Some New Sufficient Conditions for Graphs to be (a, b, k)-Critical Graphs *†

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Abstract

Let G be a graph, and let a, b and k be nonnegative integers with $1 \leq a \leq b$. An $[a, b]$-factor of graph G is defined as a spanning subgraph F of G such that $a \leq d_F(x) \leq b$ for each $x \in V(G)$. Then a graph G is called an (a, b, k)-critical graph if after any k vertices of G are deleted the remaining subgraph has an $[a, b]$-factor. In this paper, three sufficient conditions for graphs to be (a, b, k)-critical graphs are given. Furthermore, it is shown that the results in this paper are best possible in some sense.

Keywords: graph, stability number, degree condition, neighborhood union, $[a, b]$-factor, (a, b, k)-critical graph

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1 Introduction

All graphs considered in this paper will be finite and undirected simple graphs. Let G be a graph. We denote by $V(G)$ and $E(G)$ the set of vertices and the set of edges, respectively. For any $x \in V(G)$, we denote by $d_G(x)$ the degree of x in G and by $N_G(x)$ the set of vertices adjacent to x in G. For $X \subseteq V(G)$, we define $N_G(X) = \cup_{x \in X} N_G(x)$, and $G[X]$ is the subgraph of G induced by X. We denote by $G - X$ the subgraph obtained from G by deleting vertices in X together with the edges incident to vertices in X. Let S and T be disjoint subsets of $V(G)$. We denote by $e_G(S,T)$ the number of edges joining S and T. Denote by $\alpha(G)$ the stability number of a graph G, by $\delta(G)$ the minimum degree of vertices in G, by $\Delta(G)$ the maximum degree of vertices in G. We define the distance $d(x,y)$ between two vertices x and y as the minimum of the lengths of the (x,y)-paths of G.

Let $1 \le a \le b$ be integers. An $[a,b]$-factor of graph G is defined as a spanning subgraph F of G such that $a \le d_F(x) \le b$ for for each $x \in V(G)$ (Where of course d_F denotes the degree in F). And if $a = b = r$, then an $[a,b]$-factor of G is called an r-factor of G. A graph G is called an (a,b,k)-critical graph if after deleting any k vertices of G the remaining subgraph of G has an $[a,b]$-factor. If G is an (a,b,k)-critical graph, then we also say that G is (a,b,k)-critical. If $a = b = r$, then an (a,b,k)-critical graph is simply called an (r,k)-critical graph. In particular, a $(1,k)$-critical graph is simply called a k-critical graph. The other terminologies and notations not given in this paper can be found in [1].

Zhou [2,3] investigated (g,f)-factors of graphs. Favaron [4] studied the properties of k-critical graphs. Liu and Yu [5] gave the characterization of (r,k)-critical graphs. Liu and Wang [6] gave a necessary and sufficient condition for a graph to be an (a,b,k)-critical graph. Recently, Zhou [7] obtained a sufficient condition for a graph to be an (a,b,k)-critical graph. In this paper, we give three new sufficient conditions for graphs to be (a,b,k)-critical graphs.

Niessen [8] proved the following sufficient condition for a k-factor depending on $\delta(G)$ and $\alpha(G)$.

Theorem 1 [8] *Let $k \ge 2$ be an integer and let G be a graph with n vertices. If k is odd, then suppose that n is even and G is connected. Let G satisfy*

$$n > 4k + 1 - 4\sqrt{k+2},$$

$$\delta(G) \ge \frac{k-1}{2k-1}(n+2) \quad \textit{and}$$

$$\delta(G) > \frac{1}{2k-2}((k-2)n + 2\alpha(G) - 2).$$

Then G has a k-factor.

Matsuda [9] proved the following results for the existence of $[a, b]$-factors.

