The t-Pebbling Conjecture on Products of Complete r-Partite Graphs

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Abstract

The t-pebbling number $f_t(G)$ of a of graph G, is the least positive integer m such that however these m pebbles are placed on the vertices of G, we can move t pebbles to any vertex by a sequence of moves, each move taking two pebbles off one vertex and placing one on an adjacent vertex. In this paper, we study the generalized Graham's pebbling conjecture $f_t(G \times H) \leq f(G)f_t(H)$ for product of graphs when G is a complete r-partite graph and H has a 2t-pebbling property.

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1 Introduction

Pebbling in graphs was first considered by Chung [1]. She defines a pebbling distribution or a pebbling configuration on a connected graph as a placement of pebbles on the vertices of the graph . A pebbling move consists of the removal of two pebbles from a vertex, and the placement of one of those pebbles on an adjacent vertex. Chung [1] defined the pebbling number of a connected graph, which we denote $f(G)$, as follows: $f(G)$ is the minimum number of pebbles such that from any distribution of $f(G)$ pebbles on the vertices of G, any designated vertex can receive one pebble after a finite number of pebbling moves. She also defined the t-pebbling number of G as the smallest number $f_t(G)$ such that from any placement of $f_t(G)$ pebbles, it is possible to move t pebbles to any specified target vertex by a sequence of pebbling moves.

Chung also defined the two pebbling-property of a graph and Wang [10] extended Chung's definition to the odd two-pebbling property. In [4] we find the $2t$- pebbling property of a graph and generalization of Graham's Conjecture.

In this paper we prove that $f_t(G \times H) \leq f(G)f_t(H)$ when H has the $2t$-pebbling property and G is any complete r-partite graph.

In [4] we find the following.

Definition 1.1. *[4] Given a pebbling configuration of G, let p be the number of pebbles on G, and q be the number of vertices with at least one pebble. We say that G satisfies the $2t$-pebbling property if it is possible to move $2t$ pebbles to any specified target vertex of G starting from every configuration in which $p \geq 2f_t(G) - q + 1$ or, equivalently, $p + q > 2f_t(G)$ for all t.*

Conjecture 1.2 (Generalization of Graham's Conjecture). *[4] For any connected graphs G and H, the t-pebbling number of $G \times H$ satisfies $f_t(G \times H) \leq f(G)f_t(H)$ for all t.*

Theorem 1.3. *[4] Let P_m be a path on m vertices. When G satisfies the $2t$-pebbling property, $f_t(P_m \times G) \leq f(P_m)f_t(G)$ for all t.*

2 Star Graphs

In this section, we first show that star graphs $K_{1,n}$ satisfy the $2t$-pebbling property. We then prove Generalization of Graham's Conjecture when G is a star graph and H satisfies the $2t$-pebbling property.

We use the following theorems.

Theorem 2.1 ([8]). *Let G be a graph with diameter= 2. Then G satisfies the 2-pebbling property.*

Theorem 2.2 ([6]). *Let $K_{1,n}$ be an n-star where $n > 1$. Then $f_t(K_{1,n}) = 4t + n - 2$.*

Our next theorem shows that all star graphs have the $2t$-pebbling property.

Theorem 2.3. *The star graph $K_{1,n}$ where $n > 1$ satisfies the $2t$-pebbling property.*

Proof. Let $V(K_{1,n}) = V_1 \cup V_2$ where $V_1 = \{w\}$ and $V_2 = \{v_1, v_2, \ldots, v_n\}$. $f_t(K_{1,n}) = 4t+n-2$ (Theorem 2.2). We start with a configuration of $2f_t(K_{1,n}) - q + 1 = 8t + 2n - q - 3$ pebbles with q occupied vertices. We use induction on t to prove the result. For $t = 1$, the theorem is true by Theorem 2.1. We assume $t > 1$ and the target vertex has zero pebbles on it initially.

Case(i). Let the target vertex be w.

As q can be at most n, we start with at least $8t + n - 3$ pebbles.

