l_1–Embeddability of Hexagonal and Quadrilateral Möbius graphs *

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Abstract

A connected graph G is called l_1-embeddable, if G can be isometrically embedded into the l_1-space. The hexagonal Möbius graphs $H_{2m,2k}$ and $H_{2m+1,2k+1}$ are two classes of hexagonal tilings of a Möbius strip. The regular quadrilateral Möbius graph $Q_{p,q}$ is a quadrilateral tiling of a Möbius strip. In this note, we show that among these three classes of graphs only $H_{2,2}$, $H_{3,3}$ and $Q_{2,2}$ are l_1-embeddable.

Key words: l_1-embeddable; hypercube; hexagonal Möbius graph; quadrilateral Möbius graph

1 Introduction

All graphs considered in this note are finite, unoriented and simple. For a graph G, let $V(G)$ and $E(G)$ denote its vertex set and edge set, respectively. The *distance* $d_G(u,v)$ between two vertices u and v of G is the length of a shortest path between u and v. If the graph G is clear from the context, then we will simply use $d(u,v)$. It satisfies that (i) $d(u,v) = d(v,u)$, (ii) $d(u,v) \geq 0$ and $d(u,v) = 0$ iff $u = v$ and (iii) $d(u,v) \leq d(u,w) + d(w,v)$, for all $u, v, w \in V(G)$. So d_G is a metric on $V(G)$ and $(V(G), d_G)$ is called the *graphic metric space* associated with G [8].

Let X denote the set of all real sequences $\{\xi_k\}$, such that $\sum_{k=0}^{\infty} |\xi_k| < \infty$. Define the distance function d_1 on X as $d_1(x,y) = \sum_{k=0}^{\infty} |\xi_k - \eta_k|$, for all $x = \{\xi_1, \xi_2, \ldots, \xi_k, \ldots\}$, $y = \{\eta_1, \eta_2, \ldots, \eta_k, \ldots\} \in X$. It's known that (X, d_1) is

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a metric space and it is called the l_1*-space*. A graph G is called an l_1*-graph* (sometimes called l_1*-embeddalbe*) if and only if $(V(G), d_G)$ is isomorphic to a subspace of the l_1-space. That is, there exists a distance-preserving mapping ϕ from $V(G)$ into X such that $d_G(x, y) = d_1(\phi(x), \phi(y))$ for any two vertices x, y of G. From the view point of metric space, the l_1-embeddability of graphs was characterized in [5] and [17]: a graph is an l_1-graph if and only if it is an isometric subgraph of the Cartesian product of half-cubes and cocktail-party graphs.

The *Cartesian product* $G\Box H$ of graphs G and H is the graph with vertex set $V(G) \times V(H)$ such that the vertex (a, x) is adjacent to the vertex (b, y) whenever $ab \in E(G)$ and $x = y$, or $a = b$ and $xy \in E(H)$ [14].

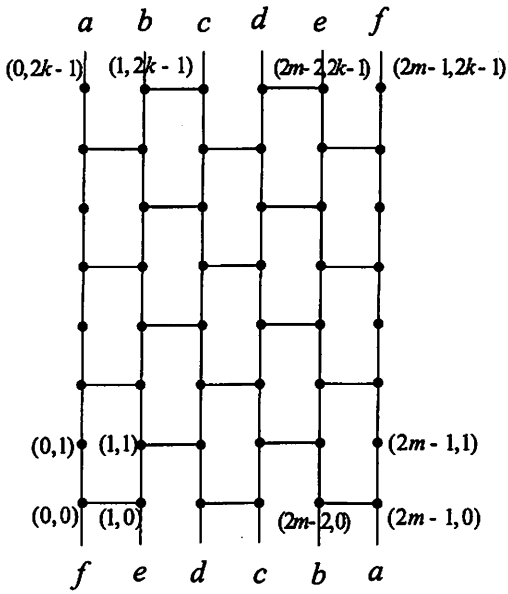
An n*-dimensional hypercube* or n*-cube* Q_n is defined as follows: the vertex set consists of all n-tuples $b_1 b_2 \cdots b_n$ with $b_i \in \{0, 1\}$, and two vertices are adjacent if and only if the corresponding n-tuples differ in precisely one place. The hypercube Q_n can be represented as the Cartesian product of n copies of the complete graph on two vertices.

