Spectral characterization of a specific class of trees *

Xiaoxia Fan, Xing Gao, Yanfeng Luo
Department of Mathematics, Lanzhou University,
Lanzhou, Gansu 730000, PR China
fanxx06@lzu.cn

Abstract

In this paper, it is shown that the graph $T_4(p,q,r)$ is determined by its Laplacian spectrum and there are no two non-isomorphic such graphs which are cospectral with respect to adjacency spectrum.

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1 Introduction

Graphs considered in this paper are undirected graphs without loops and multiple edges. Let G be a simple graph with n vertices. Denote by $A(G)$ and $D(G)$ the adjacency matrix and the diagonal matrix with the vertex degrees of G on the diagonal, respectively. The matrix $L(G) = D(G) - A(G)$ is called the *Laplacian matrix* of G. Denote by $P(G, \lambda)$ the adjacency polynomial $det(\lambda I - A(G))$ of G. The multiset of eigenvalues of $A(G)$ (resp., $L(G)$) is called the *adjacency* (resp., *Laplacian*) *spectrum* of G. Since $A(G)$ and $L(G)$ are real symmetric matrices, their eigenvalues are real numbers. So we can assume that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$ are the adjacency eigenvalues and the Laplacian eigenvalues of G, respectively. Two graphs are said to be *cospectral* with respect to the adjacency (resp. Laplacian) spectrum if they have the same adjacency (resp. Laplacian) spectrum. A graph is said to be *determined by its adjacency (resp., Laplacian) spectrum* if there is no other non-isomorphic graph with the same adjacency (resp., Laplacian) spectrum.

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Determining what kinds of graphs are determined is an old problem, which is far from resolved, in the theory of graph spectra. In their paper [14], the authors conjectured that almost all graphs are determined by their spectrum. However, it seems hard to prove a graph to be determined by its spectrum and only a few graphs have been proved to be determined by their spectrum. Therefore it would be interesting to find more examples of graphs which are determined by their spectrum. For the background on this problem and related topics, the reader can consult [14, 15]. For more recent results which have not been cited in [14, 15], we refer to [2, 9, 10, 12, 11] and their references for details.

Because the problem above is very hard to deal with, van Dam and Haemers [14] suggested a modest problem, say, "which trees are determined by their spectrum?" This paper will give a complete answer to this modified problem for a class of special trees.

As usual, we denote by P_k the path with k vertices. Let G be a graph. Denote by $\mathcal{L}(G)$ the line graph of G. We denote by $T_4(p,q,r)$ the graph shown in Fig. 1. $T_4(p,q,r)$ is a tree with 4 vertices of degree 3. For a $T_4(p,q,r)$ graph, we always assume that $1 \le p \le q \le r$. The reader is referred to [1] for any undefined notion and terminology on graphs in this paper.

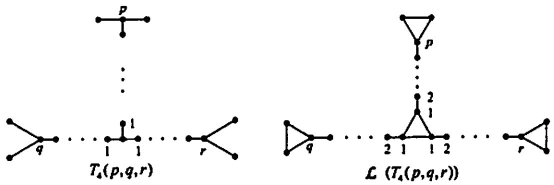

Figure 1: The graphs $T_4(p,q,r)$ and $\mathcal{L}(T_4(p,q,r))$ where $p,q,r \ge 1$.

In this paper we will show that $T_4(p,q,r)$ is determined by its Laplacian spectrum and there are no two non-isomorphic graphs which are cospectral with respect to adjacency spectrum.

2 Preliminaries

In this section, we will present some known results which will be used in this paper.

Lemma 2.1 *([1]) Two trees T and T' are cospectral with respect to the Laplacian matrix if and only if their line graphs are cospectral with respect to the adjacency matrix.*

Lemma 2.2 *([5]) If $\mathcal{L}(G) \cong \mathcal{L}(H)$ with $\{G,H\} \ne \{K_3, K_{1,3}\}$. Then $G \cong H$.*

Let W_n be the graph obtained from the path P_{n-2} (indexed in natural order $1, 2, \ldots, n-2$) by adding two pendant edges at vertices 2 and $n-3$.

Lemma 2.3 *([6]) Let G be a connected graph that is not isomorphic to W_n and G_{uv} be the graph obtained from G by subdividing the edge uv of G. If uv lies on an internal path of G, then $\lambda_1(G_{uv}) \leq \lambda_1(G)$.*

Lemma 2.4 *([14]) Let G be a graph. The following can be obtained from the adjacency spectrum and from the Laplacian spectrum:*
(i) The number of vertices,
(ii) The number of edges.
The spectrum of the adjacency matrix determines:
(iii) The number of closed walks of any length.
The Laplacian spectrum determines:
(iv) The number of spanning trees,
(v) The number of components,
(vi) The sum of squares of degrees of vertices.

Let $N_G(H)$ be the number of subgraphs of a graph G which is isomorphic to H and let $N_G(i)$ be the number of closed walks of length i of G.

Lemma 2.5 *([9]) Let G be a graph. Then*
(i) $N_G(2) = 2m,\ N_G(3) = 6N_G(K_3)$;
(ii) $N_G(4) = 2m + 4N_G(P_3) + 8N_G(C_4),\ N_G(5) = 30N_G(K_3) + 10N_G(C_5) + 10N_G(G_1)$;
(iii) $N_G(7) = 126N_G(K_3) + 84N_G(G_1) + 14N_G(G_2) + 14N_G(G_3) + 14N_G(G_4) + 28N_G(G_5) + 42N_G(G_6) + 28N_G(G_7) + 112N_G(G_8) + 70N_G(C_5) + 14N_G(C_7)$.
(see Fig. 2).

