GENERALIZED COMPOSITION OPERATORS FROM μ-BLOCH SPACES INTO MIXED NORM SPACES

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Abstract

The boundedness and compactness of the generalized composition operator from μ-Bloch spaces to mixed norm spaces are completely characterized in this paper.

1 Introduction

Let ϕ be a positive continuous function on $[0,1)$. ϕ is called normal, if there exist positive numbers s and t, $0 < s < t$, and $\delta \in [0,1)$ such that (see [11])

$$\frac{\phi(r)}{(1-r)^s} \text{ is decreasing on } [\delta,1) \text{ and } \lim_{r\to 1}\frac{\phi(r)}{(1-r)^s} = 0;$$

$$\frac{\phi(r)}{(1-r)^t} \text{ is increasing on } [\delta,1) \text{ and } \lim_{r\to 1}\frac{\phi(r)}{(1-r)^t} = \infty.$$

Let D be the unit disk of $\mathbb{C}$ and ∂D the boundary of D. Let $H(D)$ denote the space of all analytic functions in D. Let ϕ be a normal function on $[0,1)$. For $0 < p, q < \infty$, the mixed norm space $H(p,q,\phi) = H(p,q,\phi)(D)$ consists of all $f \in H(D)$ such that

$$\|f\|_{H(p,q,\phi)} = \left(\int_0^1 M_q^p(f,r)\frac{\phi^p(r)}{1-r}dr\right)^{1/p} < \infty, \tag{1}$$

where

$$M_q(f,r) = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^q d\theta\right)^{1/q}.$$

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Let $\mu : [0,1) \to [0,\infty)$ be a normal function. An $f \in H(D)$ is said to belong to the μ-Bloch space, denoted by $\mathcal{B}_\mu = \mathcal{B}_\mu(D)$, if

$$\|f\|_{\mathcal{B}_\mu} = |f(0)| + \sup_{z\in D} \mu(|z|)|f'(z)| < \infty.$$

Under the above norm, $\mathcal{B}_\mu$ becomes a Banach space. When $\mu = 1 - r^2$ and $\mu = (1-r^2)\log\frac{e}{1-r^2}$, $\mathcal{B}_\mu$ reduces to the classical Bloch space $\mathcal{B}$ and the logarithmic Bloch space $\mathcal{B}_{\log}$ respectively.

Let φ be a holomorphic self-map of D and $f \in H(D)$. The composition operator C_φ is defined by $(C_\varphi f)(z) = (f \circ \varphi)(z)$. See [1] for the study of composition operator on various spaces in the unit disk.

Let φ be a holomorphic self-map of D and $g \in H(D)$. In [6], Li and Stević defined a linear operator as follows:

$$(C_\varphi^g f)(z) = \int_0^z f'(\varphi(\xi))g(\xi)d\xi.$$

The operator C_φ^g is called the generalized composition operator. The operator C_φ^g is a generalization of the composition operator. When $\varphi(z) = z$, we get $C_\varphi^g = L_g$. The operator L_g and it's generalization were studied in [3–5, 7–9, 12–15].

The purpose of this article is to give some sufficient and necessary conditions for the boundedness and compactness of generalized composition operators from μ-Bloch spaces to mixed norm spaces in the unit disk.

Recall that a linear operator is said to be bounded if the image of a bounded set is a bounded set, while a linear operator is compact if it takes bounded sets to sets with compact closure.

Throughout the paper, constants are denoted by C, they are positive and may not be the same in every occurrence. The notation $A \asymp B$ means that there is a positive constant C such that $B/C \le A \le CB$.

2 Main results and proofs

In this section, we give our main results and their proofs. Before stating our main results, we need some auxiliary results, which are incorporated in the lemmas which follow.