Subcase(A). $n \leq 8t - 6$.
In this case we start with at least $2n + 3$ pebbles. We claim that either there will be at least one vertex with five or more pebbles or there will be at least two vertices with three or more pebbles each. Suppose not. Then we can have a vertex with at most four pebbles but each of the other vertices with at most two pebbles. Therefore, the configuration has at most $(q-1)2+4 \leq 2n+2 < 2n+3$ pebbles, raising a contradiction. Thus two pebbles can be moved to w using four pebbles without making any of the q occupied vertices empty. This leaves us with $8t+2n-q-7 > 2f_{t-1}(K_{1,n}) - q + 1$ pebbles that have not been moved with q occupied vertices. These pebbles would suffice to put $2(t-1)$ additional pebbles on w by induction.
Subcase(B). $n > 8t - 6$.
If there is a vertex with at least five pebbles or there are two vertices with at least three pebbles each then we use induction on t to put $2t$ pebbles on w as in Subcase(A). Suppose not.Without loss of generality,we assume there is a vertex with four pebbles. Therefore, each of the other vertices will contain at most two pebbles. We claim that there will be at least $2t - 1$ vertices each with exactly two pebbles. Suppose there are at most $2t - 2$ vertices each with exactly two pebbles. Therefore the configuration has at most $(2t-2)2+4+(q-(2t-1)) = 2t + q + 1 < 8t + 2n - q - 3$ (since $q \leq n$) pebbles, which is a contradiction. Therefore we have at least $2t - 1$ vertices each with exactly two pebbles and a vertex with four pebbles and hence $2t$ pebbles can be moved to w.
Case(ii). Let the target vertex be v_i for some $i = 1, 2, \ldots, n$.

Without loss of generality, we assume w has zero pebbles on it. As q can be at most $n - 1$ we start with at least $8t + n - 2$ pebbles.
Subcase(C). $n \leq 8t - 7$.
In this case we start with at least $2n+5$ pebbles. We claim that either there will be at least one vertex with nine or more pebbles or there will be at least two vertices with five or more pebbles each or there will be at least four vertices with three or more pebbles each. Suppose not. Without loss of generality, we assume there is a vertex with eight pebbles. Therefore each of the other vertices will contain at most two pebbles. Then the number of pebbles in the configuration is at most $(q-1)2+8 = 2q+6 \leq 2n+4$, since $q \leq n-1$, raising a contradiction. Therefore, four pebbles can be moved to w and hence two pebbles can be moved to v_i using eight pebbles without making any of the q occupied vertices empty. This leaves us with $8t + 2n - q - 11 = 2f_{t-1}(K_{1,n}) - q + 1$ pebbles that have not been moved with q occupied vertices. These pebbles would suffice to put $2(t-1)$ additional pebbles on v_i by induction.
Subcase(D). $n > 8t - 7$.
If there is a vertex with at least nine pebbles or there are two vertices with at least five pebbles each or there are four vertices each with at least three pebbles then we apply induction to put $2t$ pebbles on w. Otherwise, without loss of generality, we assume there is a vertex with eight pebbles. Therefore, the other vertices will contain at most two pebbles. We claim that there will be at least $4t - 1$ vertices with exactly two pebbles. Suppose, there are at most $4t - 2$ vertices with exactly two pebbles. Therefore, the configuration has at most

$(4t-2)2+8+(q-(2t-1)) = 4t+5+q < 8t+2n-q-3$ (since $q \le n-1$) pebbles, which is a contradiction. Therefore, we have at least $4t-1$ vertices with exactly two pebbles and a vertex with eight pebbles and hence $4t$ pebbles can be moved to w and then $2t$ pebbles can be moved to v_i. This completes the proof. ∎

Next theorem proves the generalization of Graham's Conjecture when G is a star graph and H satisfies the $2t$-pebbling property. We take Lemma 2.4 from [3]. It describes how many pebbles we can transfer from one copy of H to an adjacent copy H in $G \times H$. It is also called Transfer Lemma.

Lemma 2.4 (Transfer Lemma). *Let (x_i, x_j) be an edge in G. Suppose that in $G \times H$, we have p_i pebbles occupying q_i vertices of $\{x_i\} \times H$. If $q_i - 1 \le k \le p_i$ and if k and p_i have the same parity then k pebbles can be retained on $\{x_i\} \times H$ while moving $\frac{p_i - k}{2}$ pebbles on to $\{x_j\} \times H$. If k and p_i have opposite parity we must leave $k+1$ pebbles on $\{x_i\} \times H$, so we can only move $\frac{p_i-(k+1)}{2}$ pebbles on to $\{x_j\} \times H$.*

In particular, we can always move at least $\frac{p_i - q_i}{2}$ pebbles onto $\{x_j\} \times H$.