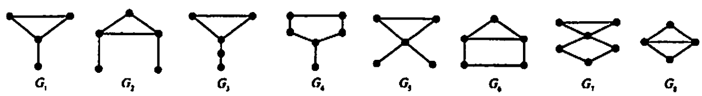

Figure 2: The graphs G_i, i=1,...,8.

Lemma 2.6 *([8]) Let G be a graph with $V(G) \neq \emptyset$ and $E(G) \neq \emptyset$. Then*

$$\Delta(G) + 1 \leq \mu_1 \leq max\{\frac{d_u(d_u + m_u) + d_v(d_v + m_v)}{d_u + d_v}, uv \in E(G)\}$$

where $\Delta(G)$ denote the maximum vertex degree of G, and m_v the average of degrees of the vertices adjacent to the vertex v in G.

Lemma 2.7 *([3], [13]) Let v be a vertex of a graph G and let $C(v)$ denote the collection of cycles containing v. Then the characteristic polynomial of G satisfies*

$$P(G,\lambda) = \lambda P(G\backslash\{v\},\lambda) - \sum_{u\sim v} P(G\backslash\{u,v\},\lambda) - 2\sum_{Z\in C(v)} P(G\backslash V(Z),\lambda).$$

For the sake of convenience, denote $P(P_r,\lambda)$ by $p_r = p_r(\lambda)$ and set $p_0 = 1, p_{-1} = 0$ and $p_{-2} = -1$.

Lemma 2.8 *([11]) $p_r = \frac{x^{2r+2}-1}{x^{r+2}-x^r}$ and $p_r(2) = r+1$, where x satisfies $x^2 - \lambda x + 1 = 0$.*

A centipede is a graph obtained by appending a pendant vertex to each vertex of degree 2 of a path.

Lemma 2.9 *([2]) The centipede is determined by its Laplacian spectrum.*

Lemma 2.10 *([15]) For bipartite graphs, the sum of cubes of degrees is determined by the Laplacian spectrum.*

3 $T_4(p,q,r)$ is determined by its Laplacian spectra

In this section, we will show that $T_4(p,q,r)$ is determined by its Laplacian spectrum. To this aim, we need to compute the characteristic polynomial of the line graph $\mathcal{L}(T_4(p,q,r)$ of $T_4(p,q,r)$. By using Lemma 2.7 with v being the vertices of degree three, we have

$$\begin{aligned} P(\mathcal{L}(T_4(p,q,r)),\lambda) &= f(q,r)(\lambda h_{p-1} - h_{p-2}) - h_{p-1}(h_{q-1}h_r - \\ &\quad h_q h_{r-1} - 2h_{q-1}h_{r-1}), \\ P(\mathcal{L}(T_4(1,q,r)),\lambda) &= f(q,r)(\lambda p_2 - 2\lambda - 2) - p_2(h_{q-1}h_r + \\ &\quad h_q h_{r-1} + 2h_{q-1}h_{r-1}), \end{aligned}$$

where $f(q,r) = h_r(\lambda h_{q-1} - h_{q-2}) - h_{q-1}h_r - 1$ and $h_k = \lambda p_{k-1}(p_2 - 2) - p_2 p_{k-2} - 2p_{k-1}$. Combining with Lemma 2.8 and using Maple, we have

$$(x^2-1)^3 x^{n+5} P(\mathcal{L}(T_4(p,q,r)),\lambda) = C_0(n;x) + W(p,q,r;x), \quad (3.1)$$
$$(x^2-1)^2 x^{n+2} P(\mathcal{L}(T_4(1,q,r)),\lambda) = C_0'(n;x) + W(1,q,r;x), \quad (3.2)$$

where $n = p+q+r+7$, x satisfies $x^2 - \lambda x + 1 = 0$ and

$$\begin{aligned} C_0(n;x) &= x^{2n+9} - 6x^{2n+7} - 8x^{2n+6} + 9x^{2n+5} + 36x^{2n+4} + 29x^{2n+3} \\ &\quad -30x^{2n+2} - 87x^{2n+1} - 72x^{2n} + 9x^{2n-1} + 78x^{2n-2} \\ &\quad +84x^{2n-3} + 48x^{2n-4} + 15x^{2n-5} + 2x^{2n-6} - 2x^{20} - 15x^{19} \\ &\quad -48x^{18} - 84x^{17} - 78x^{16} - 9x^{15} + 72x^{14} + 87x^{13} + 30x^{12} \end{aligned}$$

