Strong Distances in Strong Oriented Complete k-Partite Graphs *

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ABSTRACT

For two vertices u and v in a strong oriented graph D, the strong distance $sd(u,v)$ between u and v is the minimum size (the number of arcs) of a strong sub-digraph of D containing u and v. For a vertex v of D, the strong eccentricity $se(v)$ is the strong distance between v and a vertex farthest from v. The strong radius $srad(D)$ is the minimum strong eccentricity among the vertices of D. The strong diameter $sdiam(D)$ is the maximum strong eccentricity among the vertices of D. In this paper, we investigate the strong distances in strong oriented complete k-partite graphs. For any integers δ, r, d with $0 \leq \delta \leq \lceil \frac{k}{2} \rceil - 1$, $3 \leq r \leq \lfloor \frac{k}{2} \rfloor + 1$, $4 \leq d \leq k$, we have shown that there are strong oriented complete k-partite graphs K', K'', K''' such that $sdiam(K') - srad(K') = \delta$, $srad(K'') = r$, and $sdiam(K''') = d$.

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1 Introduction

The familiar distance $d(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest (u, v)-path in G. Equivalently, the distance is the minimum size of a connected subgraph of G containing u and v. Using this equivalent formulation of distance, this concept was extended by Chartrand *et al.* [2] to strongly connected digraphs, in particular to strong oriented graphs. In this paper, we consider only strong oriented graphs, and refer to [1] for graph theory notation and terminology not described here.

Let u, v be vertices of a strong oriented graph D. The *strong distance* $sd_D(u, v)$(or simply $sd(u, v)$) between u and v is defined as the minimum size of a strong sub-digraph of D containing u and v. A (u, v)-*geodesic* is a strong sub-digraph of D of size $sd(u, v)$ containing u and v. In some sense, this definition can be considered as an alternative definition of Steiner distance between two vertices in digraphs, as the *Steiner distance* of a vertex set S in a graph G is the minimum size of a connected subgraph which contains all the vertices of S. It was shown by Chartrand *et al.* [2] that the strong distance is a metric on the vertex set of D. If $u \neq v$, then $sd(u, v) \geq 3$. And $sd(u, v) = 3$ if and only if u and v belong to a directed 3-cycle in D. In the strong oriented graph of Figure 1, $sd(w, v) = 3$, $sd(u, w) = 5$, and $sd(u, x) = 6$.

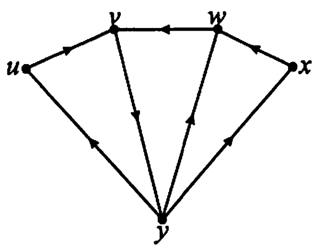

Figure 1: Strong distance in a strong oriented graph.

The *strong eccentricity* $se_D(v)$(or simply $se(v)$) of a vertex v in a strong oriented graph D is

$$se(v) = \max\{sd(v, x) \mid x \in V(D)\}.$$

The *strong radius* $srad(D)$ of D is

$$srad(D) = \min\{se(v) \mid v \in V(D)\};$$

while the *strong diameter* $sdiam(D)$ of D is

$$sdiam(D) = \max\{se(v) \mid v \in V(D)\}.$$

Chartrand *et al.* [2] showed that the strong radius and strong diameter of a strong oriented graph satisfy the following inequality.

Theorem 1.1 (Chartrand *et al.* [2]) *For every strong oriented graph D,*

$$srad(D) \le sdiam(D) \le 2srad(D).$$

Chartrand *et al.* [3] also showed that, for any integers δ, r, d with $0 \le \delta \le k$, $3 \le r \le k+1$, $3 \le d \le 2k+1$, there are strong tournaments T_1, T_2 and T_3 of order $2k+1$ such that $sdiam(T_1) - srad(T_1) = \delta$, $srad(T_2) = r$, and $sdiam(T_3) = d$.