Theorem 2.5. *Let $K_{1,n}$ be an n-star ($n > 1$). If G satisfies the $2t$-pebbling property then*

$$f_t(K_{1,n} \times G) \le f(K_{1,n}) f_t(G) \quad \text{for all } t.$$

Proof. Let $V(K_{1,n}) = U \cup W$ where $U = \{u\}$ and $W = \{w_1, w_2, \ldots, w_n\}$. The pebbling number of $K_{1,n}$ is $f(K_{1,n}) = n+2$ [2]. We use induction on n to prove

$$f_t(K_{1,n} \times G) \le (n+2) f_t(G).$$

For $n = 2$, $K_{1,2} = P_3$, a path on three vertices u, w_1 and w_2. Therefore by Theorem 1.3

$$f_t(K_{1,2} \times G) \le 4 f_t(G)$$

We assume $n > 2$. Let p be the numbers of pebbles on $\{u\} \times G$ with q occupied vertices and p_i be the number of pebbles on $\{w_i\} \times G$ with q_i occupied vertices. Let $y \in G$.

Case(i). Suppose the target vertex is (u, y).

In this case, we fix some $w_i \in W$. If $\frac{p_i + q_i}{2} > f_t(G)$, then $2t$ pebbles can be placed on (w_i, y) and hence t pebbles can be moved to (u, y). Otherwise, we transfer $\frac{p_i - q_i}{2}$ pebbles from $\{w_i\} \times G$ to $\{u\} \times G$ by Lemma 2.4. The subgraph $(K_{1,n} - \{w_i\}) \times G$ is isomorphic to $K_{1,n-1} \times G$.

Therefore, if $\left(\sum_{k=1}^{n} p_k\right) - p_i + p + \frac{p_i - q_i}{2} \ge (n+1) f_t(G)$, then t pebbles can be placed on (u, y) by induction.

Thus if we are unable to put t pebbles on the desired target (u, y) then both the inequalities

$$\frac{p_i + q_i}{2} \leq f_t(G) \quad \text{and}$$

$$\left(\sum_{k=1}^{n} p_k\right) - p_i + p + \frac{p_i - q_i}{2} < (n+1)f_t(G) \quad \text{hold.}$$

Adding them together gives

$$p + p_1 + p_2 + \ldots + p_n < (n+2)f_t(G).$$

Thus any distribution of pebbles from which we may not put t pebbles on (u, y) must begin with fewer than $(n+2)f_t(G)$ pebbles.

Case(ii). Let (w_j, y) be the target vertex for some $j = 1, 2, \ldots, n$.

Without loss of generality we assume that (w_n, y) is the target vertex. We take the $n+1$ copies of G i.e., $\{u\} \times G$, $\{w_1\} \times G$, $\{w_2\} \times G$, $\ldots$, $\{w_n\} \times G$, respectively G_u, G_1, $\ldots$, G_n. We claim that $\overset{n-1}{\underset{i=1}{\cup}} G_i$ contains at least $(n+1)f_t(G) + 1$ pebbles. Suppose not.

Then, $\overset{n-1}{\underset{i=1}{\cup}} G_i$ contains at most $(n+1)f_t(G)$ pebbles. We may assume every vertex of $\overset{n-1}{\underset{i=1}{\cup}} G_i$ contains an odd number of pebbles (This is the worst case scenario). Then by Lemma 2.4 we could move at least $\frac{(n+1)f_t(G) - m(n-1)}{2}$ pebbles to the vertices of G_u where m denotes the number of vertices of G. After this process the number of pebbles on $(K_{1,n} - \overset{n-1}{\underset{i=1}{\cup}}\{w_i\}) \times G$ will be

$$\begin{aligned}
&\geq \frac{(n+1)f_t(G) - m(n-1)}{2} + f_t(G). \\
&= 2f_t(G) + \frac{(n-1)}{2}(f_t(G) - m). \\
&\geq 2f_t(G) \quad \text{as} \quad f_t(G) \geq m.
\end{aligned}$$

Since the subgraph $(K_{1,n} - \overset{n-1}{\underset{i=1}{\cup}}\{w_i\}) \times G$ isomorphic to $P_2 \times G$, we are done by Theorem 1.3.

Therefore, we may assume that $p + p_n < f_t(G)$. We assume that $p + p_n = \alpha_0 f_t(G)$ where $0 \leq \alpha_0 < 1$ and that $\{w_j\} \times G$ contains $(k_j + \alpha_j)f_t(G)$ pebbles where k_j is a non-negative integer and $0 \leq \alpha_j < 1$ for $j = 1, 2, \ldots, n-1$.