$$
\begin{aligned}
& -29x^{11} - 36x^{10} - 9x^9 + 8x^8 + 6x^7 - x^5, \\
W(p,q,r;x) \;=\; & x^{2p+7} + x^{2q+7} + x^{2r+7} + 4x^{2p+8} + 4x^{2q+8} + 4x^{2r+8} \\
& +4x^{2p+9} + 4x^{2q+9} + 4x^{2r+9} - 8x^{2p+10} - 8x^{2q+10} \\
& -8x^{2r+10} - 29x^{2p+11} - 29x^{2q+11} - 29x^{2r+11} - 34x^{2p+12} \\
& -34x^{2q+12} - 34x^{2r+12} - x^{2p+13} - x^{2q+13} - x^{2r+13} + \\
& 52x^{2p+14} + 52x^{2q+14} + 52x^{2r+14} + 79x^{2p+15} + 79x^{2q+15} \\
& +79x^{2r+15} + 58x^{2p+16} + 58x^{2q+16} + 58x^{2r+16} + 15x^{2p+17} \\
& +15x^{2q+17} + 15x^{2r+17} - 12x^{2p+18} - 12x^{2q+18} - 12x^{2r+18} \\
& -14x^{2p+19} - 14x^{2q+19} - 14x^{2r+19} - 6x^{2p+20} - 6x^{2q+20} \\
& -6x^{2r+20} - x^{2p+21} - x^{2q+21} - x^{2r+21} + x^{2p+2q+7} \\
& +6x^{2p+2q+8} + 14x^{2p+2q+9} + 12x^{2p+2q+10} - 15x^{2p+2q+11} \\
& -58x^{2p+2q+12} - 79x^{2p+2q+13} - 52x^{2p+2q+14} + x^{2p+2q+15} \\
& +34x^{2p+2q+16} + 29x^{2p+2q+17} + 8x^{2p+2q+18} - 4x^{2p+2q+19} \\
& -4x^{2p+2q+20} - x^{2p+2q+21} + x^{2p+2r+7} + 6x^{2p+2r+8} \\
& +14x^{2p+2r+9} + 12x^{2p+2r+10} - 15x^{2p+2r+11} - 58x^{2p+2r+12} \\
& -79x^{2p+2r+13} - 52x^{2p+2r+14} + x^{2p+2r+15} + 34x^{2p+2r+16} \\
& +29x^{2p+2r+17} + 8x^{2p+2r+18} - 4x^{2p+2r+19} - 4x^{2p+2r+20} \\
& -x^{2p+2r+21} + x^{2q+2r+7} + 6x^{2q+2r+8} + 14x^{2q+2r+9} \\
& +12x^{2q+2r+10} - 15x^{2q+2r+11} - 58x^{2q+2r+12} \\
& -79x^{2q+2r+13} - 52x^{2q+2r+14} + x^{2q+2r+15} \\
& +34x^{2q+2r+16} + 29x^{2q+2r+17} + 8x^{2q+2r+18} \\
& -4x^{2q+2r+19} - 4x^{2q+2r+20} - x^{2q+2r+21}, \\
C_0'(n;x) \;=\; & x^{2n+5} - 5x^{2n+3} - 8x^{2n+2} + 3x^{2n+1} + 24x^{2n} \\
& +28x^{2n-1} + 2x^{2n-2} - 30x^{2n-3} - 36x^{2n-4} - 20x^{2n-5} \\
& -10x^{2n-6} - 15x^{2n-7} - 20x^{2n-8} - 15x^{2n-9} - 6x^{2n-10} \\
& -x^{2n-11} - x^{19} - 6x^{18} - 15x^{17} - 20x^{16} - 15x^{15} - 10x^{14} \\
& -20x^{13} - 36x^{12} - 30x^{11} + 2x^{10} + 28x^9 + 24x^8 + 3x^7 \\
& -8x^6 - 5x^5 + x^3, \\
W(1,q,r;x) \;=\; & -x^{2q+5} - x^{2r+5} - 4x^{2q+6} - 4x^{2r+6} - 6x^{2q+7} - 6x^{2r+7} \\
& -2x^{2q+8} - 2x^{2r+8} + 9x^{2q+9} + 9x^{2r+9} + 20x^{2q+10} \\
& +20x^{2r+10} + 25x^{2q+11} + 25x^{2r+11} + 26x^{2q+12} + 26x^{2r+12} \\
& +25x^{2q+13} + 25x^{2r+13} + 20x^{2q+14} + 20x^{2r+14} + 9x^{2q+15} \\
& +9x^{2r+15} - 2x^{2q+16} - 2x^{2r+16} - 6x^{2q+17} - 6x^{2r+17} \\
& -4x^{2q+18} - 4x^{2r+18} - x^{2q+19} - x^{2r+19}.
\end{aligned}
$$

In view of point above, if two graphs $T_4(p,q,r)$ and $T_4(p',q',r')$ are cospectral with respect to Laplacian spectrum, then $\mathcal{L}(T_4(p,q,r))$ and $\mathcal{L}(T_4(p',q',r'))$ are cospectral with respect to adjacency spectrum, hence $p+q+r = p'+q'+r'$ and so $W(p,q,r;x) = W(p',q',r';x)$.

Lemma 3.1 *No two non-isomorphism graphs $T_4(p,q,r)$ are cospectral with respect to Laplacian spectrum.*

Proof. Suppose that $G = T_4(p,q,r)$ and $G' = T_4(p',q',r')$ are cospectral with respect to Laplacian spectrum. Then G and G' have the same number of vertices and so $p+q+r = p'+q'+r'$. On the other hand, by Lemma 2.1, $\mathcal{L}(G)$ and $\mathcal{L}(G')$ are cospectral with respect to adjacency spectrum, so they have the same number of closed walks of any length, especially of length 5. Hence $\mathcal{L}(G)$ and $\mathcal{L}(G')$ have the same number of G_1 in it by Lemma 2.5 (ii).

Clearly, for $2 \le p \le q \le r$, $2 \le q' \le r'$, $2 \le r''$, $N_{\mathcal{L}(T_4(p,q,r))}(G_1) = 6$, $N_{\mathcal{L}(T_4(1,q',r'))}(G_1) = 8$, $N_{\mathcal{L}(T_4(1,1,r''))}(G_1) = 10$. Hence $\mathcal{L}(T_4(p,q,r))$, $\mathcal{L}(T_4(1,q',r'))$ and $\mathcal{L}(T_4(1,1,r''))$ are non-cospectral with each other with respect to adjacency spectrum. It follows from Lemma 2.1 that $T_4(p,q,r)$, $T_4(1,q',r')$ and $T_4(1,1,r'')$ are non-cospectral with each other with respect to Laplacian spectrum.

