NEW PROOFS FOR WHIPPLE'S TRANSFORMATION AND WATSON'S q-WHIPPLE TRANSFORMATION

[1]CHUANAN WEI AND [2]DIANXUAN GONG

[1]*Department of Information Technology*
Hainan Medical College, Haikou 571101, China
[2]*College of Sciences*
Hebei United University, Tangshan 063009, China

ABSTRACT. By means of inversion techniques, new proofs for Whipple's transformation and Watson's q-Whipple transformation are offered.

Several years ago, Chu [4, 5, 6, 7, 8, 9, 10, 11] studied systemically summation formulas for hypergeometric series and q-series in the light of inversion techniques. Following the work just mentioned, we shall give new proofs for Whipple's transformation and Watson's q-Whipple transformation in the same method.

1. GOULD-HSU INVERSIONS AND WHIPPLE'S TRANSFORMATION

For a complex number x, define the shifted factorial by

$$(x)_0 \equiv 1 \quad \text{and} \quad (x)_n = x(x+1)\cdots(x+n-1) \quad \text{for} \quad n = 1, 2, \cdots.$$

The fraction form of it reads as

$$\begin{bmatrix} a, & b, & \cdots, & c \\ \alpha, & \beta, & \cdots, & \gamma \end{bmatrix}_n = \frac{(a)_n(b)_n\cdots(c)_n}{(\alpha)_n(\beta)_n\cdots(\gamma)_n}.$$

Following Bailey [2], define the hypergeometric series by

$${}_{1+r}F_s\left[\begin{matrix} a_0, & a_1, & \cdots, & a_r \\ & b_1, & \cdots, & b_s \end{matrix}\middle| z\right] = \sum_{k=0}^{\infty} \frac{(a_0)_k(a_1)_k\cdots(a_r)_k}{k!(b_1)_k\cdots(b_s)_k} z^k.$$

Then Whipple's transformation (cf. Bailey [2, p. 25]) can be stated as follows.

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Email addresses: weichuanan@yahoo.com.cn(C. Wei), dxgong@heut.edu.cn (D. Gong).
The Corresponding author: Chuanan Wei.

Theorem 1. *For five complex numbers {a,b,c,d,e}, there holds:*

$$ {}_7F_6\left[\begin{matrix} a, & 1+a/2, & b, & c, & d, & e, & -n \\ a/2, & 1+a-b, & 1+a-c, & 1+a-d, & 1+a-e, & 1+a+n \end{matrix}\middle| 1\right] $$
$$ = \left[\begin{matrix} 1+a, & 1+a-b-c \\ 1+a-b, & 1+a-c \end{matrix}\right]_n {}_4F_3\left[\begin{matrix} 1+a-d-e, & b, & c, & -n \\ 1+a-d, & 1+a-e, & b+c-a-n \end{matrix}\middle| 1\right]. $$

For two nice proofs of this transformation, the reader may refer to Andrews, Askey and Roy [1, p. 633] and Bailey [2, p. 25] respectively. Now we give a new proof of Theorem 1.

Proof. Manipulating Saalschütz's summation formula (cf. Bailey [2, p. 9])

$$ {}_3F_2\left[\begin{matrix} 1+a-d-e, & a+n, & -n \\ & 1+a-d, & 1+a-e \end{matrix}\middle| 1\right] = \left[\begin{matrix} d, & e \\ 1+a-d, & 1+a-e \end{matrix}\right]_n \quad (1) $$

as the following equation

$$ \sum_{j=0}^{n}\left[\begin{matrix} b, & c, & 1+a-d-e, & a+n, & -n \\ 1, & 1+a-b, & 1+a-c, & 1+a-d, & 1+a-e \end{matrix}\right]_j \left[\begin{matrix} 1-b-n, & c+j \\ c-a-n, & 1+a-b+j \end{matrix}\right]_{n-j} $$
$$ = \left[\begin{matrix} b, & c, & d, & e \\ 1+a-b, & 1+a-c, & 1+a-d, & 1+a-e \end{matrix}\right]_n, $$