Now, we claim that $\sum_{j=1}^{n-1} q_j > (n - 2 + \alpha_0)f_t(G)$.

For suppose not. Then $\sum_{j=1}^{n-1} q_j \leq (n-2+\alpha_0)f_t(G)$. Then we could move at least

$$\frac{(n+2)f_t(G) - \alpha_0 f_t(G) - (n-2+\alpha_0)f_t(G)}{2} = (2 - \alpha_0)f_t(G)$$

pebbles to the vertices of G_u and hence after this process, the number of pebbles on the subgraph $(K_{1,n} - \bigcup_{i=1}^{n-1}\{w_i\}) \times G$ will be at least $2f_t(G)$ and hence we are done.

Now, let $\sum_{j=0}^{n-1} \alpha_j = s \leq n-1$. Hence $\sum_{j=1}^{n-1} k_j = (n+2) - s$.
Note that $\alpha_j f_t(G) + q_j < 2f_t(G)$ for $1 \leq j \leq n-1$. We claim that there exists $j_1, j_2, \ldots, j_s$ such that $j_i \geq 1$ and $\alpha_{j_i} f_t(G) + q_{j_i} > f_t(G)$, $i = 1, 2, \ldots, s$. For suppose not, then

$$\sum_{j=1}^{n-1} \left(\alpha_j f_t(G) + q_j\right) \leq 2(s-1)f_t(G) + \left((n-1) - (s-1)\right)f_t(G)$$
$$= (n-2)f_t(G) + sf_t(G)$$

$$\text{But } \sum_{j=1}^{n-1} \left(\alpha_j f_t(G) + q_j\right) = \sum_{j=1}^{n-1} \alpha_j f_t(G) + \sum_{j=1}^{n-1} q_j$$
$$> \sum_{j=1}^{n-1} \alpha_j f_t(G) + (n-2+\alpha_0)f_t(G)$$
$$= \sum_{j=0}^{n-1} \alpha_j f_t(G) + (n-2)f_t(G)$$
$$= sf_t(G) + (n-2)f_t(G)$$

and this is a contradiction.
Therefore, we may assume (after relabeling if necessary) that $\alpha_j f_t(G) + q_j > f_t(G)$ for $1 \leq j \leq s$. Assume that for some $1 \leq j \leq s$ we had $k_j = 0$, then G_j contains $\alpha_j f_t(G) < f_t(G)$ pebbles. Hence $(K_{1,n} - \{w_j\}) \times G$ has at least $(n+1)f_t(G)$ pebbles at its vertices. As the subgraph $(K_{1,n} - \{w_j\}) \times G$ is isomorphic to $K_{1,n-1} \times G$, we are done by induction.

Therefore, we may assume that $k_j \geq 1$ for $1 \leq j \leq s$. Now, in $G_j (1 \leq j \leq s)$ we have $\alpha_j f_t(G) + q_j > f_t(G)$ and $k_j \geq 1$. Hence by the $2t$-pebbling property we can move at least $(k_j+1)t$ pebbles to (w_j, y) in G_j for $i \leq j \leq s$. In $G_j (j > s)$ we can move at least $k_j t$ pebbles to (w_j, y). By the above pebbling moves we see that, at least

$$\sum_{j=1}^{s} (k_j + 1)t + \sum_{j=s+1}^{n-1} k_j t = st + (n+2-s)t = (n+2)t \geq f_t(K_{1,n})$$

pebbles can be moved to the copy $K_{1,n} \times \{y\}$ of $K_{1,n}$ and we are done $\big((w_n, y) \in K_{1,n} \times \{y\}\big)$. ∎

Theorem 2.6. *Let $K_{1,n}$ be an n-star $(n > 1)$. Then $f_t(K_{1,n} \times K_{1,m}) \leq f(K_{1,n}) f_t(K_{1,m})$.*

Proof. The proof follows from Theorem 2.5 and Theorem 2.3. ■

3 Complete r-Partite Graphs.

Definition 3.1. *For $s_1 \geq s_2 \geq \cdots \geq s_r$, $s_1 > 1$ and if $r = 2$, $s_2 > 1$, let $K_{s_1,s_2,\ldots,s_r}$ be the complete r-partite graph with $s_1, s_2, \ldots, s_r$ vertices in vertex classes $C_1, C_2, \ldots, C_r$ respectively. Let $n = \sum_{i=1}^{r} s_i$.*

We prove Theorem 3.4 in order to apply the principle of induction to prove the Generalization of Graham's Conjecture when G is a complete r-partite graph and H is a graph with the 2t-pebbling property. First we prove Proposition 3.2 and Theorem 3.3 which will be used in the proof of Theorem 3.4.