Suppose that $G = T_4(p,q,r)$ with $p > 1$. Then $G' = T_4(p',q',r')$ with $p' > 1$. From (3.1), $W(p,q,r;x) = W(p',q',r';x)$. Note that $p \le q \le r$, $p' \le q' \le r'$ and $p+q+r = p'+q'+r'$. It follows that $p = p'$, $q = q'$ and $r = r'$. Therefore G is isomorphic to G'.

Let $G = T_4(1,q,r)$ with $q > 1$. Then $G' = T_4(1,q',r')$ and $q' > 1$. By (3.2), $W(1,q,r;x) = W(1,q',r';x)$. It follows that $q = q'$ and $r = r'$. Therefore G is isomorphic to G'.

If $G = T_4(1,1,r)$, then $G' = T_4(1,1,r')$. It is easy to see that $r = r'$ since G and G' have the same number of vertices. Hence G is isomorphic to G'.

Up to now, we have completed the proof of the lemma. □

Lemma 3.2 *Let G be a tree and H be a graph cospectral to G with respect to Laplacian spectrum. If $\mu_1(G) \le 5$, then the degree sequence of H is determined by the shared spectrum.*

Proof. Let H be any graph cospectral to G with respect to Laplacian spectrum. Then by Lemma 2.4 (i) and (ii), H is also a tree. Clearly, $\Delta(G) \le 4$ by Lemmas 2.6. Let x_i and y_i be the numbers of vertices of degree i in G and H, respectively. It follows from Lemmas 2.4 and 2.10 that

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = y_1 + y_2 + y_3 + y_4, \\ x_1 + 2x_2 + 3x_3 + 4x_4 = y_1 + 2y_2 + 3y_3 + 4y_4, \\ x_1 + 4x_2 + 9x_3 + 16x_4 = y_1 + 4y_2 + 9y_3 + 16y_4, \\ x_1 + 8x_2 + 27x_3 + 64x_4 = y_1 + 8y_2 + 27y_3 + 64y_4. \end{cases}$$

It implies that $y_i = x_i$ for $i = 1, 2, 3, 4$. Hence the degree sequence of H is determined by its Laplacian spectrum. □

Corollary 3.3 *Let $G = T_4(p, q, r)$ and H be a graph cospectral to G with respect to Laplacian spectrum. Then H has the same degree sequence as G.*

Proof. Since G is a tree and $\mu_1(G) < 4.9$ by Lemma 2.6, the result is followed immediately from Lemma 3.2. □

Lemma 3.4 *Let $G = T_4(p, q, r)$ and H be a graph cospectral to G with respect to Laplacian spectrum. Then $H = H_1$ or $H = H_2$ (see Fig. 3) for some $l_i, k_i \geq 1$ for $i = 1, \ldots, 6$ and $s_j, t_j \geq 0$ for $j = 1, 2, 3$. In particular, $\mathcal{L}(H) = \mathcal{L}(H_1)$ or $\mathcal{L}(H) = \mathcal{L}(H_2)$ (see Fig. 3).*

Proof. From Lemma 2.4 and Corollary 3.3, we know H is a tree, having 4 vertices of degree 3, 6 vertices of degree 1 and other vertices of degree 2. So either all vertices of degree 3 lie on a path or exactly 3 vertices of degree 3 lie on a path and no cycle. Hence $H = H_1$ or $H = H_2$ (see Fig. 3) for some $l_i, k_i \geq 1$ for $i = 1, \ldots, 6$ and $s_j, t_j \geq 0$ for $j = 1, 2, 3$. □

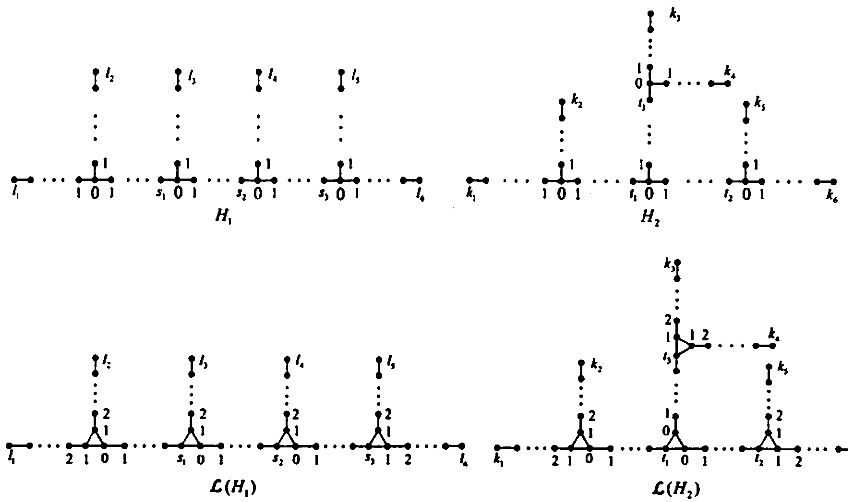

Figure 3: The graphs H_i and $\mathcal{L}(H_i)$, i=1,2, where $l_i, k_i \geq 1$ for $i = 1, \ldots, 6$ and $s_j, t_j \geq 0$ for $j = 1, 2, 3$.

