sum_{k=0}^{2n+1}(-1)^k \begin{bmatrix} 2n+1 \\ k \end{bmatrix}(a,\ b;q)_k(a,\ b;q)_{2n+1-k}q^{k(2n-k+3)/2} \\ = \ &(a,\ b,\ q^{n+1};q)_{n+1}(abq^{n+1};q)_n. \end{aligned} \tag{5}$$

In this paper, we use q-exponential operator technique to these two identities and obtain several q-series identities.

To make the paper self-contained, we introduce the concept of q-exponential operator. The q-difference operator D_q is defined by

$$D_q\{f(a)\} = \frac{1}{a}(f(a)-f(aq)).$$

The q-exponential operator due to Chen and Liu (see [1] and [8]) can be restated as follows:

$$T(bD_q) = \sum_{n=0}^{\infty} \frac{(bD_q)^n}{(q;q)_n}. \tag{6}$$

Recently, Zhang and Wang [9] obtained the following operator identity:

$$\begin{aligned} & T(dD_q)\left\{\frac{(av;q)_\infty}{(as,at,aw;q)_\infty}\right\} \\ = & \frac{(av,dv,adstw/v;q)_\infty}{(as,at,aw,ds,dt,dw;q)_\infty}\, {}_3\phi_2\left(\begin{matrix} \frac{v}{s}, & \frac{v}{t}, & \frac{v}{w} \\ & av, & dv \end{matrix};q,\frac{adstw}{v}\right), \end{aligned} \tag{7}$$

where $|adstw/v| < 1$, which is a generalization of the following result due to Chen and Liu (see [1] and [8]):

$$T(bD_q)\left\{\frac{1}{(as,at;q)_\infty}\right\} = \frac{(abst;q)_\infty}{(as,at,bs,bt;q)_\infty}. \tag{8}$$

In the context of this paper, convergence of basic hypergeometric series is no issue at all because they are terminating q-series.

2 The main results

Theorem 2.1

$$\begin{aligned} & \sum_{k=0}^{2n}(-1)^k \begin{bmatrix} 2n \\ k \end{bmatrix} (a,\ b,\ c;q)_k (a,\ b,\ c;q)_{2n-k} q^{k(2n-k+1)/2} \\ & \qquad \times {}_3\phi_2\left(\begin{matrix} q^{-k}, & q^{-(2n-k)}, & q^{-n}/b \\ & a, & c \end{matrix};q,abcq^{3n}\right) \\ = & (a,\ b,\ c,\ q^{n+1},\ abq^n,\ bcq^n;q)_n. \end{aligned} \tag{9}$$

Proof. We rewrite (4) as

$$\sum_{k=0}^{2n}(-1)^k\begin{bmatrix}2n\\k\end{bmatrix}(b;q)_k(b;q)_{2n-k}q^{\frac{k(2n-k+1)}{2}}\frac{(a;q)_\infty}{(aq^k,\ aq^{2n-k},\ abq^n;q)_\infty}$$
$$=(b,\ q^{n+1};q)_n\cdot\frac{1}{(aq^n,\ abq^{2n};q)_\infty}. \tag{10}$$

Applying the operator $T(cD_q)$ to both sides of equation (10) with respect to the variable a, then we get

$$\begin{aligned}&\sum_{k=0}^{2n}(-1)^k\begin{bmatrix}2n\\k\end{bmatrix}(b;q)_k(b;q)_{2n-k}q^{k(2n-k+1)/2}\\&\qquad\times T(cD_q)\left\{\frac{(a;q)_\infty}{(aq^k,\ aq^{2n-k},\ abq^n;q)_\infty}\right\}\\=\ &(b,\ q^{n+1};q)_n\cdot T(cD_q)\left\{\frac{1}{(aq^n,\ abq^{2n};q)_\infty}\right\}.\end{aligned} \tag{11}$$

By applying (7) and (8), we have

$$\begin{aligned}&T(cD_q)\left\{\frac{(a;q)_\infty}{(aq^k,\ aq^{2n-k},\ abq^n;q)_\infty}\right\}\\=\ &(a,c;q)_\infty\frac{(abcq^{3n};q)_\infty}{(aq^k,aq^{2n-k},abq^n,cq^k,cq^{2n-k},bcq^n;q)_\infty}\\&\times{}_3\phi_2\left(\begin{matrix}q^{-k},\ q^{-(2n-k)},\ q^{-n}/b\\a,\ c\end{matrix};q,abcq^{3n}\right)\end{aligned} \tag{12}$$

and

$$T(cD_q)\left\{\frac{1}{(aq^n,\ abq^{2n};q)_\infty}\right\}=\frac{(abcq^{3n};q)_\infty}{(aq^n,abq^{2n},cq^n,bcq^{2n};q)_\infty}. \tag{13}$$