Proposition 3.2. *Suppose G satisfies the 2t-pebbling property and consider the graph $P_3 \times G$. To t-pebble a target vertex on the middle copy of G, it suffices to start with $3f_t(G)$ pebbles on $P_3 \times G$.*

Proof. Label the vertices of P_3 by x_1, x_2 and x_3 in order. Let p_i denote the number of pebbles on $\{x_i\} \times G$ with q_i occupied vertices. Since G has the $2t$-pebbling property we can put $2t$ pebbles on (x_1, y) unless $\frac{p_1 + q_1}{2} \leq f_t(G)$. By Lemma 2.4 and Theorem 1.3 we can t-pebble (x_2, y) directly by transferring pebbles from $\{x_1\} \times G$ unless $\frac{p_1 - q_1}{2} + p_2 + p_3 < 2f_t(G)$. But if both these inequalities hold, then adding them together gives $p_1 + p_2 + p_3 < 3f_t(G)$. Thus any distribution of pebbles from which we may not put t pebbles on some vertex on the middle copy of G, must begin with fewer than $3f_t(G)$ pebbles. ■

Theorem 3.3. *Let $K_{2,2}$ be a bipartite graph and G be a graph with the 2t-pebbling property.*

$$\text{Then} \quad f_t(K_{2,2} \times G) \leq f(K_{2,2}) f_t(G)$$
$$\text{i.e.,} \quad f_t(K_{2,2} \times G) \leq 4f_t(G) \quad \text{since } f(K_{2,2}) = 4 \text{ [2]}.$$

Proof. Let $V(K_{2,2}) = V_1 \cup V_2$ where $V_1 = \{u_1, u_2\}$ and $V_2 = \{u_3, u_4\}$. Let $y \in G$. Without loss of generality we assume the target vertex on $K_{2,2} \times G$ is (u_1, y). We denote the four copies of G in $K_{2,2} \times G$, i.e., $\{u_i\} \times G$, $i = 1, 2, 3, 4$ respectively by G_i. Let p_i denote the number of pebbles on G_i with q_i occupied vertices. Suppose we start with a configuration of $4f_t(G)$ pebbles. We consider the following cases.

Case 1. Suppose $\frac{p_2 + q_2}{2} \leq f_t(G)$.

In this case we use Lemma 2.4 to transfer pebbles from G_2 to the rest of the graph. The subgraph formed by the three copies G_1, G_3 and G_4 of G is isomorphic to $P_3 \times G$. By Proposition 3.2, if we put $3f_t(G)$ pebbles on this subgraph we can t-pebble the target as it lies on the middle copy of the above subgraph.

Therefore we can put t pebbles on (u_1, y) unless

$$\frac{p_2 - q_2}{2} + p_1 + p_3 + p_4 < 3f_t(G). \tag{1}$$

Adding $\frac{p_2 + q_2}{2} \leq f_t(G)$ to (1) we get $p_1 + p_2 + p_3 + p_4 < 4f_t(G)$ and so the original configuration had fewer than $4f_t(G)$ pebbles.

Case 2. If $\frac{p_2 + q_2}{2} > f_t(G)$.

We put $2t$ pebbles on (u_2, y) using $2f_t(G) - q_2 + 2$ pebbles and we transfer $\frac{p_2 - (2f_t(G) - q_2 + 2)}{2}$ pebbles to G_4 by Lemma 2.4. If we can put t pebbles on (u_4, y), then the target can be t-pebbled as t more pebbles can be added to (u_4, y) from (u_2, y). If this is not possible then

$$\frac{p_2 - (2f_t(G) - q_2 + 2)}{2} + p_4 < f_t(G).$$

$$\text{i.e.,} \quad \frac{p_2 + q_2}{2} + p_4 < 2f_t(G) + 1.$$

$$\text{i.e.,} \quad \frac{p_2 + q_2}{2} + p_4 \leq 2f_t(G). \tag{2}$$

Alternatively, if we can put $2f_t(G)$ pebbles on the subgraph formed by G_1 and G_3 then the target can be t-pebbled as the above subgraph is isomorphic to $P_2 \times G$ and $f_t(P_2 \times G) \leq 2f_t(G)$ by Theorem 1.3. If this is not possible then

$$\frac{p_2 - q_2}{2} + p_1 + p_3 < 2f_t(G). \tag{3}$$

Suppose (2) and (3) hold. Then adding (2) and (3) we get $p_1 + p_2 + p_3 + p_4 < 4f_t(G)$ and hence the original configuration had fewer than $4f_t(G)$ pebbles.

