Up-embeddability and independent number of simple graphs*

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Abstract: Let G be a $k(k \leq 3)$-edge connected simple graph with minimal degree $\delta \geq 3$ and girth g, $r = \lfloor \frac{g-1}{2} \rfloor$. If the independent number $\alpha(G)$ of G satisfies

$$\alpha(G) < \frac{6(\delta-1)^{\lfloor \frac{g}{2} \rfloor} - 6}{(4-k)(\delta-2)} - \frac{6(g-2r-1)}{4-k},$$

then G is up-embeddable.

Keywords: Up-embeddability; Maximum genus; Independent number.
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1 Introduction

The *maximum genus*, $\gamma_M(G)$, of a connected graph G is the largest integer k such that there exists a cellular embedding of G in the orientable surface with genus k. Recall that any cellular embedding of G has at least one region. By the Euler polyhedral equation, the maximum genus $\gamma_M(G) \leq \lfloor \frac{\beta(G)}{2} \rfloor$, where $\beta(G) = |E(G)| - |V(G)| + 1$ is the cycle rank or Betti number of G. A graph G is *up-embeddable* if $\gamma_M(G) = \lfloor \frac{\beta(G)}{2} \rfloor$ exactly.

For a spanning tree T in graph G, $\xi(G,T)$ denotes the number of components of $G \setminus E(T)$ with odd number of edges. $\xi(G) = \min_T \xi(G,T)$ is called the *Betti deficiency number* of G, where the minimum is taken over all spanning trees T of G.

Theorem 1.1(Xuong [9], Liu [3]) *Let G be a graph, then*
(1) $\gamma_M(G) = \frac{1}{2}(\beta(G) - \xi(G))$;

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(2) *G is up-embeddable if and only if* $\xi(G) \leq 1$.

Let A be an edge subset of $E(G)$. $c(G \setminus A)$ denotes the number of components of $G\backslash A$, when $b(G\backslash A)$ denotes the number of components of $G\backslash A$ with odd Betti number. In 1981, Nebesky [7] obtained an combinatorial expression of $\xi(G)$ in terms of the edge set.

Theorem 1.2(Nebesky [7]) *Let G be a graph, then*

$$\xi(G) = \max_{A\subseteq E(G)}\{c(G\backslash A) + b(G\backslash A) - |A| - 1\}.$$

Let $F_{i_1}, \cdots, F_{i_l}$ be l distinct components of $G\backslash A$. $E(F_{i_1}, \cdots, F_{i_l})$ denotes the set of edges whose end vertices are in different components F_{i_m} and $F_{i_n}(1 \leq m < n \leq l)$. For an induced subgraph F of G, $E(F, G) = E(F, G \setminus E(F))$. An *independent set* is the set of vertices in a graph, no two of which are adjacent. The cardinality of a maximum independent set is called the *independent number* of a graph G and is denoted by $\alpha(G)$. For more graphical notations without explanation, see [1].

Theorem 1.3(Huang and Liu [4]) *Let G be a graph. If G is not up-embeddable, i.e.,* $\xi(G) \geq 2$*, then there exists an edge subset* $A \subseteq E(G)$ *satisfying the following properties:*

(1) $c(G\backslash A) = b(G\backslash A) \geq 2$;

(2) *for any component F of* $G\backslash A$*, F is an induced subgraph of G;*

(3) *for any* $l \geq 2$ *distinct components* $F_{i_1}, \cdots, F_{i_l}$ *of* $G\backslash A$, $|E(F_{i_1}, \cdots, F_{i_l})| \leq 2l - 3$;

(4) $\xi(G) = 2c(G\backslash A) - |A| - 1$.

The study of the maximum genus was inaugurated by Nordhaus, Stewart and White[8]. From then on, various classes of graphs have been proved up-embeddable. A formerly known result[9] states that every 4-edge connected graph is up-embeddable. But, there exists 3-edge connected graphs(see [2]) which are not up-embedabble. Based on this, what kind of restrictions, under which a graph is up-embeddable, are studied extensively. Huang and Liu[5] proved that the maximum genus of a connected 3-regular graph G is equal to the maximum nonseparating independent number of G. In this paper, we study the up-embeddability of simple graphs via the independent number and obtain the following results.

Theorem 1.4 *Let G be a* $k(k \leq 3)$*-edge connected simple graph with minimal degree* $\delta \geq 3$ *and girth g,* $r = \lfloor \frac{g-1}{2} \rfloor$. *If the independent number* $\alpha(G)$ *of G satisfies*

$$\alpha(G) < \frac{6(\delta-1)^{\lfloor \frac{g}{2} \rfloor} - 6}{(4-k)(\delta-2)} - \frac{6(g-2r-1)}{4-k},$$

then G is up-embeddable.