Theorem 1.2 (Chartrand *et al.* [3]) *For every integer $k \ge 2$, there exists a strong tournament T of order $2k+1$ in which $sdiam(T) - srad(T) = \delta$ for every δ with $0 \le \delta \le k$.*

Theorem 1.3 (Chartrand *et al.* [3]) *For every integer $k \ge 2$, there exists a strong tournament T of order $2k+1$ with $srad(T) = r$ for every integer r with $3 \le r \le k+1$.*

Theorem 1.4 (Chartrand *et al.* [3]) *For every integer $k \ge 2$, there exists a strong tournament T of order $2k+1$ in which $sdiam(T) = d$ for every integer d with $3 \le d \le 2k+1$.*

Let $K(m_1, m_2, \ldots, m_k)$ be a complete k-partite graph with vertex partition of cardinalities m_1, m_2, $\ldots$, m_k. In this paper, we consider strong distances in strong oriented complete k-partite graphs $K(m_1, m_2, \ldots, m_k)$ with $1 \le m_1 \le m_2 \le \cdots \le m_k$, $m_k \ge 2$ and $k \ge 3$. It is shown that, for any integers δ, r, d with $0 \le \delta \le \lceil k/2 \rceil - 1$, $3 \le r \le \lfloor k/2 \rfloor + 1$, $4 \le d \le k$, there are complete k-partite digraphs K', K'', K''' such that $sdiam(K') - srad(K') = \delta$, $srad(K'') = r$, and $sdiam(K''') = d$.

2 Main Results

It is shown by Chartrand *et al.* [3] that there exists a strong tournament T of order n with $srad(T) = sdiam(T) = 3$ except $n = 4$. The only strong tournament of order 4 contains two vertices of strong eccentricity 3 and two vertices of strong eccentricity 4.

Theorem 2.1 (Chartrand *et al.* [3]) *For any integer $n \geq 3$ except $n = 4$, there exists a strong tournament T of order n with $sdiam(T) = 3$.*

In the following, we will show that when $k \geq 3$, $1 = m_1 \leq \cdots \leq m_k$ and $m_k \geq 2$, there exists a strong orientation K of $K(m_1, m_2, \ldots, m_k)$ such that $srad(K) = 3$.

Lemma 2.2 *For every integer $k \geq 3$, let $1 = m_1 \leq m_2 \leq \cdots \leq m_k$ and $m_k \geq 2$. Then there exists a strong orientation K of $K(m_1, m_2, \ldots, m_k)$ such that*
when $m_1 + m_2 + \cdots + m_{k-1} > m_k$, $srad(K) = 3$ and $sdiam(K) = 4$;
when $m_1 + m_2 + \cdots + m_{k-1} \leq m_k$, $srad(K) = 3$ and $sdiam(K) = 5$.

Proof. Let $(V_1, V_2, \ldots, V_k)$ be a k-partition of $K(m_1, m_2, \ldots, m_k)$, where $V_i = \{v_j^{(i)} \mid j = 1, \ldots, m_i\}$ for $1 \leq i \leq k$.

If $m_1 + m_2 + \cdots + m_{k-1} > m_k$, let K be the strong orientation of $K(m_1, m_2, \ldots, m_k)$ such that

$$\begin{aligned} A(K) &= \{(v_i^{(s)}, v_j^{(t)}) \mid 1 \leq s < t \leq k-1, 1 \leq i \leq m_s, 1 \leq j \leq m_t\} \\ &\bigcup \{(v_j^{(k)}, v_1^{(1)}) \mid 1 \leq j \leq m_k\} \\ &\bigcup \{(v_i^{(s)}, v_j^{(k)}) \mid 2 \leq s \leq k-1, 1 \leq i \leq m_s, \\ &\qquad\qquad j \equiv m_1 + \cdots + m_{s-1} + i - 1 (\text{mod } m_k)\} \\ &\bigcup \{(v_j^{(k)}, v_i^{(s)}) \mid 2 \leq s \leq k-1, 1 \leq i \leq m_s, \\ &\qquad\qquad j \not\equiv m_1 + \cdots + m_{s-1} + i - 1 (\text{mod } m_k)\}. \end{aligned}$$

Consider the vertex $v_1^{(1)} \in V_1$. For any vertex $v_i^{(s)} \in V_s$ with $2 \leq s \leq k-1$ and $1 \leq i \leq m_s$, the directed 3-cycle $v_1^{(1)} v_i^{(s)} v_j^{(k)} v_1^{(1)}$ is a $(v_1^{(1)}, v_i^{(s)})$-geodesic, where $j \equiv m_1 + \cdots + m_{s-1} + i - 1 (\text{mod } m_k)$. So $sd(v_1^{(1)}, v_i^{(s)}) = 3$. For any vertex $v_j^{(k)} \in V_k$ with $1 \leq j \leq m_k$, there exists a vertex $v_i^{(s)} \in V_s$

for some s, $2 \le s \le k-1$, such that the directed 3-cycle $v_1^{(1)}v_i^{(s)}v_j^{(k)}v_1^{(1)}$ is a $(v_1^{(1)}, v_j^{(k)})$-geodesic, where $j \equiv m_1 + \cdots + m_{s-1} + i - 1 (\text{mod } m_k)$. So $sd(v_1^{(1)}, v_j^{(k)}) = 3$. Hence, $se(v_1^{(1)}) = 3$.