A *scale* λ *embedding* (or $\lambda-$*embedding*) ϕ of a graph G into a hypercube Q_n is a mapping $V(G) \to V(Q_n)$, such that $\lambda d_G(x, y) = d_{Q_n}(\phi(x), \phi(y))$. Assouad and Deza [1] showed that a graph G is an l_1-graph if and only if it admits a λ-embedding into a cube Q_k for some integers λ and k. According to [8, Lemma 21.1.2], λ is 1 or an even number. Prisăcaru, Soltan and Chepoi [16] have shown that any planar graph in which all interior faces have size larger than four and the interior vertices have degree larger than three is l_1-embeddable. Later, Chepoi, Deza and Grishukhin [2] gave a criterion to decide whether a planar graph is an l_1-graph. It is well known that there are six famous surfaces except the plane: sphere, torus, Klein bottle, projective plane, cylinder, and the Möbius strip. The first four are closed, while the last two are not. Recently, Deza and Shpectorov [10] determined all finite trivalent surface graphs with hexagonal faces on the torus and the Klein bottle which are l_1-graphs. In this note we consider l_1-embeddability of hexagonal and quadrilateral lattices on Möbius strips. For other results on l_1-graphs we refer to [3, 6, 7, 11, 12].

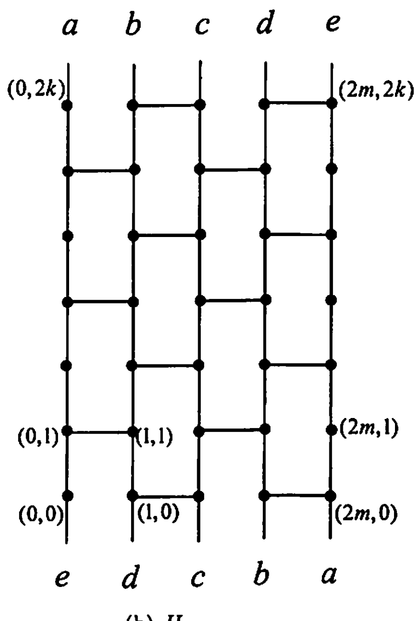
The path P_k is a graph with vertex-set $\{0, 1, \ldots, k-1\}$ and edges $i(i+1)$ with $0 \leq i \leq k-2$. By the definition of the cartesian product of graphs, $P_p \Box P_q$ is the $p \times q$ grid and $V(P_p \Box P_q) = \{(i, j) | 0 \leq i \leq p-1, 0 \leq j \leq q-1\}$.

The *hexagonal Möbius graph* of length $2k$ and breadth $2m$ is defined as the graph obtained from $P_{2m} \Box P_{2k}$ by removing the edges $\{(2i, 2j+1), (2i+1, 2j+1)\}$ and $\{(2i+1,2j),(2i+2,2j)\}$ with $0 \leq i \leq (m-1)$, $0 \leq j \leq (k-1)$ and adding the edges $\{(n, 0), (2m-1-n, 2k-1)\}$ with $0 \leq n \leq 2k-1$, denoted by $H_{2m,2k}$ (see Figure 1(a)).

The *hexagonal Möbius graph* of length $2k+1$ and breadth $2m+1$ is obtained from $P_{2m+1} \Box P_{2k+1}$ by deleting the edges $\{(2i, 2j), (2i+1, 2j)\}$ and $\{(2i+1, 2j+1), (2i+2, 2j+1)\}$ $(0 \leq i \leq m-1, 0 \leq j \leq k)$ and add



(a) $H_{2m,2k}$



(b) $H_{2m+1,2k+1}$

Figure 1. Hexagonal Möbius graphs.

the edges $\{(n,0),(2m-n,2k)\}$ with $0 \le n \le 2m$, denoted by $H_{2m+1,2k+1}$ (see Figure 1(b)).

The graph $H_{2m,2k}$ (or $H_{2m+1,2k+1}$) can be seen as a hexagonal tiling of a Möbius strip. Every hexagon of the tiling is called a *face cycle* of $H_{2m,2k}$ (or $H_{2m+1,2k+1}$).

The *quadrilateral Möbius graph* of length q and breadth p is obtained from $P_p \Box P_q$ by adding the edges $\{(i,0),(p-1-i,q-1)\}$ with $0 \le i \le p-1$, denoted by $Q_{p,q}$ (see Figure 2). The graph $Q_{p,q}$ can be seen as a quadrilateral tiling of a Möbius strip. When $p=1$, $Q_{p,q}$ is exactly a cycle and it is seen as a degenerative quadrilateral Möbius graph. Every quadrilateral of the tiling is called a *face cycle* of $Q_{p,q}$.