we have, according to (1), the identity

$$ \sum_{j=0}^{n}\left[\begin{matrix} b, & c, & 1+a-d-e, & a+n, & -n \\ 1, & 1+a-b, & 1+a-c, & 1+a-d, & 1+a-e \end{matrix}\right]_j $$
$$ \times\ {}_3F_2\left[\begin{matrix} 1+a-b-c, & a+n+j, & -n+j \\ & 1+a-b+j, & 1+a-c+j \end{matrix}\middle| 1\right] $$
$$ = \left[\begin{matrix} b, & c, & d, & e \\ 1+a-b, & 1+a-c, & 1+a-d, & 1+a-e \end{matrix}\right]_n $$

which is exactly the following double sum

$$ \sum_{j=0}^{n}\left[\begin{matrix} b, & c, & 1+a-d-e, & a+n, & -n \\ 1, & 1+a-b, & 1+a-c, & 1+a-d, & 1+a-e \end{matrix}\right]_j $$
$$ \times \sum_{k=0}^{n-j}\left[\begin{matrix} 1+a-b-c, & a+n+j, & -n+j \\ 1, & 1+a-b+j, & 1+a-c+j \end{matrix}\right]_k $$
$$ = \left[\begin{matrix} b, & c, & d, & e \\ 1+a-b, & 1+a-c, & 1+a-d, & 1+a-e \end{matrix}\right]_n. $$

Preforming the replacement $k \to k - j$ for the last equation and then exchanging the order of the double sum, we obtain the following expression

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}(a+k)_n\left[\begin{matrix} a, & 1+a-b-c \\ 1+a-b, & 1+a-c \end{matrix}\right]_k$$
$$\times {}_4F_3\left[\begin{matrix} 1+a-d-e, & b, & c, & -k \\ 1+a-d, & 1+a-e, & b+c-a-k \end{matrix}\bigg| 1\right]$$
$$= \left[\begin{matrix} a, & b, & c, & d, & e \\ 1+a-b, & 1+a-c, & 1+a-d, & 1+a-e \end{matrix}\right]_n. \tag{2}$$

For a complex variable x and two complex sequences $\{a_k, b_k\}_{k\geq 0}$, define a polynomial sequence by

$$\phi(x;0) \equiv 1, \quad \phi(x;n) = \prod_{i=0}^{n-1}(a_i + xb_i), \quad n = 1, 2, \cdots .$$

Then a pair of inverse series relations due to Gould-Hsu [13] (see Chu [5] also) can be stated as

$$f(n) = \sum_{k=0}^{n}(-1)^k\binom{n}{k}\phi(k;n)\, g(k), \tag{3a}$$

$$g(n) = \sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{a_k + kb_k}{\phi(n;k+1)}\, f(k). \tag{3b}$$

Equation (2) matches with (3a) perfectly and (3b) creats the following dual relation

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{a+2k}{(a+n)_{k+1}}\left[\begin{matrix} a, & b, & c, & d, & e \\ 1+a-b, & 1+a-c, & 1+a-d, & 1+a-e \end{matrix}\right]_k$$
$$= \left[\begin{matrix} a, & 1+a-b-c \\ 1+a-b, & 1+a-c \end{matrix}\right]_n {}_4F_3\left[\begin{matrix} 1+a-d-e, & b, & c, & -n \\ 1+a-d, & 1+a-e, & b+c-a-n \end{matrix}\bigg| 1\right]$$

which is even the transformation that appears in Theorem 1. □

2. Carlitz inversions and Watson's q-Whipple's transformation

With two complex numbers q and x, define the q-shifted factorial by

$$(x;q)_0 \equiv 1 \quad \text{and} \quad (x;q)_n = \prod_{k=0}^{n-1}(1 - xq^k) \quad \text{for} \quad n = 1, 2, \cdots .$$