Therefore, if we cannot put t pebbles on (u_1, y) then the original configuration has fewer than $4f_t(G)$ pebbles. ∎

Theorem 3.4. *Let $K_{s_1,2}$ be a bipartite graph with $s_1 \geq 2$ and G be a graph with the $2t$-pebbling property. Then*

$$f_t(K_{s_1,2} \times G) \leq (s_1 + 2)f_t(G). \qquad [f(K_{s_1,2}) = s_1 + 2]\ [2]$$

Proof. We use induction on s_1 to prove the result. The result is true for $s_1 = 2$ by Theorem 3.3. We assume $s_1 > 2$. Let $V(K_{s_1,2}) = V_1 \cup V_2$ where $V_1 = \{v_1, v_2, \ldots, v_{s_1}\}$ and $V_2 = \{u_1, u_2\}$. Let p_{1j} be the number of pebbles on $\{v_j\} \times G$ with q_{1j} occupied vertices and p_{2i} be the number of pebbles on $\{u_i\} \times G$ with q_{2i} occupied vertices. Let $y \in G$. Suppose we start with a configuration of $(s_1 + 2)f_t(G)$ pebbles on $K_{s_1,2} \times G$.

Case(i). Suppose the target vertex is (u_i, y) for some $i = 1, 2$.

Without loss of generality, we take the target vertex is (u_1, y). We choose $v_j \in V_1$ for some $j = 1, 2, \ldots, s_1$. Since G satisfies the $2t$-pebbling property, if

$\frac{p_{1j}+q_{1j}}{2} > f_t(G)$, then $2t$ pebbles can be placed on (v_j, y) and hence t-pebbles can be moved to (u_1, y). Otherwise, we transfer $\frac{p_{1j}-q_{1j}}{2}$ pebbles to the vertices of $\{u_1\} \times G$.

If $\left(\sum_{k\neq j} p_{1k}\right) + p_{21} + p_{22} + \frac{p_{1j}-q_{1j}}{2} \geq (s_1+1)f_t(G)$ then by induction the result follows since $(K_{s_1,2} - \{v_j\}) \times G$ is isomorphic to $K_{s_1-1,2} \times G$. Therefore, if we cannot t-pebble (u_1, y) then both the inequalities

$$\frac{p_{1j}+q_{1j}}{2} \leq f_t(G) \quad \text{and}$$

$$\Big(\sum_{k\neq j} p_{1k}\Big) + p_{21} + p_{22} + \frac{p_{1j}-q_{1j}}{2} < (s_1+1)f_t(G) \quad \text{hold.}$$

Adding these together gives

$$\Big(\sum_{k=1}^{s_1} p_{1k}\Big) + p_{21} + p_{22} < (s_1+2)f_t(G).$$

Thus the original configuration has fewer than $(s_1+2)f_t(G)$ pebbles.

Case(ii). Suppose the target vertex is (v_i, y) for some $v_i \in V_1$.

If there exists some $v_k \in V_1$, $k \neq i$ such that $p_{1k} \leq f_t(G)$, then the subgraph $(K_{s_1,2} - \{v_k\}) \times G$ which is isomorphic to $K_{s_1-1,2} \times G$ will contain at least $(s_1+1)f_t(G)$ pebbles and hence the result follows by induction. Therefore we assume that $p_{1k} > f_t(G)$ for every $k \neq i$.

Subcase (A). If there exists some j and k such that $\frac{p_{1j}+q_{1j}}{2} > f_t(G)$ and $\frac{p_{1k}+q_{1k}}{2} > f_t(G)$ then since G satisfies the $2t$-pebbling property, $2t$ pebbles can be placed on each (v_j, y) and (v_k, y) and hence $2t$ pebbles can be placed either on (u_1, y) or on (u_2, y) and thus t pebbles can be moved to (v_i, y).