Lemma 3.5 *Let $G = T_4(p, q, r)$ with $p \geq 2$. Then G is determined by its Laplacian spectrum.*

Proof. Let H be a graph cospectral to G with respect to Laplacian spectrum. Then $\mathcal{L}(H)$ and $\mathcal{L}(G)$ are cospectral with respect to adjacency spectrum by Lemma 2.1. So $\mathcal{L}(H)$ and $\mathcal{L}(G)$ have the same number of vertices, edges and triangles. Obviously, $\Delta(\mathcal{L}(G)) = 3$ and $\Delta(\mathcal{L}(H)) \leq 4$. Let y_i be the number of vertices of degree i in $\mathcal{L}(H)$. Note that $\mathcal{L}(G)$ has $m = p + q + r + 6$ vertices, where 6 of them have degree 3 and others have degree 2. It follows from Lemma 2.4 that

$$\begin{cases} y_1 + y_2 + y_3 + y_4 = m, \\ y_1 + 2y_2 + 3y_3 + 4y_4 = 2(m+3), \\ y_2 + \binom{3}{2} y_3 + \binom{4}{2} y_4 = 6\binom{3}{2} + m - 6. \end{cases}$$

Solving this system of linear equation, we obtain $(y_1, y_2, y_3, y_4) = (-y_4, m-6+3y_4, 6-3y_4, y_4)$. Hence $y_1 = y_4 = 0$ since $y_i \geq 0$ for $i = 1,2,3,4$. Therefore $(y_1, y_2, y_3, y_4) = (0, m-6, 6, 0)$. By Lemma 3.4, there are two cases.

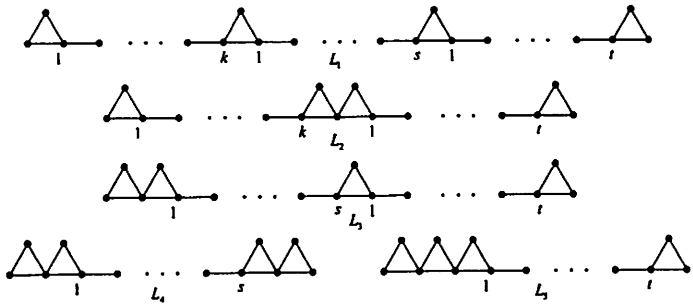

Figure 4: The graphs L_i, $i = 1, \ldots, 5$, where $k, s, t > 1$.

If $\mathcal{L}(H) = \mathcal{L}(H_1)$, then $l_i = 1$ and $s_j > 0$ for $i = 1, \ldots, 6$ and $j = 1,2,3$ since $\mathcal{L}(H)$ has no vertex of degree 1 and 4. Hence $\mathcal{L}(H) \cong L_1$ (see Fig. 4). Obviously, $N_{\mathcal{L}(G)}(G_1) = N_{\mathcal{L}(H)}(G_1) = 6$, $N_{\mathcal{L}(G)}(G_2) = 3$, $N_{\mathcal{L}(H)}(G_2) = 2$, $N_{\mathcal{L}(G)}(G_3) = 6$, $N_{\mathcal{L}(H)}(G_3) = 6$ or 8 or 10 or 12, $N_{\mathcal{L}(G)}(K_3) = N_{\mathcal{L}(H)}(K_3) = 4$, $N_{\mathcal{L}(G)}(C_k) = N_{\mathcal{L}(H)}(C_k) = 0$ for $k = 5, 7$ and $N_{\mathcal{L}(G)}(G_i) = N_{\mathcal{L}(H)}(G_i) = 0$ for $i = 4,5,6,7,8$. It follows from Lemma 2.5 (iii) that $N_{\mathcal{L}(G)}(7) \neq N_{\mathcal{L}(H)}(7)$. This contradicts the fact that $\mathcal{L}(H)$ and $\mathcal{L}(G)$ are cospectral with respect to adjacency spectrum.

If $\mathcal{L}(H) = \mathcal{L}(H_2)$, then $k_i = 1$ and $t_j > 0$ for $i = 1, \ldots, 6$ and $j = 1,2,3$ since $\mathcal{L}(H)$ has no vertex of degree 1 and 4. It implies that $\mathcal{L}(H) \cong \mathcal{L}(T_4(p', q', r'))$ for some $p', q', r' \geq 2$. Hence $H \cong T_4(p', q', r')$ by Lemma 2.2. It follows from Lemma 3.1 that $H \cong T_4(p, q, r) = G$. □

Lemma 3.6 *Let $G = T_4(1, q, r)$ with $q > 1$. Then G is determined by its Laplacian spectrum.*

Proof. Let H be a graph cospectral to G with respect to Laplacian spectrum. Then $\mathcal{L}(H)$ and $\mathcal{L}(G)$ are cospectral with respect to adjacency spectrum by Lemma 2.1. So $\mathcal{L}(H)$ and $\mathcal{L}(G)$ have the same number of vertices, edges and triangles. Obviously, $\Delta(\mathcal{L}(G)) = 4$ and $\Delta(\mathcal{L}(H)) \leq 4$. Let y_i be the number of vertices of degree i in $\mathcal{L}(H)$. It follows from Lemma 2.4 that

$$\begin{cases} y_1 + y_2 + y_3 + y_4 = m, \\ y_1 + 2y_2 + 3y_3 + 4y_4 = 2(m+3), \\ y_2 + \binom{3}{2} y_3 + \binom{4}{2} y_4 = \binom{4}{2} + 4\binom{3}{2} + m - 5, \end{cases}$$

Solving this system of linear equation, we obtain $(y_1, y_2, y_3, y_4) = (1 - y_4, m - 8 + 3y_4, 7 - 3y_4, y_4)$. Hence either $y_4 = 0$ or $y_4 = 1$ since $y_i \geq 0$ for $i = 1, 2, 3, 4$.