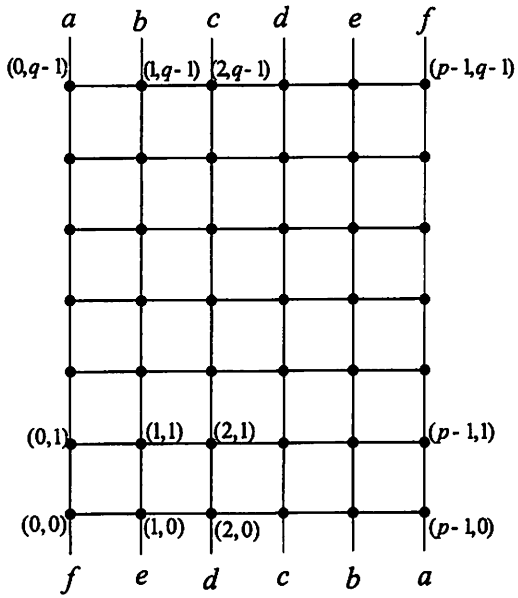


Figure 2. Quadrilateral Möbius graph $Q_{p,q}$.

Some of these structures such as $H_{2,2k}$, $H_{3,2k+1}$ and $Q_{p,q}$ appeared in [13, 18] to classify the hexagonal tilings or the quadrilateral tilings of the torus and the Klein bottle. In this note, our results show that in $H_{2m,2k}, H_{2m+1,2k+1}$ and $Q_{p,q}$ only $H_{2,2}$, $H_{3,3}$ and $Q_{2,2}$ are l_1-graphs.

2 Labels of l_1-graphs

We introduce the labels on l_1-graphs firstly used in [9, 17] and later in [10, 12, 15].

An n-dimensional hypercube Q_n can also be constructed as follows: Let $\Omega = \{1, 2, \ldots, n\}$. The vertices of Q_n are all subsets of Ω. Two vertices A and B are adjacent if and only if $|A \Delta B| = 1$, where Δ denotes the symmetric difference of sets, i.e., $A \Delta B = (A \setminus B) \cup (B \setminus A)$. Then the distance between any two vertices A and B in Q_n is equal to $|A \Delta B|$.

Let G be a finite l_1-graph and ϕ be a scale λ embedding of G in Q_n. Now we assign to each edge uv of G a *label* $l(uv)$ as follows: $l(uv) = \phi(u) \Delta \phi(v)$. For each edge $e = uv$ of G, $|\phi(u) \Delta \phi(v)| = d_{Q_n}(\phi(u), \phi(v)) = \lambda \cdot d_G(u, v) = \lambda$. We see that every edge label consists of precisely λ elements from {1,2,..., n}. The following two useful lemmas about labels can be proved by utilizing the associativity and commutativity of symmetric difference.

Lemma 2.1. *[9, 15] Let v_0, v_n be two vertices of an l_1-graph G and ϕ a scale λ embedding of G into a hypercube. The following statements hold:*

1. *If $\gamma = uu_1u_2 \ldots u_{k-1}v$ is a path from u to v, then $\phi(u) \Delta \phi(v) = l(uu_1) \Delta l(u_1u_2) \Delta \ldots$ $\Delta l(u_{k-1}v)$, and*

2. *If γ is geodesic, then the labels $l(uu_1), l(u_1u_2), \ldots, l(u_{k-1}v)$ are pairwise disjoint and $\phi(u) \Delta \phi(v) = l(uu_1) \cup l(u_1u_2) \cup \ldots \cup l(u_{k-1}v)$. In particular, every edge label on every shortest path from u to v is contained in $\phi(u) \Delta \phi(v)$.*

A subgraph H of G is called *isometric* if $d_H(u, v) = d_G(u, v)$ for any two vertices u, v of H. Let $C_k = v_1v_2 \ldots v_kv_1$ be a cycle. Two edges v_iv_{i+1} and v_jv_{j+1} of C_k with $1 \leq i, j \leq k$ are *opposite* if $d_{C_k}(v_i, v_j) = d_{C_k}(v_{i+1}, v_{j+1})$ and equal to the diameter of C_k, where $v_{k+1} = v_1$. Thus, every edge has a unique opposite edge if k is even; and has two opposite edges, otherwise.