2 Characterizations of induced subgraphs

The *distance* between two vertices u and v in a graph G, denoted by $d_G(u,v)$, is the length of a shortest (u,v)-path in G. The *distance* between the edge ab and vertex v in a graph G is $d_G(ab,v) = \min\{d_G(a,v), d_G(b,v)\}$. Clearly, $d_G(uv,u) = d_G(uv,v) = d_G(u,u) = 0$. The $i(i \geq 0)$*neighbor set* of a vertex or an edge x in a graph G is $N_i(x) = \{v \mid d_G(x,v) = i, v \in V(G)\}$. For an induced subgraph F of a graph G, the vertex $v \in V(F)$ is called a *t-touching vertex* or simply *touching vertex* of F, if v is the end vertex of $t(t \geq 1)$ edges in $E(F,G)$. In paper [6], we obtain the following Proposition 1 and Proposition 2.

Proposition 1 *Let G be a simple graph with minimal degree ≥ 3, girth g, $r = \lfloor \frac{g-1}{2} \rfloor$. H is a connected induced subgraph of G, $\beta(H) \geq 1$. If $\{u,v\} \subseteq V(H)$ contains all the touching vertices of H, then,*

(1) *when $g = 2r+2$, there exists an edge $ab \in E(H)$ such that* $\min\{d_H(ab,u), d_H(ab,v)\} \geq r$;

(2) *when $g = 2r+1$, there exists a vertex $a \in V(H)$ such that* $\min\{d_H(a,u), d_H(a,v)\} \geq r$.

Proposition 2 *Let G be a simple graph with minimal degree ≥ 3, girth g, $r = \lfloor \frac{g-1}{2} \rfloor$. H is a connected induced subgraph of G, $\beta(H) \geq 1$. If H has exactly three 1-touching vertices u,v,w, then,*

(1) *when $g = 2r+2$, there exists an edge $ab \in E(H)$ such that* $\min\{d_H(ab,u), d_H(ab,v)\} \geq r-1$, $\max\{d_H(ab,u), d_H(ab,v)\} \geq r$, $d_H(ab,w) \geq r$;

(2) *when $g = 2r+1$, there exists a vertex $a \in V(H)$ such that* $\min\{d_H(a,u), d_H(a,v)\} \geq r-1$, $\max\{d_H(a,u), d_H(a,v)\} \geq r$, $d_H(a,w) \geq r$.

Lemma 2.1 *Let G be a simple graph with minimal degree ≥ 3, girth $g \geq 4$, $r = \lfloor \frac{g-1}{2} \rfloor$. H is a connected induced subgraph of G, $\beta(H) \geq 1$. If $|E(H,G)| \leq 3$, then there exists an independent set A of H, which has no touching vertex of H, such that*

$$|A| \geq \frac{(\delta-1)^{\lfloor \frac{g}{2} \rfloor} - 1}{\delta - 2} - g + 2r + 1.$$

Proof Firstly, when H has exactly three 1-touching vertices $\{u,v,w\}$, by Proposition 2, there exists an edge or a vertex x in H such that $\min\{d_H(x,u), d_H(x,v)\} \geq r-1$ and $\min\{\max\{d_H(x,u), d_H(x,v)\}, d_H(x,w)\} \geq r$. Suppose $d_H(x,u) = \min\{d_H(x,u), d_H(x,v)\} \geq r-1$ and $\min\{d_H(x,v), d_H(x,w)\} \geq r$.

Case 1 When $g = 2r+1 \geq 5$, then x is a vertex in H. As $N_i(x)(0 \leq i \leq r-2)$ has no touching vertices of H, thus

$$N_i(x) \geq \delta \cdot (\delta-1)^{i-1}, \quad 1 \leq i \leq r-1.$$

Define

$$A' = \begin{cases} N_{r-1}(x) \cup N_{r-3}(x) \cup \cdots \cup N_2(x) \cup N_0(x), & r \text{ is odd;} \\ N_{r-1}(x) \cup N_{r-3}(x) \cup \cdots \cup N_3(x) \cup N_1(x), & r \text{ is even.} \end{cases}$$