Suppose that $y_4 = 0$. Then $(y_1, y_2, y_3, y_4) = (1, m-8, 7, 0)$, that is, $\mathcal{L}(H)$ has exactly one vertex of degree 1, $m - 8$ vertices of degree 2, 7 vertices of degree 3 and no vertex of degree 4. Whether $\mathcal{L}(H) = \mathcal{L}(H_1)$ or $\mathcal{L}(H) = \mathcal{L}(H_2)$ (see Fig. 3), we always have $N_{\mathcal{L}(H)}(G_1) = 7$, $N_{\mathcal{L}(H)}(K_3) = 4$ and $N_{\mathcal{L}(H)}(C_5) = 0$. However, $N_{\mathcal{L}(G)}(G_1) = 8$, $N_{\mathcal{L}(G)}(K_3) = 4$ and $N_{\mathcal{L}(G)}(C_5) = 0$. It follows from Lemma 2.5 (ii) that $N_{\mathcal{L}(G)}(5) \neq N_{\mathcal{L}(H)}(5)$. This contradicts the fact that $\mathcal{L}(H)$ and $\mathcal{L}(G)$ are cospectral with respect to adjacency spectrum.

Suppose that $y_4 = 1$. Then $(y_1, y_2, y_3, y_4) = (0, m-5, 4, 1)$. If $\mathcal{L}(H) = \mathcal{L}(H_1)$, then $\mathcal{L}(H) \cong L_2$ or L_3 (see Fig. 4). Clearly,

$$\begin{aligned} &N_{\mathcal{L}(G)}(G_1) = N_{L_2}(G_1) = N_{L_3}(G_1) = 8, \\ &N_{\mathcal{L}(G)}(K_3) = N_{L_2}(K_3) = N_{L_3}(K_3) = 4, \\ &N_{\mathcal{L}(G)}(G_5) = N_{L_2}(G_5) = N_{L_3}(G_5) = 2, \\ &N_{\mathcal{L}(G)}(C_i) = N_{L_2}(C_i) = N_{L_3}(C_i) = 0,\ i = 5, 7, \\ &N_{\mathcal{L}(G)}(G_i) = N_{L_2}(G_i) = N_{L_3}(G_i) = 0,\ i = 4, 6, 7, 8. \end{aligned}$$

However,

$$\begin{aligned} &N_{\mathcal{L}(G)}(G_2) = 5,\ N_{L_2}(G_2) = 4,\ N_{L_3}(G_2) = 3,\ N_{\mathcal{L}(G)}(G_3) = 10 \text{ or } 12 \text{ or } 14, \\ &N_{L_2}(G_3) = 10 \text{ or } 12 \text{ or } 14,\ N_{L_3}(G_3) = 9 \text{ or } 11 \text{ or } 13. \end{aligned}$$

It follows from Lemma 2.5 (iii) that $N_{\mathcal{L}(G)}(7) \neq N_{\mathcal{L}(H)}(7)$. This contradicts the fact that $\mathcal{L}(H)$ and $\mathcal{L}(G)$ are cospectral with respect to adjacency spectrum.

If $\mathcal{L}(H) = \mathcal{L}(H_2)$, then $\mathcal{L}(H) \cong \mathcal{L}(T_4(1, q', r'))$ for some $q', r' \geq 2$. Hence $H \cong T_4(1, q', r')$ by Lemma 2.2. Therefore $H \cong T_4(1, q, r)$ by Lemma 3.1. □

Lemma 3.7 *Let $G = T_4(1, 1, r)$ with $r \geq 2$. Then G is determined by its Laplacian spectrum.*

Proof. Let H be a graph cospectral to G with respect to Laplacian spectrum. Then $\mathcal{L}(H)$ and $\mathcal{L}(G)$ are cospectral with respect to adjacency spectrum by Lemma 2.1. So $\mathcal{L}(H)$ and $\mathcal{L}(G)$ have the same number of vertices, edges and triangles.

Obviously, $\Delta(\mathcal{L}(G)) = 4$ and $\Delta(\mathcal{L}(H)) \leq 4$. Let y_i be the number of vertices of degree i in $\mathcal{L}(H)$. It follows from Lemma 2.4 that

$$\begin{cases} y_1 + y_2 + y_3 + y_4 = m, \\ y_1 + 2y_2 + 3y_3 + 4y_4 = 2(m+3), \\ y_2 + \binom{3}{2} y_3 + \binom{4}{2} y_4 = 2\binom{4}{2} + 2\binom{3}{2} + m - 4. \end{cases}$$

Solving this system of linear equation, we obtain $(y_1, y_2, y_3, y_4) = (2 - y_4, m - 10 + 3y_4, 8 - 3y_4, y_4)$. Hence $y_4 = 0$ or 1 or 2 since $y_i \geq 0$ for $i = 1, 2, 3, 4$.

Suppose that $y_4 = 0$. Then $(y_1, y_2, y_3, y_4) = (2, m - 10, 8, 0)$, that is, $\mathcal{L}(H)$ has 2 vertices of degree 1 , $m - 10$ vertices of degree 2, 8 vertices of degree 3 and no vertex of degree 4. Whether $\mathcal{L}(H) = \mathcal{L}(H_1)$ or $\mathcal{L}(H) = \mathcal{L}(H_2)$, we always have $N_{\mathcal{L}(G)}(K_3) = N_{\mathcal{L}(H)}(K_3) = 4$, $N_{\mathcal{L}(G)}(C_5) = N_{\mathcal{L}(H)}(C_5) = 0$, $N_{\mathcal{L}(H)}(G_1) = 8$ and $N_{\mathcal{L}(G)}(G_1) = 10$. It follows from Lemma 2.5 (ii) that $N_{\mathcal{L}(H)}(5) \neq N_{\mathcal{L}(G)}(5)$. This contradicts the fact that $\mathcal{L}(H)$ and $\mathcal{L}(G)$ are cospectral with respect to adjacency spectrum.