For any two vertices $v_i^{(s)}, v_j^{(s)} \in V_s$ (if $m_s \ge 2$) with $2 \le s \le k-1$ and $1 \le i < j \le m_s$, $v_i^{(s)}$ and $v_j^{(s)}$ are contained in the directed 4-cycle $v_i^{(s)}v_l^{(k)}v_j^{(s)}v_r^{(k)}v_i^{(s)}$, where $l \equiv m_1 + \cdots + m_{s-1} + i - 1 (\text{mod } m_k)$ and $r \equiv m_1 + \cdots + m_{s-1} + j - 1 (\text{mod } m_k)$. Hence, $sd(v_i^{(s)}, v_j^{(s)}) \le 4$.

For any two vertices $v_i^{(k)}, v_j^{(k)} \in V_k$ with $1 \le i < j \le m_k$, there exist two vertices $v_l^{(s)} \in V_s$ and $v_r^{(t)} \in V_t$, $2 \le s, t \le k-1$ such that $v_i^{(k)}$ and $v_j^{(k)}$ are contained in the directed 4-cycle $v_i^{(k)}v_l^{(s)}v_j^{(k)}v_r^{(t)}v_i^{(k)}$, where $j \equiv m_1 + \cdots + m_{s-1} + l - 1 (\text{mod } m_k)$ and $i \equiv m_1 + \cdots + m_{t-1} + r - 1 (\text{mod } m_k)$. Hence, $sd(v_i^{(k)}, v_j^{(k)}) \le 4$.

For any two vertices $v_i^{(s)} \in V_s$, $v_j^{(t)} \in V_t$ with $2 \le s < t \le k-1$, $1 \le i \le m_s$ and $1 \le j \le m_t$, $v_i^{(s)}$ and $v_j^{(t)}$ are contained in the directed 4-cycle $v_1^{(1)}v_i^{(s)}v_j^{(t)}v_r^{(k)}v_1^{(1)}$, where $r \equiv m_1 + \cdots + m_{t-1} + j - 1 (\text{mod } m_k)$. Hence, $sd(v_i^{(s)}, v_j^{(t)}) \le 4$.

For any two vertices $v_i^{(s)} \in V_s$, $v_j^{(k)} \in V_k$ with $2 \le s \le k-1$, $1 \le i \le m_s$ and $1 \le j \le m_k$, if $j \equiv m_1 + \cdots + m_{s-1} + i - 1 (\text{mod } m_k)$, then $v_i^{(s)}$ and $v_j^{(k)}$ are contained in the directed 3-cycle $v_1^{(1)}v_i^{(s)}v_j^{(k)}v_1^{(1)}$; if $j \not\equiv m_1 + \cdots + m_{s-1} + i - 1 (\text{mod } m_k)$, then $v_i^{(s)}$ and $v_j^{(k)}$ are contained in the directed 4-cycle $v_i^{(s)}v_l^{(k)}v_r^{(t)}v_j^{(k)}v_i^{(s)}$, where $l \equiv m_1 + \cdots + m_{s-1} + i - 1 (\text{mod } m_k)$ and $j \equiv m_1 + \cdots + m_{t-1} + r - 1 (\text{mod } m_k)$. Hence, $sd(v_i^{(s)}, v_j^{(k)}) \le 4$.

Therefore, $srad(K) = 3$ and $sdiam(K) \le 4$. On the other hand, any two vertices in the same partite set cannot be contained in any directed 3-cycle, which implies that $sdiam(K) \ge 4$. Hence, $srad(K) = 3$ and $sdiam(K) = 4$.

If $m_1 + m_2 + \cdots + m_{k-1} \le m_k$, contract each partite set V_i to a new vertex v_i. Denote T_k by the strong tournament induced by $\{v_1, v_2, \ldots, v_k\}$. If $k = 4$, let T_4 be the unique strong tournament with $se(v_1) = 3$. If $k \ne 4$, by Theorem 2.1, let T_k be the strong tournament with $sdiam(T_k) = 3$. Let K be a strong orientation of $K(m_1, m_2, \ldots, m_k)$ such that $(v_s, v_t) \in A(T_k)$ if and only if $(v_i^{(s)}, v_j^{(t)}) \in A(K)$ for $1 \le s < t \le k$, $1 \le i \le m_s$ and

$1 \leq j \leq m_t$.