Lemma 1. *Let $\mu : [0,1) \to [0,\infty)$ be a nonincreasing radial weight function and normal on the interval $[0,1)$. Assume that $0 < p, q < \infty$, ϕ is normal on $[0,1)$, $g \in H(D)$ and φ is an analytic self-map of D. Then $C_\varphi^g : \mathcal{B}_\mu \to H(p,q,\phi)$ is compact if and only if $C_\varphi^g : \mathcal{B}_\mu \to H(p,q,\phi)$ is bounded and for any bounded sequence $(f_k)_{k\in\mathbb{N}}$ in $\mathcal{B}_\mu$ which converges to zero uniformly on compact subsets of D as $k \to \infty$, we have $\|C_\varphi^g f_k\|_{H(p,q,\phi)} \to 0$ as $k \to \infty$.*

The proof of Lemma 1 follows by standard arguments (see, for example, Proposition 3.11 of [1], as well as the proof of the corresponding result in [4]). Here we omit the details.

The following lemma can be found in [10].

Lemma 2. *Let $\mu : [0,1) \to [0,\infty)$ be a nonincreasing radial weight function and normal on the interval $[0,1)$. Then there are two functions $f_1, f_2 \in \mathcal{B}_\mu$ such that*

$$|f_1'(z)| + |f_2'(z)| \geq \frac{C}{\mu(|z|)}, \quad z \in D, \tag{2}$$

for some positive constant $C > 0$.

Now we are in a position to state and prove our main results.

Theorem 1. *Let $\mu : [0,1) \to [0,\infty)$ be a nonincreasing radial weight function and normal on the interval $[0,1)$. Assume that $0 < p, q < \infty$, ϕ is normal on $[0,1)$, $g \in H(D)$ and φ is an analytic self-map of D. Then the following statements are equivalent:*
(i) $C_\varphi^g : \mathcal{B}_\mu \to H(p,q,\phi)$ is a bounded operator;
(ii) $C_\varphi^g : \mathcal{B}_\mu \to H(p,q,\phi)$ is a compact operator;
(iii)

$$\int_0^1 \Big(\int_0^{2\pi} \frac{|g(re^{i\theta})|^q}{\mu(|\varphi(re^{i\theta})|)} d\theta \Big)^{p/q} \frac{\phi^p(r)}{(1-r)^{1-p}} dr < \infty; \tag{3}$$

(iv)

$$\lim_{t \to 1} \int_0^1 \Big(\int_{|\varphi(re^{i\theta})| > t} \frac{|g(re^{i\theta})|^q}{\mu(|\varphi(re^{i\theta})|)} d\theta \Big)^{p/q} \frac{\phi^p(r)}{(1-r)^{1-p}} dr = 0. \tag{4}$$

Proof. $(ii) \Rightarrow (i)$. This implication is obvious.

$(i) \Rightarrow (iii)$. Assume that $C_\varphi^g : \mathcal{B}_\mu \to H(p,q,\phi)$ is bounded. From Lemma 2, there are two functions $f_1, f_2 \in \mathcal{B}_\mu$ such that

$$|f_1'(z)| + |f_2'(z)| \geq \frac{C}{\mu(|z|)}, \quad z \in D. \tag{5}$$

By using the following well-known asymptotic formula (see, e.g. [2])

$$\int_0^1 M_q^p(f,r) \frac{\phi^p(r)}{1-r} dr \asymp |f(0)|^q + \int_0^1 M_q^p(f',r) \frac{\phi^p(r)}{(1-r)^{1-p}} dr,$$

(5) implies

$$\int_0^1 \Big(\int_0^{2\pi} |g(re^{i\theta})|^q |(f_j' \circ \varphi)(re^{i\theta})|^q d\theta \Big)^{p/q} \frac{\phi^p(r)}{(1-r)^{1-p}} dr < \infty, j = 1,2. \tag{6}$$

From (5), (6) and the elementary inequality

$$(a+b)^p \le \begin{cases} a^p+b^p & , \quad p\in(0,1) \\ 2^p(a^p+b^p) & , \quad p\ge 1 \end{cases}, \quad a>0, \quad b>0,$$