It's easy to see that A' is an independent set of H, and

$$|A'| \geq \frac{(\delta-1)^r - 1}{\delta - 2}.$$

Subcase 1.1 $d_H(x,u) = r-1$, then $u \in N_{r-1}(x) \subseteq A'$. If $N_1(u) \cap \{v,w\} \neq \emptyset$, there exists another vertex $u' \in N_{r-1}(x)$ such that $N_1(u') \cap \{u,v,w\} = \emptyset$ and $N_1(u') \cap N_r(x) \geq 2$, because $d_H(x) \geq 3$ and $d_H(u') \geq 3$. If $N_1(u) \cap \{v,w\} = \emptyset$, let $u' = u$, then $N_1(u') \cap N_r(x) \geq 1$ because u is just a 1-touching vertex and $d_H(u) \geq 2$. Since all vertices in $N_{r-1}(x)$ have no common neighbors in $N_r(x)$, the set $A = A' \setminus \{u,u'\} \cup (N_1(u') \cap N_r(x))$ is also an independent set of H, which has no touching vertex of H, and

$$|A| \geq |A'| \geq \frac{(\delta-1)^r - 1}{\delta - 2}.$$

Subcase 1.2 $d_H(x,u) \geq r$, then, $u \notin A'$. Clearly, $A = A'$ is an independent set of H without touching vertex such that

$$|A| = |A'| \geq \frac{(\delta-1)^r - 1}{\delta - 2}.$$

Case 2 When $g = 2r+2 \geq 4$, then x is an edge in H. Suppose $x = ab$ and $d_H(x,v) = d_H(a,v) \geq r$.

Subcase 2.1 $d_H(x,w) = d_H(a,w) \geq r$. This means $\{v,w\} \cap N_r(b) \cap N_r(x) = \emptyset$, for otherwise, a cycle with length $2r+1 < g$ is formed. Define, when r is odd,

$$\begin{cases} B_1 = (N_0(a) \cap N_0(x)) \cup (N_2(a) \cap N_2(x)) \cup \cdots \cup (N_{r-1}(a) \cap N_{r-1}(x)), \\ B_2 = (N_1(b) \cap N_1(x)) \cup (N_3(b) \cap N_3(x)) \cup \cdots \cup (N_r(b) \cap N_r(x)), \end{cases}$$

when r is even,

$$\begin{cases} B_1 = (N_1(a) \cap N_1(x)) \cup (N_3(a) \cap N_3(x)) \cup \cdots \cup (N_{r-1}(a) \cap N_{r-1}(x)), \\ B_2 = (N_0(b) \cap N_0(x)) \cup (N_2(b) \cap N_2(x)) \cup \cdots \cup (N_r(b) \cap N_r(x)). \end{cases}$$

For $0 \leq i \leq r-1$, all vertices in $N_i(x)$ have no common neighbors in $N_{i+1}(x)$, and all vertices in $N_{i+1}(a)$ (or $N_{i+1}(b)$) are not adjacent, for otherwise a cycle with length $2(i+1)+1 < g$ is formed. Hence, $A = (B_1 \cup B_2) \setminus \{u\}$ is an independent set of H without touching vertex. It's easy to see that $B_1 \cap B_2 = \emptyset$, and

$$|N_i(a) \cap N_i(x)| \geq (\delta-1)^i, \quad |N_i(b) \cap N_i(x)| \geq (\delta-1)^i, \quad 0 \leq i \leq r-1.$$

In addition, if $u \in N_{r-1}(b) \cap N_{r-1}(x)$, then $u \notin B_1 \cup B_2$, $|N_r(b) \cap N_r(x)| \geq (\delta - 1)^r - 1$. Thus, through simple calculation,

$$|A| = |B_1| + |B_2| \geq \frac{(\delta - 1)^{r+1} - 1}{\delta - 2} - 1.$$

If $u \in B_1 \cup B_2$, then $u \notin N_{r-1}(b) \cap N_{r-1}(x)$, $|N_r(b) \cap N_r(x)| \geq (\delta - 1)^r$. Thus,

$$|A| = |B_1| + |B_2| - 1 \geq \frac{(\delta - 1)^{r+1} - 1}{\delta - 2} - 1.$$

Subcase 2.2 $d_H(x, w) = d_H(b, w) \geq r$. Furthermore, suppose $d_H(x, u) = d_H(a, u)$. As $d_H(b) \geq 3$, there exists a vertex $a' \in N_1(b) \cap N_1(x)$ such that $a' \neq a$ and $d_H(ba', y) \geq r$, $y \in \{u, v, w\}$. Now, for $0 \leq i, j \leq r$, i is odd, j is even, define

$$B_3 = \bigcup_i (N_i(a') \cap N_i(x)) \cup \bigcup_j (N_j(b) \cap N_j(x)),$$

$$B_4 = \bigcup_i (N_i(b) \cap N_i(x)) \cup \bigcup_j (N_j(a') \cap N_j(x)).$$

Clearly, B_3, B_4 both are independent set of H and $B_3 \cap B_4 = \emptyset$. So, one of B_3 and B_4 has at most one touching vertex of H, suppose $|B_3 \cap \{u, v, w\}| \leq 1$. As

$$|N_i(a') \cap N_i(x)| \geq (\delta - 1)^i, \quad |N_i(b) \cap N_i(x)| \geq (\delta - 1)^i, \quad 0 \leq i \leq r,$$

hence $A = B_3 \setminus \{u, v, w\}$ is an independent set without touching vertex of H, and

$$|A| = |B_3 \setminus \{u, v, w\}| \geq |B_3| - 1 \geq \frac{(\delta - 1)^{r+1} - 1}{\delta - 2} - 1.$$