Suppose that $y_4 = 1$. Then $(y_1, y_2, y_3, y_4) = (1, m - 7, 5, 1)$, that is, $\mathcal{L}(H)$ has 1 vertex of degree 1 , $m - 10$ vertices of degree 2, 8 vertices of degree 3 and 1 vertex of degree 4. Whether $\mathcal{L}(H) = \mathcal{L}(H_1)$ or $\mathcal{L}(H) = \mathcal{L}(H_2)$, we always have $N_{\mathcal{L}(H)}(5) \neq N_{\mathcal{L}(G)}(5)$, contradiction.

Suppose that $y_4 = 2$. Then $(y_1, y_2, y_3, y_4) = (0, m - 4, 2, 2)$. If $\mathcal{L}(H) = \mathcal{L}(H_1)$, then $\mathcal{L}(H) \cong L_4$ or L_5 (see Fig. 4). Clearly,

$$\begin{aligned} &N_{\mathcal{L}(G)}(G_1) = N_{L_4}(G_1) = N_{L_5}(G_1) = 10, \\ &N_{\mathcal{L}(G)}(G_5) = N_{L_4}(G_5) = N_{L_5}(G_5) = 4. \\ &N_{\mathcal{L}(G)}(K_3) = N_{L_4}(K_3) = N_{L_5}(K_3) = 4, \\ &N_{\mathcal{L}(G)}(C_k) = N_{L_4}(C_k) = N_{L_5}(C_k) = 0, k = 5, 7, \\ &N_{\mathcal{L}(G)}(G_i) = N_{L_4}(G_i) = N_{L_5}(G_i) = 0, i = 4, 6, 7, 8. \end{aligned}$$

However,

$$\begin{array}{lll} N_{\mathcal{L}(G)}(G_2) = 8, & N_{L_4}(G_2) = 4, & N_{L_5}(G_2) = 6, \\ N_{\mathcal{L}(G)}(G_3) = 16 \text{ or } 18, & N_{L_4}(G_3) = 12 \text{ or } 14, & N_{L_5}(G_3) = 15 \text{ or } 17. \end{array}$$

It follows from Lemma 2.5 (iii) that $N_{\mathcal{L}(G)}(7) \neq N_{\mathcal{L}(H)}(7)$. This contradicts the fact that $\mathcal{L}(H)$ and $\mathcal{L}(G)$ are cospectral with respect to adjacency spectrum.

If $\mathcal{L}(H) = \mathcal{L}(H_2)$, then $\mathcal{L}(H) \cong \mathcal{L}(T_4(1, 1, r'))$ for some $r' \geq 2$. Hence $H \cong T_4(1, 1, r')$ by Lemma 2.2. Therefore $H \cong T_4(1, 1, r)$ by Lemma 3.1. □

Lemma 3.8 *Let* $G = T_4(1, 1, 1)$. *Then* G *is determined by its Laplacian spectrum.*

Proof. Let H be a graph cospectral to G with respect to Laplacian spectrum. By Lemma 3.2, the degree sequence of H is $(3, 3, 3, 3, 1, 1, 1, 1, 1, 1)$, so H is isomorphic to a centipede graph or $T_4(1,1,1)$. By Lemma 2.9, the centipede is determined by its Laplacian spectrum. Hence $H \cong T_4(1,1,1)$. □

Now we may give our main result in this section.

Theorem 3.9 *$T_4(p,q,r)$ is determined by its Laplacian spectrum.*

Proof. It follows from Lemmas 3.5, 3.6, 3.7 and 3.8. □

Recall from [7] that the Laplacian eigenvalues of the complement of a graph G are completely determined by the Laplacian eigenvalues of G. As a direct consequence of Theorem 3.9, we have

Corollary 3.10 *The complement of $T_4(p,q,r)$ is determined by its Laplacian spectrum.*

4 Adjacency spectral characterization of $T_4(p,q,r)$

In this section, we will study the adjacency spectral characterization of $T_4(p,q,r)$. It will be shown that there is no two non-isomorphism graphs $T_4(p,q,r)$ are cospectral with respect to adjacency spectrum.

Using Lemma 2.7 with v being the vertices of degree 3, we can compute the characteristic polynomial of $T_4(p,q,r)$ in terms of the characteristic polynomials of paths. Put $P(T_4) = P(T_4(p,q,r),\lambda)$ and $f_r = \lambda(p_{r+1} - p_{r-1})$ for any integer r. Then we have

$$P(T_4) = \begin{cases} \lambda p_2^3 - 3\lambda^2 p_2^2, & if\ p = q = r = 1, \\ \lambda p_2 p_2 f_r - 2\lambda^2 p_2 f_r - p_2^2 f_{r-1}, & if\ 1 = p = q < r, \\ \lambda p_2 f_q f_r - \lambda^2 f_q f_r - p_2 f_{q-1} f_r - p_2 f_q f_{r-1}, & if\ 1 = p < q \le r, \\ \lambda f_q f_p f_r - f_{q-1} f_p f_r - f_q f_{p-1} f_r - f_q f_p f_{r-1}, & if\ 2 \le p \le q \le r. \end{cases}$$