If $k = 4$, $v_1^{(1)}$ and $v_i^{(s)}$ are contained in a directed 3-cycle for $2 \leq s \leq 4$, $1 \leq i \leq m_s$; $v_i^{(s)}$ and $v_j^{(t)}$ are contained in a directed 4-cycle for $2 \leq s < t \leq 4$, $1 \leq i \leq m_s$ and $1 \leq j \leq m_t$. If $k \neq 4$, any two vertices $v_i^{(s)} \in V_s$ and $v_j^{(t)} \in V_t$ are contained in a directed 3-cycle for $1 \leq s < t \leq k$, $1 \leq i \leq m_s$ and $1 \leq j \leq m_t$. For any two vertices $v_i^{(s)}, v_j^{(s)} \in V_s$ with $2 \leq s \leq k$, $1 \leq i < j \leq m_s$, $v_i^{(s)}$ and $v_j^{(s)}$ cannot be contained in any directed 3-cycle or 4-cycle, since $N^+(v_i^{(s)}) = N^+(v_j^{(s)})$. So $sd(v_i^{(s)}, v_j^{(s)}) \geq 5$. On the other hand, v_1 and v_s are contained in a directed 3-cycle, we may assume $v_1v_sv_tv_1$. Then the sub-digraph induced by $\{(v_1^{(1)}, v_i^{(s)}), (v_i^{(s)}, v_1^{(t)}), (v_1^{(t)}, v_1^{(1)}), (v_1^{(1)}, v_j^{(s)}), (v_j^{(s)}, v_1^{(t)})\}$ is a strong sub-digraph containing $v_i^{(s)}$ and $v_j^{(s)}$. So $sd(v_i^{(s)}, v_j^{(s)}) = 5$. Hence, $srad(K) = 3$ and $sdiam(K) = 5$.□

Let $K_{m,n} = (U, V)$ be a complete bipartite graph, where $U = \{u_1, u_2, \ldots, u_m\}$, $V = \{v_1, v_2, \ldots, v_n\}$ and $2 \leq m \leq n$. Let $K^*_{m,n}$ be a strong orientation of $K_{m,n}$ such that

$$\begin{aligned} A(K^*_{m,n}) &= \{(v_i, u_i) \mid 1 \leq i < m\} \\ &\bigcup \{(v_j, u_m) \mid m \leq j \leq n\} \\ &\bigcup \{(u_i, v_j) \mid j \neq i, 1 \leq i < m, 1 \leq j \leq n\} \\ &\bigcup \{(u_m, v_j) \mid 1 \leq j < m\}. \end{aligned}$$

Lai *et al.*[4] showed that if $m = n$, then $srad(K^*_{m,n}) = sdiam(K^*_{m,n}) = 4$; if $m < n$, then $srad(K^*_{m,n}) = se(u_i) = se(v_j) = 4$, where $1 \leq i \leq m$ and $1 \leq j < m$, and $sdiam(K^*_{m,n}) = se(v_j) = 6$, where $m \leq j \leq n$.

Figure 2 shows the strong orientation $K^*_{3,4}$ of $K_{3,4}$ such that $srad(K^*_{3,4}) = 4$ and $sdiam(K^*_{3,4}) = 6$.

For any strong orientation K of $K(m_1, m_2, \ldots, m_k)$, where $m_i \geq 2$ for $1 \leq i \leq k$, any two vertices of the same partite set cannot be contained in any directed 3-cycle. So $srad(K) \geq 4$. Now we will show that the lower bound is sharp.

Lemma 2.3 *For every integer $k \geq 3$, let $2 \leq m_1 \leq m_2 \leq \cdots \leq m_k$. Then there exists a strong orientation K of $K(m_1, m_2, \ldots, m_k)$ such that*

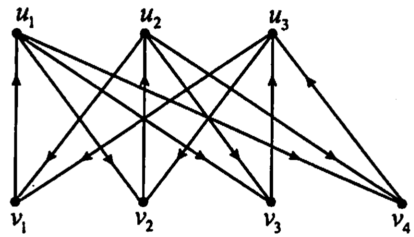

Figure 2: The strong orientation $K^*_{3,4}$ of $K_{3,4}$.

when $m_1 + m_2 + \cdots + m_{k-1} \geq m_k$, $\quad srad(K) = sdiam(K) = 4$;
when $m_1 + m_2 + \cdots + m_{k-1} < m_k$, $\quad srad(K) = 4$ and $sdiam(K) = 5$.