Its fraction form reads as

$$\begin{bmatrix} a, & b, & \cdots, & c \\ \alpha, & \beta, & \cdots, & \gamma \end{bmatrix} q \Big]_n = \frac{(a;q)_n(b;q)_n\cdots(c;q)_n}{(\alpha;q)_n(\beta;q)_n\cdots(\gamma;q)_n}.$$

Following Gasper and Rahman [12], define q-series by

$${}_{1+r}\phi_s\left[\begin{matrix} a_0, & a_1, & \cdots, & a_r \\ & b_1, & \cdots, & b_s \end{matrix}\,\Big|\, z\right] = \sum_{k=0}^{\infty}\left[\begin{matrix} a_0, & a_1, & \cdots, & a_r \\ q, & b_1, & \cdots, & b_s \end{matrix}\,\Big|\, q\right]_k \left\{(-1)^k q^{\binom{k}{2}}\right\}^{s-r} z^k.$$

Then Watson's q-Whipple transformation (cf. Gasper and Rahman [12, p. 43]) can be stated as follows.

Theorem 2. *For six complex numbers $\{q, a, b, c, d, e\}$, there holds:*

$${}_8\phi_7\left[\begin{matrix} a, & q\sqrt{a}, & -q\sqrt{a}, & b, & c, & d, & e, & q^{-n} \\ & \sqrt{a}, & -\sqrt{a}, & qa/b, & qa/c, & qa/d, & qa/e, & aq^{n+1} \end{matrix}\,\Big|\, q; \frac{a^2q^{2+n}}{bcde}\right]$$
$$= \left[\begin{matrix} qa, & qa/bc \\ qa/b, & qa/c \end{matrix}\,\Big|\, q\right]_n {}_4\phi_3\left[\begin{matrix} b, & c, & qa/de, & q^{-n} \\ & qa/d, & qa/e, & q^{-n}bc/a \end{matrix}\,\Big|\, q; q\right].$$

For two beautiful proofs of this transformation, the reader may refer to Andrews, Askey and Roy [1, p. 587] and Gasper and Rahman [12, p. 43] respectively. Here we provide a new proof of Theorem 2.

Proof. Reformulating q-Saalschütz's summation formula (cf. Gasper and Rahman [12, p. 17])

$${}_3\phi_2\left[\begin{matrix} qa/de, & aq^n, & q^{-n} \\ & qa/d, & qa/e \end{matrix}\,\Big|\, q; q\right] = \left[\begin{matrix} d, & e \\ qa/d, & qa/e \end{matrix}\,\Big|\, q\right]_n \left(\frac{qa}{de}\right)^n \tag{4}$$

as the following equation

$$\sum_{j=0}^{n}\left[\begin{matrix} b, & c, & qa/de, & aq^n, & q^{-n} \\ q, & qa/b, & qa/c, & qa/d, & qa/e \end{matrix}\,\Big|\, q\right]_j \left(\frac{q^2a}{bc}\right)^j \left[\begin{matrix} q^{1-n}/b, & q^jc \\ q^{-n}c/a, & q^{1+j}a/b \end{matrix}\,\Big|\, q\right]_{n-j}$$
$$= \left[\begin{matrix} b, & c, & d, & e \\ qa/b, & qa/c, & qa/d, & qa/e \end{matrix}\,\Big|\, q\right]_n \left(\frac{q^2a^2}{bcde}\right)^n,$$