Secondly, when H has at most two touching vertices, the proof is the same by using Proposition 1. ▯

3 The proof of Theorem 1.4

Suppose that Theorem 1.4 is not true, i.e., G is not up-embeddable. There exists an edge set $A \subseteq E(G)$ satisfying the characterizations (1)-(4) of Theorem 1.3. Define $C(G \setminus A)$ to be the set of components of $G \setminus A$, and

$$\begin{aligned} B_4 &= \{F \mid |E(F, G)| \geq 4, F \in C(G \setminus A)\}; \\ B_i &= \{F \mid |E(F, G)| = i, F \in C(G \setminus A)\}, \quad i = 1, 2, 3. \end{aligned}$$

Clearly,

$$c(G \setminus A) = |B_1| + |B_2| + |B_3| + |B_4|. \tag{1}$$

For each edge $e \in A$, the two end vertices of e must belong to two distinct components of $G \setminus A$, because any component $F \in C(G\setminus A)$ is an induced subgraph of G. On the other hand, the edge $e \in E(F, G)$ must belong to A. Thus,

$$|A| = \frac{1}{2} \sum_{F \in C(G\setminus A)} |E(F,G)| \geq 2|B_4| + \frac{3}{2}|B_3| + |B_2| + \frac{1}{2}|B_1|. \tag{2}$$

From Theorem 1.3, Equation (1) and (2), we have

$$\begin{aligned} \xi(G) &= 2c(G \setminus A) - |A| - 1 \\ &\leq 2(|B_4| + |B_3| + |B_2| + |B_1|) - (2|B_4| + \frac{3}{2}|B_3| + |B_2| + \frac{1}{2}|B_1|) - 1 \\ &= \frac{1}{2}|B_3| + |B_2| + \frac{3}{2}|B_1| - 1. \end{aligned}$$

As G is not up-embeddable, $\xi(G) \geq 2$. Hence

$$\frac{1}{2}|B_3| + |B_2| + \frac{3}{2}|B_1| \geq 3.$$

When G is $k(k \leq 3)$-edge connected, $|B_i| = 0$ for $i < k$. Through simple calculation, we have

$$|B_3| + |B_2| + |B_1| \geq \frac{6}{4-k}.$$

Without loss of generality, let $F_i (1 \leq i \leq 6/(4-k))$ be the components of $G \setminus A$ with $|E(F_i, G)| \leq 3$.

When $g \geq 4$, there exists an independent set A_i of each component $F_i (1 \leq i \leq 6/(4-k))$ satisfying the result of Lemma 2.1. Hence, we have

$$\alpha(G) \geq |\bigcup_{i=1}^{6/(4-k)} A_i| \geq \frac{6(\delta-1)^{\lfloor \frac{g}{2} \rfloor} - 6}{(4-k)(\delta-2)} - \frac{6(g-2r-1)}{4-k}.$$

When $g = 3$ and $k = 3$, $6/(4-k) = 6$. First, assume that each vertex in $F_i (1 \leq i \leq 6)$ is a touching vertex of F_i. Since $|E(F_i, G)| = 3$, $V(F_i)$ contains exactly three 1-touching vertices, denoted by $\{x_i, y_i, z_i\}$. Furthermore, suppose $\{x_6 z_5, y_6 z_4, z_6 x_3\} = E(F_6, G)$. As the vertex z_3 connects at most one vertex in $V(F_1) \cup V(F_2)$, hence there are at least 2 vertices in F_1 and F_2 respectively, which aren't adjacent with z_3, denoted them by $\{z_1, y_1\}$ and $\{z_2, y_2\}$. But, as $|E(F_1, F_2)| \leq 1$, we can assume that z_1 and

z_2 are not adjacent. Now, the vertices set $\{z_1, z_2, \cdots, z_6\}$ is clearly an independent set of G. Second, if there exists vertex u_i in $F_i (1 \le i \le 6)$ which is not a touching vertex, then by replacing z_i with u_i, we also can obtain an independent set of G with 6 vertices. For $k = 1, 2$, by similar discussions, there exists an independent set of G with $6/(4-k)$ vertices. Thus, when $g = 3$, we have $\alpha(G) \ge \frac{6}{4-k}$. But, this also contradicts with the condition. So, the graph G is up-embeddable. This completes the proof. □

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