we obtain

$$\begin{aligned}
& \int_0^1 \Big(\int_0^{2\pi} \frac{|g(re^{i\theta})|^q}{\mu(|\varphi(re^{i\theta})|)} d\theta\Big)^{p/q} \frac{\phi^p(r)}{(1-r)^{1-p}} dr \\
\le\ & C\int_0^1 \Big(\int_0^{2\pi} |g(re^{i\theta})|^q \Big(|f_1'(\varphi(re^{i\theta}))| + |f_2'(\varphi(re^{i\theta}))|\Big)^q d\theta\Big)^{p/q} \frac{\phi^p(r)}{(1-r)^{1-p}} dr \\
\le\ & C\int_0^1 \Big(\int_0^{2\pi} |g(re^{i\theta})|^q |(f_1'\circ\varphi)(re^{i\theta})|^q d\theta\Big)^{p/q} \frac{\phi^p(r)}{(1-r)^{1-p}} dr + \\
& C\int_0^1 \Big(\int_0^{2\pi} |g(re^{i\theta})|^q |(f_2'\circ\varphi)(re^{i\theta})|^q d\theta\Big)^{p/q} \frac{\phi^p(r)}{(1-r)^{1-p}} dr \\
\le\ & C\int_0^1 \Big(\int_0^{2\pi} |(C_\varphi^g f_1)'(re^{i\theta})|^q d\theta\Big)^{p/q} \frac{\phi^p(r)}{(1-r)^{1-p}} dr + \\
& C\int_0^1 \Big(\int_0^{2\pi} |(C_\varphi^g f_2)'(re^{i\theta})|^q d\theta\Big)^{p/q} \frac{\phi^p(r)}{(1-r)^{1-p}} dr \\
<\ & \infty,
\end{aligned}$$

which implies that (3) holds.

(iii) $\Rightarrow$ *(iv)*. This implication follows from the dominated convergence Theorem.

(iv) $\Rightarrow$ *(ii)*. Assume that (4) holds. To prove that $C_\varphi^g : \mathcal{B}_\mu \to H(p,q,\phi)$ is compact, it suffices to prove that if $\{f_k\}$ is a bounded sequence in $\mathcal{B}_\mu$ such that $\{f_k\}$ converges to zero uniformly on compact subset of D, then

$$\|C_\varphi^g f_k\|_{H(p,q,\phi)} \to 0, \quad \text{as} \quad k\to\infty. \tag{7}$$

Take such a sequence $\{f_k\}\subset \mathcal{B}_\mu$, we have

$$\begin{aligned}
& \int_0^1 \Big(\int_{|\varphi(re^{i\theta})|>t} |g(re^{i\theta})|^q |(f_k'\circ\varphi)(re^{i\theta})|^q d\theta\Big)^{p/q} \frac{\phi^p(r)}{(1-r)^{1-p}} dr \\
\le\ & \|f_k\|_{\mathcal{B}_\mu}^p \int_0^1 \Big(\int_{|\varphi(re^{i\theta})|>t} \frac{|g(re^{i\theta})|^q}{\mu(|\varphi(re^{i\theta})|)} d\theta\Big)^{p/q} \frac{\phi^p(r)}{(1-r)^{1-p}} dr \\
\le\ & C\int_0^1 \Big(\int_{|\varphi(re^{i\theta})|>t} \frac{|g(re^{i\theta})|^q}{\mu(|\varphi(re^{i\theta})|)} d\theta\Big)^{p/q} \frac{\phi^p(r)}{(1-r)^{1-p}} dr,
\end{aligned} \tag{8}$$

for all k. Take $\varepsilon>0$. (4) and (8) imply that there exists $t_0\in(0,1)$ such that

$$\int_0^1 \Big(\int_{|\varphi(re^{i\theta})|>t_0} |g(re^{i\theta})|^q |(f_k'\circ\varphi)(re^{i\theta})|^q d\theta\Big)^{p/q} \frac{\phi^p(r)}{(1-r)^{1-p}} dr < \varepsilon, \tag{9}$$

for all k. Since $\{f_k\}$ converges to 0 on any compact subset of D, by Cauchy estimate we see that $\{f'_k\}$ converges to 0 on any compact subset of D. For the above ε, there exists a k_0 such that

$$\int_0^1 \Big(\int_{|\varphi(re^{i\theta})|\le t_0} |g(re^{i\theta})|^q |(f'_k \circ \varphi)(re^{i\theta})|^q d\theta\Big)^{p/q} \frac{\phi^p(r)}{(1-r)^{1-p}} dr < \varepsilon, \quad (10)$$