Subcase (B). Suppose $\frac{p_{1j}+q_{1j}}{2} > f_t(G)$ for a unique j. Therefore $\frac{p_{1k}+q_{1k}}{2} \leq f_t(G)$ for every $k \neq j$. We make two observations. First, if we could put t pebbles on (u_1, y) and retain $2f_t(G) + 2 - q_{1j}$ pebbles on $\{v_j\} \times G$ in order to put $2t$ pebbles on (v_j, y) then we get a path $\{(v_j, y), (u_1, y), (v_i, y)\}$ with $2t$ pebbles on (v_j, y), t pebbles on (u_1, y) and hence t pebbles could be moved to (v_i, y). Alternatively if we could put $2f_t(G)$ pebbles on the subgraph formed by the copies $\{u_2\} \times G$ and $\{v_i\} \times G$ of G then we could put t pebbles on (v_i, y) by Theorem 1.3 as this is isomorphic to $P_2 \times G$.

From Lemma 2.4 we can keep $2f_t(G) + 2 - q_{1j}$ pebbles on $\{v_j\} \times G$ and still transfer $\frac{p_{1j} - (2f_t(G) + 2 - q_{1j})}{2}$ pebbles to $\{u_1\} \times G$ and transfer $\frac{p_{1k}-q_{1k}}{2}$ pebbles from $\{v_k\} \times G$ for each $k \neq j, i$ to $\{u_1\} \times G$. We can achieve the first goal by putting $f_t(G)$ pebbles on $\{u_1\} \times G$.

We achieve the second by putting $2f_t(G)$ pebbles on the subgraph formed by the copies $\{u_2\} \times G$ and $\{v_i\} \times G$ by transferring $\frac{p_{1k} - q_{1k}}{2}$ pebbles for each $k \neq i$ to $\{u_2\} \times G$. If we cannot reach either objective, then both of the following inequalities hold:

$$\frac{p_{1j} - (2f_t(G) + 2 - q_{1j})}{2} + p_{21} + \sum_{k \neq j,i} \frac{p_{1k} - q_{1k}}{2} < f_t(G) \quad \text{and}$$

$$p_{1i} + p_{22} + \sum_{k \neq i} \frac{p_{1k} - q_{1k}}{2} < 2f_t(G).$$

Adding these inequalities together, we have

$$p_{21} + p_{22} + \Big(\sum_{k=1}^{s_1} p_{1k}\Big) - \sum_{k \neq j,i} q_{1k} \leq 4f_t(G)$$

$$\text{i.e.,} \quad (s_1 + 2)f_t(G) - \sum_{k \neq j,i} q_{1k} \leq 4f_t(G)$$

$$\text{i.e.,} \quad \sum_{k \neq j,i} q_{1k} \geq (s_1 - 2)f_t(G) \tag{4}$$

$$\text{But for} \quad k \neq j,i \quad \frac{p_{1k} + q_{1k}}{2} < f_t(G)$$

$$\text{i.e.,} \quad q_{1k} < 2f_t(G) - p_{1k}$$

$$\begin{aligned} \text{Therefore} \quad \sum_{k \neq j,i} q_{1k} &< 2(s_1 - 2)f_t(G) - \sum_{k \neq j,i} p_{1k} \\ &< 2(s_1 - 2)f_t(G) - (s_1 - 2)f_t(G) \\ &\qquad \text{(since } p_{1k} > f_t(G) \text{ for every } k \neq i\text{)} \\ &= (s_1 - 2)f_t(G) \end{aligned}$$

which contradicts (4)

Subcase (C). Suppose $\frac{p_{1k} + q_{1k}}{2} \leq f_t(G)$ for all $k = 1, 2, \ldots, s_1$.

If we can put $3f_t(G)$ pebbles on the subgraph formed by the three copies $\{u_1\} \times G$, $\{u_2\} \times G$ and $\{v_i\} \times G$ then the target can be pebbled by Proposition 3.2 as it lies on the middle copy of the above subgraph which is isomorphic to $P_3 \times G$. Therefore, if we cannot pebble the target then the inequality

$$p_{1i} + p_{21} + p_{22} + \sum_{j \neq i} \frac{p_{1j} - q_{1j}}{2} < 3f_t(G) \quad \text{holds.}$$

Also $\frac{p_{1k} + q_{1k}}{2} \leq f_t(G)$ for all $k = 1, 2, \ldots, s_1$. Adding these we get

$$p_{1i} + p_{21} + p_{22} + \sum_{j \neq i} p_{1j} < (s_1 + 2)f_t(G)$$

Thus we cannot t-pebble the target if the original configuration has fewer than $(s_1 + 2)f_t(G)$ pebbles. ∎

We now prove $K_{s_1,s_2,\ldots,s_r} \times G$ satisfies the Generalized Graham's Conjecture when G is any graph with the $2t$-pebbling property.