Substituting these two identities into (11), we complete the proof of the theorem. □

Theorem 2.2

$$\sum_{k=0}^{2n}(-1)^k\begin{bmatrix}2n\\k\end{bmatrix}(a,\ b,\ c;q)_k(a,\ b,\ c;q)_{2n-k}q^{k(2n-k+1)/2}$$

$$= (a,\ b,\ c,\ q^{n+1},\ abq^n,\ bcq^n;q)_n \times {}_3\phi_2\left(\begin{matrix} q^{-n}, & bq^n, & b \\ & abq^n, & bcq^n \end{matrix};q, acq^{2n}\right). \quad (14)$$

Proof. From (10), we have

$$\sum_{k=0}^{2n}(-1)^k \begin{bmatrix}2n\\k\end{bmatrix}(b;q)_k(b;q)_{2n-k}q^{k(2n-k+1)/2}\cdot\frac{1}{(aq^k,\ aq^{2n-k};q)_\infty}$$
$$= (b,\ q^{n+1};q)_n\cdot\frac{(abq^n;q)_\infty}{(a,\ aq^n,\ abq^{2n};q)_\infty}. \quad (15)$$

Applying the operator $T(cD_q)$ to both sides of equation (15) with respect to the variable a, we get

$$\sum_{k=0}^{2n}(-1)^k \begin{bmatrix}2n\\k\end{bmatrix}(b;q)_k(b;q)_{2n-k}q^{k(2n-k+1)/2} \times T(cD_q)\left\{\frac{1}{(aq^k,\ aq^{2n-k};q)_\infty}\right\}$$
$$= (b,\ q^{n+1};q)_n\cdot T(cD_q)\left\{\frac{(abq^n;q)_\infty}{(a,\ aq^n,\ abq^{2n};q)_\infty}\right\}. \quad (16)$$

By applying (7) and (8), we have

$$T(cD_q)\left\{\frac{1}{(aq^k,\ abq^{2n-k};q)_\infty}\right\} = \frac{(acq^{2n};q)_\infty}{(aq^k, abq^{2n-k}, cq^k, bcq^{2n-k};q)_\infty}. \quad (17)$$

and

$$T(cD_q)\left\{\frac{(abq^n;q)_\infty}{(a,\ aq^n,\ abq^{2n};q)_\infty}\right\}$$
$$= (abq^n,\ bcq^n;q)_\infty\frac{(acq^{2n};q)_\infty}{(a, aq^n, abq^{2n}, c, cq^n, bcq^{2n};q)_\infty} \times{}_3\phi_2\left(\begin{matrix} q^{-n}, & b, & bq^n \\ & abq^n, & bcq^n \end{matrix};q, acq^{2n}\right). \quad (18)$$

Substituting these two identities into (16), we complete the proof of the theorem. □

By the same argument as in the proofs of Theorems 2.1 and 2.2, from (5), we obtain the following results.

Theorem 2.3

$$\begin{aligned}
&\sum_{k=0}^{2n+1}(-1)^k\begin{bmatrix}2n+1\\k\end{bmatrix}(a,\ b,\ c;q)_k(a,\ b,\ c;q)_{2n+1-k}q^{k(2n-k+3)/2}\cdot\\
&\qquad\times{}_3\phi_2\left(\begin{matrix}q^{-k}, & q^{-(2n+1-k)}, & q^{-(n+1)}/b\\ & a, & c\end{matrix};q, abcq^{3n+2}\right)\\
&= (a,\ b,\ c,\ q^{n+1};q)_{n+1}(abq^{n+1},\ bcq^{n+1};q)_n. \qquad (19)
\end{aligned}$$

Theorem 2.4

$$\begin{aligned}
&\sum_{k=0}^{2n+1}(-1)^k\begin{bmatrix}2n+1\\k\end{bmatrix}(a,\ b,\ c;q)_k(a,\ b,\ c;q)_{2n+1-k}q^{k(2n-k+3)/2}\\
&= (a,\ b,\ c,\ q^{n+1};q)_{n+1}(abq^{n+1},\ bcq^{n+1};q)_n\\
&\qquad\times\ {}_3\phi_2\left(\begin{matrix}q^{-n}, & bq^{n+1}, & b\\ & abq^{n+1}, & bcq^{n+1}\end{matrix};q, acq^{2n+1}\right). \qquad (20)
\end{aligned}$$

Note. In [4], Guo and Zeng gave a short proof of Jackson's ${}_8\phi_7$ summation formula. But the q-exponential operator technique couldn't be used to Jackson's ${}_8\phi_7$ summation formula to obtain some new formulas.

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