we have, from (4), the identity

$$\sum_{j=0}^{n}\left[\begin{matrix} b, & c, & qa/de, & aq^n, & q^{-n} \\ q, & qa/b, & qa/c, & qa/d, & qa/e \end{matrix}\,\Big|\, q\right]_j \left(\frac{q^2a}{bc}\right)^j$$
$$\times\ {}_3\phi_2\left[\begin{matrix} qa/bc, & aq^{n+j}, & q^{-n+j} \\ & q^{1+j}a/b, & q^{1+j}a/c \end{matrix}\,\Big|\, q; q\right]$$
$$= \left[\begin{matrix} b, & c, & d, & e \\ qa/b, & qa/c, & qa/d, & qa/e \end{matrix}\,\Big|\, q\right]_n \left(\frac{q^2a^2}{bcde}\right)^n$$

which is exactly the following double sum

$$\sum_{j=0}^{n}\begin{bmatrix} b, & c, & qa/de, & aq^n, & q^{-n} \\ q, & qa/b, & qa/c, & qa/d, & qa/e \end{bmatrix} q \Big]_j \left(\frac{q^2a}{bc}\right)^j$$

$$\times \sum_{k=0}^{n-j}\begin{bmatrix} qa/bc, & aq^{n+j}, & q^{-n+j} \\ q, & q^{1+j}a/b, & q^{1+j}a/c \end{bmatrix} q \Big]_k q^k$$

$$= \begin{bmatrix} b, & c, & d, & e \\ qa/b, & qa/c, & qa/d, & qa/e \end{bmatrix} q \Big]_n \left(\frac{q^2a^2}{bcde}\right)^n.$$

Replacing k by $k-j$ for the last equation and then interchanging the order of the double sum, we get the following expression

$$\sum_{k=0}^{n}(-1)^k\begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{n-k}{2}}(aq^k;q)_n \begin{bmatrix} a, & qa/bc \\ qa/b, & qa/c \end{bmatrix} q \Big]_k$$

$$\times\ {}_4\phi_3\begin{bmatrix} qa/de, & b, & c, & q^{-k} \\ & qa/d, & qa/e, & q^{-k}bc/a \end{bmatrix} q;q \Big]$$

$$= \begin{bmatrix} a, & b, & c, & d, & e \\ & qa/b, & qa/c, & qa/d, & qa/e \end{bmatrix} q \Big]_n \left(\frac{q^2a^2}{bcde}\right)^n q^{\binom{n}{2}} \tag{5}$$

where q-binomial coefficient has been defined by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}.$$

For a complex variable x and two complex sequences $\{c_k, d_k\}_{k\geq 0}$, define a polynomial sequence by

$$\psi(x;0) \equiv 1, \quad \psi(x;n) = \prod_{i=0}^{n-1}(c_i + q^x d_i), \quad n = 1, 2, \cdots.$$

Then a pair of inverse series relations due to Carlitz [3] (see Chu [7] also) can be stated as

$$F(n) = \sum_{k=0}^{n}(-1)^k\begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{n-k}{2}}\psi(k;n)\, G(k), \tag{6a}$$

$$G(n) = \sum_{k=0}^{n}(-1)^k\begin{bmatrix} n \\ k \end{bmatrix} \frac{c_k + q^k d_k}{\psi(n;k+1)}\, F(k). \tag{6b}$$

Equation (5) fits into (6a) ideally and (6b) produces the following dual relation

$$\sum_{k=0}^{n}(-1)^k\begin{bmatrix}n\\k\end{bmatrix}\frac{1-aq^{2k}}{(aq^n;q)_{k+1}}\left[\begin{matrix}a, & b, & c, & d, & e\\ qa/b, & qa/c, & qa/d, & qa/e\end{matrix}\,\Big|\, q\right]_k\left(\frac{q^2a^2}{bcde}\right)^k q^{\binom{k}{2}}$$
$$=\left[\begin{matrix}a, & qa/bc\\ qa/b, & qa/c\end{matrix}\,\Big|\, q\right]_n {}_4\phi_3\left[\begin{matrix}qa/de, & b, & c, & q^{-n}\\ & qa/d, & qa/e, & q^{-n}bc/a\end{matrix}\,\Big|\, q;q\right]$$

which is even the transformation displayed in Theorem 2. □

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