Proof. Let $(V_1, V_2, \ldots, V_k)$ be a k-partition of $K(m_1, m_2, \ldots, m_k)$, where $V_i = \{v_j^{(i)} \mid j = 1, 2, \ldots, m_i\}$ for $1 \leq i \leq k$.

For a digraph D with disjoint vertex subsets S and T, we denote by $D[S, T]$ the bipartite sub-digraph of D induced by all the arcs with one end in S and the other end in T.

If $m_1 + m_2 + \cdots + m_{k-1} \geq m_k$, let K' be a strong orientation of $K(m_1, m_2, \ldots, m_k)$ such that $K'[V_s, V_t]$ is isomorphic to $K^*_{m_s,m_t}$ for $1 \leq s < t \leq k-1$; let $U = V_k$ and $V = \cup_{i=1}^{k-1} V_i = \{v_1 = v_1^{(k-1)}, v_2 = v_2^{(k-1)}, \ldots, v_{m_{k-1}} = v_{m_{k-1}}^{(k-1)}, v_{m_{k-1}+1} = v_1^{(k-2)}, \ldots, v_{m_1+m_2+\cdots+m_{k-1}} = v_{m_1}^{(1)}\}$, and $K'[U, V]$ is isomorphic to $K^*_{m_k, m_1+\cdots+m_{k-1}}$.

Thus, we have $sd_{K'}(v_i^{(s)}, v_j^{(s)}) \leq sd_{K'[V_s, V_{k-1}]}(v_i^{(s)}, v_j^{(s)}) \leq 4$ for $1 \leq s < k-1, 1 \leq i < j \leq m_s$; $sd_{K'}(v_i^{(k-1)}, v_j^{(k-1)}) \leq sd_{K'[U,V]}(v_i^{(k-1)}, v_j^{(k-1)}) \leq 4$ for $1 \leq i < j \leq m_{k-1}$; $sd_{K'}(v_i^{(s)}, v_j^{(t)}) \leq sd_{K'[V_s, V_t]}(v_i^{(s)}, v_j^{(t)}) \leq 4$ for $1 \leq s < t \leq k-1$, $1 \leq i \leq m_s$ and $1 \leq j \leq m_t$; $sd_{K'}(v_i^{(s)}, v_j^{(k)}) \leq sd_{K'[U,V]}(v_i^{(s)}, v_j^{(k)}) \leq 4$ for $1 \leq s \leq k$, $1 \leq i \leq m_s$ and $1 \leq j \leq m_k$. Hence, $srad(K') \leq 4$ and $sdiam(K') \leq 4$. But we also have $sdiam(K') \geq srad(K') \geq 4$. So, $srad(K') = sdiam(K') = 4$.

If $m_1 + m_2 + \cdots + m_{k-1} < m_k$, let K'' be a strong orientation of $K(m_1, m_2, \ldots, m_k)$ such that $K''[V_s, V_t]$ is isomorphic to $K^*_{m_s,m_t}$ for $1 \leq s < t \leq k-1$; let $U = \cup_{i=1}^{k-1} V_i$, $V = V_k$, and $K''[U, V]$ is isomorphic to $K^*_{m_1+\cdots+m_{k-1}, m_k}$.