Theorem 2 [9] *Let* $1 \le a < b$ *be integers and G a graph of order* $n \ge \frac{2(a-1)(a+1)(a+b)(a+b-1)}{a(b-1)} - \frac{(a+b)(ab+b-1)}{ab(b-1)}$. *Suppose that* $\delta(G) \ge a$ *and*

$$|N_G(x) \cup N_G(y)| \ge \frac{an}{a+b}$$

for any two nonadjacent vertices x and y of V(G) such that $N_G(x) \cap N_G(y) \ne \varnothing$. *Then G has an* $[a, b]$*-factor.*

Theorem 3 [9] *Let* $1 \le a < b$ *be integers and G a graph of order* $n \ge \frac{(a-1)(a+1)(a+b)(a+b-1)}{a(b-1)} - \frac{(a+b)(ab+b-1)}{ab(b-1)}$. *Suppose that* $\delta(G) \ge a$ *and*

$$\max\{d_G(x), d_G(y)\} \ge \frac{an}{a+b}$$

for any vertices x and y of G with $d(x, y) = 2$. *Then G has an* $[a, b]$*-factor.*

In this paper, we prove the following results, which are an extension of Theorems 1 and 2 and 3, respectively. We extend Theorems 1 and 2 and 3 to (a, b, k)-critical graphs, respectively.

Theorem 4 *Let a, b and k be nonnegative integers with* $2 \le a < b$, *and let G be a graph of order n. Let G satisfy*

$$n > \frac{(a+b-1)(a+b-3)}{b} + k,$$

$$\delta(G) \ge \frac{(a-1)n + a + b + bk - 1}{a+b-1} \quad \text{and}$$

$$\delta(G) > \frac{(a-2)n + 2\alpha(G) + bk - 1}{a+b-2}.$$

Then G is an (a, b, k)*-critical graph.*

Theorem 5 *Let a, b and k be nonnegative integers such that* $1 \le a < b$ *and G be a graph with order* $n \ge \frac{2(a-1)(a+1)(a+b)(a+b-1)}{a(b-1)} - \frac{(a+b)(ab+b-1)}{ab(b-1)} + k$. *If* $\delta(G) \ge a + k$, *and*

$$|N_G(x) \cup N_G(y)| \ge \frac{an + bk}{a+b}$$

for any two nonadjacent vertices x and y of V(G) such that $N_G(x) \cap N_G(y) \ne \varnothing$. *Then G is an* (a, b, k)*-critical graph.*

Theorem 6 *Let a, b and k be nonnegative integers such that $1 \leq a < b$ and G be a graph with order $n \geq \frac{(a-1)(a+1)(a+b)(a+b-1)}{a(b-1)} - \frac{(a+b)(ab+b-1)}{ab(b-1)} + k$. If $\delta(G) \geq a + k$, and*

$$\max\{d_G(x), d_G(y)\} \geq \frac{an + bk}{a + b}$$

for any vertices x and y of $V(G)$ with $d(x, y) = 2$. Then G is an (a, b, k)-critical graph.

2 The Proofs of Main Theorems

In order to prove our main theorems, we depend heavily on the following lemma.

Lemma 2.1 [6] *Let a, b and k be nonnegative integers with $a < b$, and let G be a graph of order $n \geq a + k + 1$. Then G is an (a, b, k)-critical graph if and only if for any $S \subseteq V(G)$ and $|S| \geq k$*

$$\delta_G(S, T) = b|S| + d_{G-S}(T) - a|T| \geq bk,$$

where $T = \{x : x \in V(G) \setminus S,\ d_{G-S}(x) \leq a - 1\}$.

Proof of Theorem 4 Suppose a graph G satisfies the condition of the theorem, but it is not an (a, b, k)-critical graph. Then, by Lemma2.1, there exists a subset S of $V(G)$ with $|S| \geq k$ such that

$$\delta_G(S, T) = b|S| + d_{G-S}(T) - a|T| \leq bk - 1, \tag{1}$$

where $T = \{x : x \in V(G) \setminus S,\ d_{G-S}(x) \leq a - 1\}$. We choose subsets S and T such that $|T|$ is minimum and S and T satisfy (1).

If $T = \varnothing$, then by (1), $bk - 1 \geq \delta_G(S, T) = b|S| \geq bk$, a contradiction. Hence, $T \neq \varnothing$. We choose a vertex $x_1 \in T$ with

$$h = \min\{d_{G-S}(x) : x \in T\} = d_{G-S}(x_1).$$

Obviously,

$$\delta(G) \leq d_G(x_1) \leq d_{G-S}(x_1) + |S| = h + |S|. \tag{2}$$

According to the definition of T, we have

$$0 \leq h \leq a - 1.$$

We shall consider two cases according to the value of h and derive contradictions.

Case 1 $h = 0$.