Let $n = p+q+r+7$ and $\phi(p,q,r) = x^n \left(x^2-1\right)^3 P(T_4(p,q,r),\lambda)$. By Lemma 2.8, we have

$$\phi(p,q,r)) = \begin{cases} C_1(n;x), & if\ 1 = p = q < r, \\ C_2(n;x) + U(1,q,r;x), & if\ 1 = p < q \le r, \\ C_3(n;x) + U(p,q,r;x), & if\ 2 \le p \le q \le r, \end{cases} \tag{4.1}$$

where x satisfies $x^2 - \lambda x + 1 = 0$ and

$$\begin{aligned} C_1(n;x) \quad = \quad & 2x^{2n-13} - x^{2n-12} + 2x^{2n-11} - 4x^{2n-9} + x^{2n-8} - 6x^{2n-7} \\ & +2x^{2n-6} + 6x^{2n-3} - 2x^{2n-2} + 4x^{2n-1} - x^{2n} - 2x^{2n+1} \\ & -2x^{2n+3} + x^{n+4} - 2x^{19} + x^{18} - 2x^{17} + 4x^{15} - x^{14} + 6x^{13} \end{aligned}$$

$$
\begin{aligned}
& -2x^{12} - 6x^9 + 2x^8 - 4x^7 + x^6 + 2x^5 + 2x^3 - x^2. \\
C_2(n;x) &= +x^{2n-11} + x^{2n-10} - x^{2n-9} + x^{2n-8} - 2x^{2n-7} - 3x^{2n-6} \\
& +x^{2n-5} - 3x^{2n-4} + 2x^{2n-3} + 3x^{2n-2} + x^{2n-1} + 3x^{2n} \\
& -2x^{2n+1} - x^{2n+2} - x^{2n+3} - x^{2n+4} + x^{2n+5} - x^{17} - x^{16} \\
& +x^{15} - x^{14} + 2x^{13} + 3x^{12} - x^{11} + 3x^{10} - 2x^9 - 3x^8 \\
& -x^7 - 3x^6 + 2x^5 + x^4 + x^3 + x^2 - x. \\
C_3(n;x) &= 2x^{2n-8} - x^{2n-6} - 6x^{2n-4} + 3x^{2n-2} + 6x^{2n} - 3x^{2n+2} \\
& -2x^{2n+4} + x^{2n+6} - 2x^{14} + x^{12} + 6x^{10} - 3x^8 - 6x^6 + \\
& 3x^4 + 2x^2 - 1. \\
U(1,q,r;x) &= x^{2q+4} + x^{2q+6} - 3x^{2q+8} - 3x^{2q+10} + 3x^{2q+12} + 3x^{2q+14} \\
& -x^{2q+16} - x^{2q+18} + x^{2r+4} + x^{2r+6} - 3x^{2r+8} - 3x^{2r+10} \\
& +3x^{2r+12} + 3x^{2r+14} - x^{2r+16} - x^{2r+18}. \\
U(p,q,r;x) &= x^{2p+4} - 3x^{2p+8} + 3x^{2p+12} - x^{2p+16} + x^{2q+4} - 3x^{2q+8} \\
& +3x^{2q+12} - x^{2q+16} + x^{2r+4} - 3x^{2r+8} + 3x^{2r+12} \\
& -x^{2r+16} + x^{2p+2q+4} - 3x^{2p+2q+8} + 3x^{2p+2q+12} \\
& -x^{2p+2q+16} + x^{2p+2r+4} - 3x^{2p+2r+8} + 3x^{2p+2r+12} \\
& -x^{2p+2r+16} + x^{2q+2r+4} - 3x^{2q+2r+8} + 3x^{2q+2r+12} \\
& -x^{2q+2r+16}.
\end{aligned}
$$

Theorem 4.1 *No two non-isomorphism graphs $T_4(p,q,r)$ are cospectral with respect to adjacency spectrum.*

Proof. Suppose that $G = T_4(p,q,r)$ and $G' = T_4(p',q',r')$ are cospectral with respect to adjacency spectrum. Then $p+q+r = p'+q'+r'$ and $\phi(p,q,r) = \phi(p',q',r')$, hence $U(p,q,r;x) = U(p',q',r';x)$. Obviously, for any positive integers p,q,r with $2 \le p \le q \le r$, $\phi(1,1,r), \phi(1,q,r)$ and $\phi(p,q,r)$ are three distinct polynomials. Therefore $T_4(1,1,r), T_4(1,q,r)$ and $T_4(p,q,r)$ are non-cospectral with each other with respect to adjacency spectrum.

Let $G = T_4(p,q,r)$ with $2 \le p \le q \le r$. Then $G' = T_4(p',q',r')$ with $2 \le p' \le q' \le r'$ and $U(p,q,r;x) = U(p',q',r';x)$. It follows that $p = p'$, $q = q'$ and $r = r'$. Therefore $G \cong G'$.

Let $G = T_4(1,q,r)$ with $2 \le q \le r$. Then $G' = T_4(1,q',r')$ with $2 \le q' \le r'$ and $U(1,q,r;x) = U(1,q',r';x)$. It follows that $q = q'$ and $r = r'$. Therefore $G \cong G'$.

Let $G = T_4(1,1,r)$ with $1 \le r$. Then $G' = T_4(1,1,r')$ with $1 \le r'$ and so $r = r'$. Therefore $G \cong G'$.

Up to now, we have completed the proof of the theorem. □

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