Similar to the case of K', we have $sd_{K''}(v_i^{(s)}, v_j^{(t)}) \le sd_{K''[U,V]}(v_i^{(s)}, v_j^{(t)}) \le 4$ for $1 \le s \le k-1$, $1 \le t \le k$, $1 \le i \le m_s$ and $1 \le j \le m_t$; $sd_{K''}(v_i^{(k)}, v_j^{(k)}) \le sd_{K''[U,V]}(v_i^{(k)}, v_j^{(k)}) \le 4$ for $1 \le i < m_1+m_2+\cdots+m_{k-1}$ and $1 \le j \le m_k$. For any two vertices $v_i^{(k)}, v_j^{(k)} \in V_k$ with $m_1+m_2+\cdots+m_{k-1} \le i < j \le m_k$, $v_i^{(k)}$ and $v_j^{(k)}$ cannot be contained in any directed 3-cycle or 4-cycle, since $N^+(v_i^{(k)}) = N^+(v_j^{(k)})$. So $sd_{K''}(v_i^{(k)}, v_j^{(k)}) \ge 5$. On the other hand, the sub-digraph induced by $\{(v_i^{(k)}, v_{m_{k-1}}^{(k-1)}), (v_j^{(k)}, v_{m_{k-1}}^{(k-1)}), (v_{m_{k-1}}^{(k-1)}, v_{m_1}^{(1)}), (v_{m_1}^{(1)}, v_i^{(k)}), (v_{m_1}^{(1)}, v_j^{(k)})\}$ is a strong sub-digraph containing $v_i^{(k)}$ and $v_j^{(k)}$. So $sd_{K''}(v_i^{(k)}, v_j^{(k)}) = 5$. Hence, $srad(K'') \le 4$ and $sdiam(K'') = 5$. But we also have $sdiam(K'') \ge srad(K'') \ge 4$. So, $srad(K'') = 4$ and $sdiam(K'') = 5$. □

Theorem 2.4 *For every integer $k \ge 3$, there exists a strong oriented complete k-partite graph K in which $sdiam(K) - srad(K) = \delta$ for every δ with $0 \le \delta \le \lceil k/2 \rceil - 1$.*

Proof. By Lemma 2.3, it is clear for $k = 3, 4$.

For $k \ge 5$, let $(V_1, V_2, \ldots, V_k)$ be a k-partition of $K(m_1, m_2, \ldots, m_k)$, where $V_i = \{v_j^{(i)} \mid j = 1, 2, \ldots, m_i\}$ and $m_i \ge 2$ for $1 \le i \le k$. Let $S_1 = V_1 \cup V_2 \cup \cdots \cup V_m$, $S_2 = V_{m+1} \cup \cdots \cup V_k$, where $2 \le m \le k-3$. Consider a strong orientation K of $K(m_1, m_2, \ldots, m_k)$ with the following arc set:

$$\begin{aligned}
A(K) \;=\; & \{(v_i^{(s)}, v_j^{(t)}) \mid 1 \le s < t \le m, 1 \le i \le m_s, 1 \le j \le m_t\} \\
\bigcup\; & \{(v_i^{(t)}, v_j^{(t+1)}) \mid m+1 \le t < k, 1 \le i \le m_t, 1 \le j \le m_{t+1}\} \\
\bigcup\; & \{(v_j^{(t)}, v_i^{(s)}) \mid m+1 \le s < t \le k, s \ne t-1, \\
& \qquad 1 \le i \le m_s, 1 \le j \le m_t\} \\
\bigcup\; & \{(v_j^{(k)}, v_i^{(1)}) \mid 1 \le i \le m_1, 1 \le j \le m_k\} \\
\bigcup\; & \{(v_i, v_j) \mid v_i \in S_1, v_j \in S_2\} \\
-\; & \{(v_i^{(1)}, v_j^{(k)}) \mid 1 \le i \le m_1, 1 \le j \le m_k\} \text{ (see Figure 3).}
\end{aligned}$$

For any two vertices u, v belonging to the same partite set, by the orientation of K, we know that u and v cannot be contained in any directed 3-cycle or 4-cycle. Thus, $sd(u, v) \ge 5$.

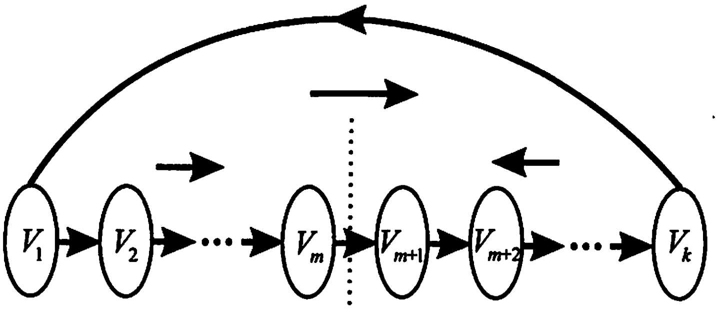

Figure 3: The strong orientation K of $K(m_1, m_2, \ldots, m_k)$.