Theorem 3.5. *Let $K_{s_1,s_2,\ldots,s_r}$ be a complete r-partite graph with $s_1, s_2, \ldots, s_r$ vertices in vertex classes $C_1, C_2, \ldots, C_r$ respectively and G be a graph with the 2t-pebbling property. Then*

$$f_t(K_{s_1,s_2,\ldots,s_r} \times G) \leq n f_t(G) \quad \textit{where}$$
$$n = f(K_{s_1,s_2,\ldots,s_r}) = s_1 + s_2 + \ldots + s_r \text{ [2]}.$$

Proof. We prove the theorem by induction on n. Suppose we start a configuration of $nf_t(G)$ pebbles on $K_{s_1,s_2,\ldots,s_r} \times G$. By Theorem 3.4, the result is true when $r = 2$ and $s_2 = 2$. Therefore we assume $r \geq 2$ and $s_2 > 2$ if $r = 2$.

Let $\{v_{i1}, v_{i2}, \ldots, v_{i_{s_i}}\}$ be the vertices of C_i for $i = 1, 2, \ldots, r$. Let p_{ik} denote the number of pebbles on $\{v_{ik}\} \times G$ with q_{ik} occupied vertices. Let $y \in G$ and (v_{ij}, y) be the target vertex.

Suppose there exists some $v_{lm} \in K_{s_1,s_2,\ldots,s_r}$ such that $v_{lm} \neq v_{ij}$ and $p_{lm} \leq f_t(G)$ then the subgraph $(K_{s_1,s_2,\ldots,s_r} - \{v_{lm}\}) \times G$ which is isomorphic to $K_{s_1,s_2,\ldots,s_l-1,\ldots s_r} \times G$ will have at least $(n-1)f_t(G)$ pebbles and hence the result follows by induction. Therefore we assume $p_{uv} > f_t(G)$ for every $u \neq i$ and $v \neq j$.

Now, we choose some element, say v_{lm} from any one of the vertex classes, say C_l, other than C_i. If $\frac{p_{lm} + q_{lm}}{2} > f_t(G)$ then $2t$ pebbles can be placed on (v_{lm}, y) and hence t pebbles can be moved to (v_{ij}, y).

Suppose $\frac{p_{lm} + q_{lm}}{2} \leq f_t(G)$. Then by Lemma 2.4 we move $\frac{p_{lm} - q_{lm}}{2}$ pebbles to the copy $\{v_{ij}\} \times G$ of G.

$$\text{If} \quad \left(\sum_{w=1}^{r}\left(\sum_{k=1}^{s_w} p_{wk}\right) - p_{lm}\right) + \frac{p_{lm} - q_{lm}}{2} \geq (n-1)f_t(G),$$

then the result follows by induction. Therefore, we cannot t-pebble the target if both the inequalities

$$\frac{p_{lm} + q_{lm}}{2} \leq f_t(G) \quad \text{and}$$
$$\left(\sum_{w=1}^{r}\left(\sum_{k=1}^{s_w} p_{wk}\right) - p_{lm}\right) + \frac{p_{lm} - q_{lm}}{2} < (n-1)f_t(G) \quad \text{hold.}$$

Adding these together gives

$$\sum_{w=1}^{r}\sum_{k=1}^{s_w} p_{wk} < nf_t(G).$$

Thus the original configuration has fewer than $nf_t(G)$ pebbles. This completes the proof. ∎

We proved Theorem 3.6 in [5] which will be used in proving Theorem 3.7.

Theorem 3.6. *Any complete r-partite graph satisfies the 2t-pebbling property.*

We conclude this paper with the following theorem.

Theorem 3.7. *Let $K_{s_1,s_2,\ldots,s_r}$ be a complete r-partite graph. Then*

$$f_t(K_{s_1,s_2,\ldots,s_r} \times K_{m_1,m_2,\ldots,m_n}) \leq f(K_{s_1,s_2,\ldots,s_r})f_t(K_{m_1,m_2,\ldots,m_n}).$$

Proof. Follows from Theorem 3.5 and Theorem 3.6. ∎

4 Conclusion.

We have proved generalized Graham's pebbling conjecture for the product of complete r-partite graphs. Producing algorithms on graph pebbling will be another interesting area of research.

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