Let $X = \{x \in T : d_{G-S}(x) = 0\}$, $Y = \{x \in T : d_{G-S}(x) = 1\}$, $Y_1 = \{x \in Y : N_{G-S}(x) \subseteq T\}$ and $Y_2 = Y - Y_1$. Then the graph induced by Y_1 in $G - S$ has maximum degree at most 1. Let Z be a maximum independent set of this graph. Clearly, $|Z| \geq \frac{1}{2}|Y_1|$. In view of our definitions, $X \cup Z \cup Y_2$ is an independent set of G. Thus, we obtain

$$\alpha(G) \geq |X| + |Z| + |Y_2| \geq |X| + \frac{1}{2}|Y_1| + \frac{1}{2}|Y_2| = |X| + \frac{1}{2}|Y|. \quad (3)$$

By (1) and (2) and (3), we get

$$\begin{aligned}
\alpha(G) &\geq |X| + \frac{1}{2}|Y| \\
&\geq b|S| + d_{G-S}(T) - a|T| - bk + 1 + |X| + \frac{1}{2}|Y| \\
&= b|S| + d_{G-S}(T \setminus (X \cup Y)) - a|T| - bk + 1 + |X| + \frac{3}{2}|Y| \\
&\geq b|S| + 2|T - (X \cup Y)| - a|T| - bk + 1 + |X| + \frac{3}{2}|Y| \\
&= b|S| + 2|T| - a|T| - bk + 1 - (|X| + \frac{1}{2}|Y|) \\
&\geq b\delta(G) - (a-2)|T| - bk + 1 - \alpha(G),
\end{aligned}$$

which implies

$$(a-2)|T| \geq b\delta(G) - 2\alpha(G) - bk + 1. \quad (4)$$

If $a = 2$, then (4) is equivalent to $\delta(G) \leq \frac{2\alpha(G)+bk-1}{b}$, which contradicts $\delta(G) > \frac{(a-2)n+2\alpha(G)+bk-1}{a+b-2} = \frac{2\alpha(G)+bk-1}{b}$.

If $a \geq 3$, then by (2) and (4), we have

$$\begin{aligned}
0 &\leq n - |S| - |T| \\
&\leq n - \delta(G) - \frac{b\delta(G) - 2\alpha(G) - bk + 1}{a-2} \\
&= \frac{(a-2)n - (a+b-2)\delta(G) + 2\alpha(G) + bk - 1}{a-2}
\end{aligned}$$

Thus, we obtain

$$\delta(G) \leq \frac{(a-2)n + 2\alpha(G) + bk - 1}{a+b-2},$$

that contradicts $\delta(G) > \frac{(a-2)n+2\alpha(G)+bk-1}{a+b-2}$.

Case 2 $1 \le h \le a-1$.

By $|S| + |T| \le n$ and (1), we have

$$\begin{aligned} bk-1 &\ge \delta_G(S,T) = b|S| + d_{G-S}(T) - a|T| \\ &\ge b|S| - (a-h)|T| \\ &\ge b|S| - (a-h)(n-|S|) \\ &= (a+b-h)|S| - (a-h)n. \end{aligned}$$

This inequality implies

$$|S| \le \frac{(a-h)n + bk - 1}{a+b-h}.$$

Combining this with (2) gives

$$\delta(G) \le |S| + h \le \frac{(a-h)n + bk - 1}{a+b-h} + h. \tag{5}$$

If $h = 1$, then by (5), $\delta(G) \le \frac{(a-1)n+bk-1}{a+b-1} + 1 = \frac{(a-1)n+a+b+bk-2}{a+b-1}$, a contradiction. Hence, we may assume that $2 \le h \le a-1$, and let

$$f(h) = \frac{(a-h)n + bk - 1}{a+b-h} + h.$$

According to $n > \frac{(a+b-1)(a+b-3)}{b} + k$, we obtain $f'(h) < 0$. Then, $f(h)$ attains its maximum value at $h = 2$. By (5) and $\delta(G) \ge \frac{(a-1)n+a+b+bk-1}{a+b-1}$, we have

$$\frac{(a-1)n + a + b + bk - 1}{a+b-1} \le \delta(G) \le f(h) \le f(2) = \frac{(a-2)n + bk - 1}{a+b-2} + 2,$$

implying $n \le \frac{(a+b-1)(a+b-3)}{b} + k$, which is a contradiction.