For any two vertices $v_i^{(1)}, v_j^{(1)} \in V_1$, the sub-digraph induced by the arc set $\{(v_i^{(1)}, v_1^{(2)}), (v_1^{(2)}, v_1^{(k)}), (v_1^{(k)}, v_j^{(1)}), (v_j^{(1)}, v_1^{(2)}), (v_1^{(k)}, v_i^{(1)})\}$ is a strong sub-digraph containing $v_i^{(1)}$ and $v_j^{(1)}$. So $sd(v_i^{(1)}, v_j^{(1)}) = 5$.

For any two vertices $v_i^{(t)}, v_j^{(t)} \in V_t$ with $2 \leq t \leq m$, the sub-digraph induced by the arc set $\{(v_i^{(t)}, v_1^{(k)}), (v_1^{(k)}, v_1^{(1)}), (v_1^{(1)}, v_j^{(t)}), (v_j^{(t)}, v_1^{(k)}), (v_1^{(1)}, v_i^{(t)})\}$ is a strong sub-digraph containing $v_i^{(t)}$ and $v_j^{(t)}$. So $sd(v_i^{(t)}, v_j^{(t)}) = 5$.

For any two vertices $v_i^{(m+1)}, v_j^{(m+1)} \in V_{m+1}$, the sub-digraph induced by the arc set $\{(v_i^{(m+1)}, v_1^{(m+2)}), (v_1^{(m+2)}, v_1^{(m+3)}), (v_1^{(m+3)}, v_j^{(m+1)}), (v_j^{(m+1)}, v_1^{(m+2)}), (v_1^{(m+3)}, v_i^{(m+1)})\}$ is a strong sub-digraph containing $v_i^{(m+1)}$ and $v_j^{(m+1)}$. So $sd(v_i^{(m+1)}, v_j^{(m+1)}) = 5$.

For any two vertices $v_i^{(t)}, v_j^{(t)} \in V_t$ with $m+2 \leq t < k$, the sub-digraph induced by the arc set $\{(v_i^{(t)}, v_1^{(t+1)}), (v_j^{(t)}, v_1^{(t+1)}), (v_1^{(t-1)}, v_i^{(t)}), (v_1^{(t-1)}, v_j^{(t)}), (v_1^{(t+1)}, v_1^{(t-1)})\}$ is a strong sub-digraph containing $v_i^{(t)}$ and $v_j^{(t)}$. So $sd(v_i^{(t)}, v_j^{(t)}) = 5$.

For any two vertices $v_i^{(k)}, v_j^{(k)} \in V_k$, the sub-digraph induced by the arc set $\{(v_i^{(k)}, v_1^{(k-2)}), (v_1^{(k-2)}, v_1^{(k-1)}), (v_1^{(k-1)}, v_j^{(k)}), (v_j^{(k)}, v_1^{(k-2)}), (v_1^{(k-1)}, v_i^{(k)})\}$ is a strong sub-digraph containing $v_i^{(k)}$ and $v_j^{(k)}$. So $sd(v_i^{(k)}, v_j^{(k)}) = 5$.

Thus for any u, v belonging to the same partite set, $sd(u, v) = 5$.

For any two vertices $u \in V_1$ and $v \in \cup_{i=2}^{m} V_i$, u and v are contained in a directed 3-cycle $uvv_1^{(k)}u$. So $sd(u, v) \leq 3$.

For any two vertices $v_i^{(s)} \in V_s$ and $v_j^{(t)} \in V_t$ with $2 \leq s < t \leq m$, $v_i^{(s)}$ and $v_j^{(t)}$ are contained in a directed 4-cycle $v_i^{(s)}v_j^{(t)}v_1^{(k)}v_1^{(1)}v_i^{(s)}$. So $sd(v_i^{(s)}, v_j^{(t)}) \leq 4$.

For any two vertices $v_i^{(s)} \in V_s$ and $v_j^{(t)} \in V_t$ with $m+1 \leq s < t \leq k$, the directed $(t-s+1)$-cycle $v_i^{(s)}v_1^{(s+1)} \cdots v_j^{(t)}v_i^{(s)}$ is a $(v_i^{(s)}, v_j^{(t)})$-geodesic. So $sd(v_i^{(s)}, v_j^{(t)}) = t - s + 1$.

For any two vertices $v_i^{(1)} \in V_1$ and $v_j^{(k)} \in V_k$, $v_i^{(1)}$ and $v_j^{(k)}$ are contained in a directed 3-cycle $v_i^{(1)}v_1^{(2)}v_j^{(k)}v_i^{(1)}$. So $sd(v_i^{(1)}, v_j^{(k)}) \leq 3$.