for all $k > k_0$. Hence by (9) and (10) we have

$$\begin{aligned} & \|C_\varphi^g f_k\|_{H(p,q,\phi)} \\ \asymp\ & \int_0^1 \Big(\int_0^{2\pi} |g(re^{i\theta})|^q |(f'_k \circ \varphi)(re^{i\theta})|^q d\theta\Big)^{p/q} \frac{\phi^p(r)}{(1-r)^{1-p}} dr \\ =\ & \int_0^1 \Big(\int_{|\varphi(re^{i\theta})|> t_0} |g(re^{i\theta})|^q |(f'_k \circ \varphi)(re^{i\theta})|^q d\theta\Big)^{p/q} \frac{\phi^p(r)}{(1-r)^{1-p}} dr + \\ & \int_0^1 \Big(\int_{|\varphi(re^{i\theta})|\le t_0} |g(re^{i\theta})|^q |(f'_k \circ \varphi)(re^{i\theta})|^q d\theta\Big)^{p/q} \frac{\phi^p(r)}{(1-r)^{1-p}} dr \\ \le\ & 2\varepsilon, \quad \text{as} \quad k > k_0, \end{aligned}$$

from which we obtain $\lim_{k\to\infty} \|C_\varphi^g f_k\|_{H(p,q,\phi)} = 0$. Thus $C_\varphi^g : \mathcal{B}_\mu \to H(p,q,\phi)$ is compact by Lemma 1. The proof of this theorem is completed.

References

[1] C. C. Cowen and B. D. MacCluer, *Composition Operators on Spaces of Analytic Functions*, Studies in Advanced Math., CRC Press, Boca Raton, 1995.

[2] Z. Hu, Extended Cesàro operators on mixed norm spaces, *Proc. Amer. Math. Soc.* **131**(2003), 2171-2179.

[3] S. Li, Riemann-Stieltjes operators from $F(p,q,s)$ to Bloch space on the unit ball, Vol. 2006, Article ID 27874 *J. Inequal. Appl.* (2006), 14 pages.

[4] S. Li and S. Stević, Riemann-Stieltjes type integral operators on the unit ball in $\mathbb{C}^n$, *Complex Var. Elliptic Equ.* **52**(2007), 495-517.

[5] S. Li and S. Stević, Riemann-Stieltjes operators on Hardy spaces in the unit ball of $\mathbb{C}^n$, *Bull. Belg. Math. Soc. Simon Stevin*, **14**(2007), 621-628.

[6] S. Li and S. Stević, Generalized composition operators on Zygmund spaces and Bloch-type spaces, *J. Math. Anal. Appl.* **338**(2008), 1282-1295.

[7] S. Li and S. Stević, Riemann-Stieltjes operators on weighted Bergman spaces in the unit ball of $\mathbb{C}^n$, *Bull. Belg. Math. Soc. Simon Stevin*, **15**(2008), 677-686.

[8] S. Li and S. Stević, Compactness of Riemann-Stieltjes operators between $F(p,q,s)$ and α-Bloch spaces, *Publ. Math. Debrecen*, **72/1-2**(2008), 111-128.

[9] S. Li and S. Stević, Cesàro type operators on some spaces of analytic functions on the unit ball, *Appl. Math. Comput.* **208**(2009), 378-388.

[10] C. Yang, W. Xu and M. Kotilainen, Composition operators from Bloch type spaces into Q_K type spaces, *J. Math. Anal. Appl.* (2011), to appear.

[11] A. L. Shields and D. L. Williams, Bounded projections, duality, and multipliers in spaces of analytic functions, *Trans. Amer. Math. Soc.* **162**(1971), 287-302.

[12] W. Yang, On an integral-type operator between Bloch-type spaces, *Appl. Math. Comput.* **215**(2009), 954-960.

[13] W. Yang and X. Meng, Generalized composition operators from $F(p,q,s)$ spaces to Bloch-type spaces, *Appl. Math. Comput.* **217**(2010), 2513-2519.

[14] X. Zhu, Generalized composition operators from generalized weighted Bergman spaces to Bloch type spaces, *J. Korea Math. Soc.* **46**(2009), 1219-1232.

[15] X. Zhu, Generalized composition operators and Volterra composition operators on Bloch spaces in the unit ball, *Complex Var. Elliptic Equ.* **54**(2009), 95-102.

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