From the argument above, we deduce the contradictions, so the hypothesis can not hold. Hence, G is an (a,b,k)-critical graph.
Completing the proof of Theorem 4.

Proof of Theorem 5 Suppose that G is not an (a,b,k)-critical graph. Then, by Lemma 2.1, there exist disjoint subsets S and T of $V(G)$ satisfying

$$\delta_G(S,T) = b|S| + d_{G-S}(T) - a|T| \le bk - 1. \tag{6}$$

We choose such subsets S and T so that $|T|$ is as small as possible.

Claim 1. $|T| \geq b+1$.

Proof. If $|T| \leq b$, then by (6) and $|S| + d_{G-S}(x) \geq d_G(x) \geq \delta(G) \geq a+k$, we get that

$$-1 \geq b|S| + d_{G-S}(T) - a|T| - bk \geq \sum_{x \in T}(|S| + d_{G-S}(x) - a - k) \geq 0,$$

a contradiction.

Claim 2. $|S| \geq k+1$.

Proof. If $|S| = k$, then by (6), we obtain

$$\begin{aligned} bk - 1 &\geq \delta_G(S,T) = b|S| + d_{G-S}(T) - a|T| \\ &= bk + \sum_{x \in T}(d_{G-S}(x) - a) \\ &\geq bk + \sum_{x \in T}(d_G(x) - |S| - a) \\ &\geq bk + \sum_{x \in T}(\delta(G) - k - a) \geq bk, \end{aligned}$$

that is a contradiction.

Claim 3. $|S| < \frac{an+bk}{a+b}$.

Proof. According to (6) and $|S| + |T| \leq n$, we have

$$\begin{aligned} bk - 1 &\geq \delta_G(S,T) = b|S| + d_{G-S}(T) - a|T| \\ &\geq b|S| - a|T| \geq b|S| - a(n - |S|) \\ &= (a+b)|S| - an, \end{aligned}$$

implying $|S| < \frac{an+bk}{a+b}$.

Claim 4. $a|T| \geq b|S| - bk + 1$.

Proof. In view of (6), we get

$$bk - 1 \geq \delta_G(S,T) = b|S| + d_{G-S}(T) - a|T| \geq b|S| - a|T|,$$

which implies $a|T| \geq b|S| - bk + 1$.

Claim 5. $|S| < \frac{an+bk}{a+b} - 2(a-1)$.

Proof. If $|S| \geq \frac{an+bk}{a+b} - 2(a-1)$, that is to say, $an - (a+b)|S| \leq 2(a-1)(a+b) - bk$. By (6) and $|S| + |T| \leq n$, we have

$$\begin{aligned} d_{G-S}(T) &\leq a|T| - b|S| + bk - 1 \\ &\leq a(n - |S|) - b|S| + bk - 1 \\ &= an - (a+b)|S| + bk - 1 \\ &\leq 2(a-1)(a+b) - 1. \end{aligned}$$

According to $n \geq \frac{2(a-1)(a+1)(a+b)(a+b-1)}{a(b-1)} - \frac{(a+b)(ab+b-1)}{ab(b-1)} + k$ and Claim 4, we obtain

$$\begin{aligned} \frac{d_{G-S}(T)}{a|T|} &\leq \frac{2(a-1)(a+b)-1}{a|T|} \leq \frac{2(a-1)(a+b)-1}{b|S|-bk+1} \\ &\leq \frac{2(a-1)(a+b)-1}{b(an+bk)/(a+b)-2b(a-1)-bk+1} \\ &\leq \frac{1}{a}(1-\frac{1}{b}). \end{aligned}$$

Combining this with Claim 1, we get

$$d_{G-S}(T) \leq (1-\frac{1}{b})|T| < |T|-1. \tag{7}$$

Let $T_0 = \{x \in T : d_{G-S}(x) = 0\}$. Note that $|T_0| \geq 2$ holds by (7). According to Claim 3, for any two vertices $x, y \in T_0$, we obtain

$$|N_G(x) \cup N_G(y)| \leq |S| < \frac{an+bk}{a+b}.$$

Since T_0 is an independent set of G and G satisfies the hypothesis of Theorem 5, then the neighborhoods of the vertices in T_0 are disjoint. Hence, we get that

$$|S| \geq |\cup_{x \in T_0} N_G(x)| \geq \delta(G)|T_0| \geq (a+k)|T_0|. \tag{8}$$

In view of (7), we have

$$(1-\frac{1}{b})|T| \geq d_{G-S}(T) \geq |T|-|T_0|,$$

which implies $|T_0| \geq \frac{|T|}{b}$. Combining this with (8), we have

$$|S| \geq (a+k)|T_0| \geq \frac{a+k}{b}|T|. \tag{9}$$

By Claims 1 and 4 and (9), we obtain

$$b|S| \geq (a+k)|T| \geq b|S|-bk+1+k|T| \geq b|S|-bk+1+k(b+1) = b|S|+k+1,$$

which is a contradiction. Completing the proof of Claim 5.