For any two vertices $v_i^{(1)} \in V_1$ and $v_j^{(t)} \in V_t$ with $m+1 \leq t \leq k-1$, the directed $(k-t+2)$-cycle $v_i^{(1)}v_j^{(t)}v_1^{(t+1)} \cdots v_1^{(k)}v_i^{(1)}$ is a $(v_i^{(1)}, v_j^{(t)})$-geodesic. So $sd(v_i^{(1)}, v_j^{(t)}) = k - t + 2$.

For any two vertices $v_i^{(s)} \in V_s$ and $v_j^{(t)} \in V_t$ with $2 \leq s \leq m$, $m + 1 \leq t \leq k$, the directed $(k-t+3)$-cycle $v_i^{(s)}v_j^{(t)}v_1^{(t+1)} \cdots v_1^{(k)}v_1^{(1)}v_i^{(s)}$ is a $(v_i^{(s)}, v_j^{(t)})$-geodesic. So $sd(v_i^{(s)}, v_j^{(t)}) = k - t + 3$. Therefore,

$$\begin{aligned}
se(v_i^{(1)}) &= \max\{5, k-m+1\}, && \text{for} v_i^{(1)} \in V_1; \\
se(v_i^{(s)}) &= \max\{5, k-m+2\} = k-m+2, && \text{for } v_i^{(s)} \in V_s \subseteq S_1 - V_1; \\
se(v_j^{(t)}) &= \max\{5, k-t+2, k-t+3, t-m, k-t+1\} \\
&= \max\{5, k-t+3, t-m\}, && \text{for } v_j^{(t)} \in V_t \subseteq S_2.
\end{aligned}$$

With respect to t, $k-t+3$ decreases and $t-m$ increases, so the vertices in S_2 with the smallest strong eccentricity are $v_j^{(t)} \in V_t$, $1 \leq j \leq m_t$ with $k-t+3 = t-m$. However, this implies that $t = (k+m+3)/2$, which may not be an integer. So $v_j^{(\lfloor (k+m+3)/2 \rfloor)} \in V_{\lfloor (k+m+3)/2 \rfloor}$, $1 \leq j \leq m_{\lfloor (k+m+3)/2 \rfloor}$ or $v_j^{(\lceil (k+m+3)/2 \rceil)} \in V_{\lceil (k+m+3)/2 \rceil}$, $1 \leq j \leq m_{\lceil (k+m+3)/2 \rceil}$ are the vertices in S_2 with the smallest strong eccentricity. Moreover, we have

$$se(v_j^{(\lfloor (k+m+3)/2 \rfloor)}) = se(v_j^{(\lceil (k+m+3)/2 \rceil)}) = \max\{5, \lceil \frac{k-m+3}{2} \rceil\}.$$

Certainly, since $m+1 \leq t \leq k$, for $2 \leq s \leq m$,

$$se(v_1^{(s)}) = k - m + 2 \geq \max\{k-t+3, t-m\}.$$

Therefore, $sdiam(K) = k - m + 2$, $srad(K) = \max\{5, \lceil (k-m+3)/2 \rceil\}$.

If $m = k - 3$, $sdiam(K) - srad(K) = 5 - 5 = 0$.

If $m = k - 4$, $sdiam(K) - srad(K) = 6 - 5 = 1$.

If $m = k-5$, $sdiam(K) - srad(K) = 7 - 5 = 2$.

If $2 \leq m \leq k-6$, $sdiam(K) - srad(K) = k - m + 2 - \lceil (k-m+3)/2 \rceil = \lfloor (k-m+1)/2 \rfloor$, which implies $3 \leq sdiam(K) - srad(K) \leq \lceil k/2 \rceil - 1$.□

The strong oriented complete k-partite graphs constructed in the proof of the preceeding theorem and Lemmas 2.2 and 2.3 provide us with the following results.

Theorem 2.5 *For every integer $k \geq 4$, there exists a strong oriented complete k-partite graph K with $srad(K) = r$ for every integer r with $3 \leq r \leq \lfloor k/2 \rfloor + 1$ and for $k = 3$, $r = 3$.* □

Theorem 2.6 *For every integer $k \geq 4$, there exists a strong oriented complete k-partite graph K with $sdiam(K) = d$ for every integer d with $4 \leq d \leq k$ and for $k = 3$, $d = 4$.* □

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