Claim 6. $e_G(S,T) \leq a|S|$.

Proof. By Claim 1 and the definition of T, then there exist at least two independent vertices $x, y \in T$. Moreover, by the definition of T and Claim 5, we have

$$|N_G(x) \cup N_G(y)| \leq |S| + d_{G-S}(x) + d_{G-S}(y) < \frac{an+bk}{a+b} \tag{10}$$

for any two vertices $x, y \in T$.

According to (10) and the assumption of Theorem 5, $G[N_G(u) \cap T]$ is complete for any $u \in S$. By Claim 2 and the definition of T, we have

$$e_G(u,T) \leq \Delta(G[T]) + 1 \leq a.$$

This inequality above implies $e_G(S,T) \leq a|S|$.

According to (6), and Claims 1 and 6, and $\delta(G) \geq a + k$, we have

$$\begin{aligned} bk - 1 &\geq \delta_G(S,T) = b|S| + d_{G-S}(T) - a|T| \\ &= b|S| + d_G(T) - a|T| - e_G(S,T) \\ &\geq b|S| + \delta(G)|T| - a|T| - a|S| \\ &\geq b|S| + k|T| - a|S| > k|T| \geq bk. \end{aligned}$$

This is a contradiction.
This completes the proof.

The proof of Theorem 6 is quite similar to that of Theorem 5. The proof of Theorem 6 is omitted.

Remark Let us show that the condition $|N_G(x) \cup N_G(y)| \geq \frac{an+bk}{a+b}$ in Theorem 5 can not be replaced by $|N_G(x) \cup N_G(y)| \geq \frac{an+bk}{a+b} - 1$. Let $G = (A, B)$ be a complete bipartite graph such that $|A| = at + k$ and $|B| = bt + 1$, where t is any positive integer. Then it follows that $n = |A| + |B| = (a+b)t + k + 1$ and

$$\frac{an+bk}{a+b} > |N_G(x) \cup N_G(y)| > \frac{an+bk}{a+b} - 1$$

for any subset $\{x, y\}$ of B. However, let $U \subseteq A$ with $|U| = k$, it is easy to see that $G - U$ has no $[a, b]$-factor since $b|A - U| < a|B|$. According to the definition of the (a, b, k)-critical graph, G is not an (a, b, k)-critical graph. In this sense, the condition $|N_G(x) \cup N_G(y)| \geq \frac{an+bk}{a+b}$ is the best possible.

We may adopt the similar way to argue the condition $\max\{d_G(x), d_G(y)\} \geq \frac{an+bk}{a+b}$ in Theorem 6, and the condition $\max\{d_G(x), d_G(y)\} \geq \frac{an+bk}{a+b}$ in Theorem 6 is the best possible.

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On the spectral radius of unicyclic graphs with fixed maximum degree

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Abstract: Let $\Delta(G)$ be the maximum degree of a graph G, and $\mathcal{U}(n,\Delta)$ be the set of all unicyclic graphs on n vertices with fixed maximum degree Δ. Among all the graphs in $\mathcal{U}(n,\Delta)$ $(\Delta \geq \frac{n+3}{2})$, we characterize the graph with the maximal spectral radius. We also prove that the spectral radius of a unicyclic graph G on $n\,(n \geq 30)$ vertices strictly increases with its maximum degree when $\Delta(G) \geq \lceil \frac{7n}{9} \rceil + 1$.

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Keywords: Unicyclic graph; Maximum degree; Spectral radius

1 Introduction

Let $G = (V(G), E(G))$ be a simple graph on n vertices. When vertices u and v are endpoints of an edge e, we write $e = uv$. Denote the set of all the neighbors of a vertex v in G by $N_G(v)$ and the degree of v in

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