Lemma 2.2. *[9, 15] Suppose that C_k is an isometric cycle of G and uv and xy are a pair of opposite edges of C_k. If k is even, then $l(uv) = l(xy)$, while if k is odd, then $|l(xy) \cap l(uv)| = \frac{\lambda}{2}$. Furthermore if k is odd and vw is the second edge opposite to xy then $l(xy) \subset l(uv) \cup l(vw)$. The labels of nonopposite edges are disjoint.*

Lemma 2.3. *If a simple graph G is an l_1-graph, then the labels on adjacent edges of G are never equal.*

Proof. Let $e_1 = uv$ and $e_2 = vw$ be two adjacent edges of G and ϕ a scale λ embedding of G into a hypercube. Since G is simple, $1 \leq d(u, w) \leq$

2. Suppose to the contrary that $l(uv) = l(vw)$, then by Lemma 2.1(1), $\lambda d(u,w) = |\phi(u)\Delta\phi(w)| = |l(uv)\Delta l(vw)| = |\emptyset| = 0$. Therefore $d(uw) = 0$, a contradiction. □

3 l_1-embeddability of hexagonal Möbius graphs

A necessary condition for l_1-embeddability of graphs was given in [4]. We introduce it as the following lemma.

Lemma 3.1. *[4] For a graph G, if it is an l_1-graph, d_G must satisfy the following 5-gonal inequality: for any five vertices x, y, a, b, c of G,*

$$\begin{aligned} d(x,y) + (d(a,b) + d(a,c) + d(b,c)) \leq & \ (d(x,a) + d(x,b) + d(x,c)) \\ & + (d(y,a) + d(y,b) + d(y,c)). \end{aligned}$$

Recall that a (finite) planar graph G is *outerplanar* if there is an embedding of G in the Euclidean plane such that all vertices of G lie on the exterior face. Chepoi, Deza and Grishukhin proved in [2]:

Lemma 3.2. *Any outerplanar graph is an l_1-graph.*

Denote K_n the complete graph on n vertices. The set of neighbors of a vertex x in G is denoted by $N_G(x)$.

Theorem 3.3. *$H_{2m,2k}$ with $m \geq 1$ and $k \geq 1$ is l_1 if and only if $m = k = 1$.*

Proof. Since the graph $H_{2,2}$ is an outerplanar graph, by Lemma 3.2, it is an l_1-graph and its scale 2 embedding into Q_4 is shown in Figure 3. For any of the other graphs $H_{2m,2k}$, we either find its five vertices which do not satisfy the 5-gonal inequality, or show that its edges do not possess l_1-labels by reductio ad absurdum.

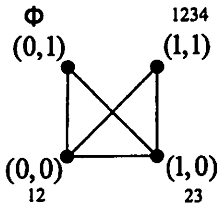


Figure 3. $H_{2,2}$.

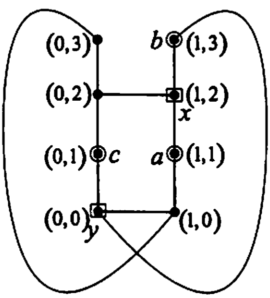


Figure 4. $H_{2,4}$.

Next we prove the theorem by distinguishing several cases with respect to the values of m and k.

Case 1. $m = 1$.

1. $k = 1$. See Figure 3.

   It is clear that $H_{2,2}$ is isomorphic to $K_4 - e$, which stands for the graph obtained from K_4 by deleting one edge e. It has been pointed out above that it is an l_1-graph.

2. $k = 2$. See Figure 4.

   Let $x = (1,2), y = (0,0), a = (1,1), b = (1,3)$ and $c = (0,1)$. Then $d(a,b) = 2, d(a,c) = 3, d(b,c) = 2, d(x,y) = 2$ and $d(x,a) = 1, d(x,b) = 1, d(x,c) = 2, d(y,a) = 2, d(y,b) = 1, d(y,c) = 1$.

   So $d(x,y) + (d(a,b) + d(a,c) + d(b,c)) = 9$, while $(d(x,a) + d(x,b) + d(x,c)) + (d(y,a) + d(y,b) + d(y,c)) = 8$. These five vertices violate the 5-gonal inequality in Lemma 3.1. Hence $H_{2,4}$ is not an l_1-graph.

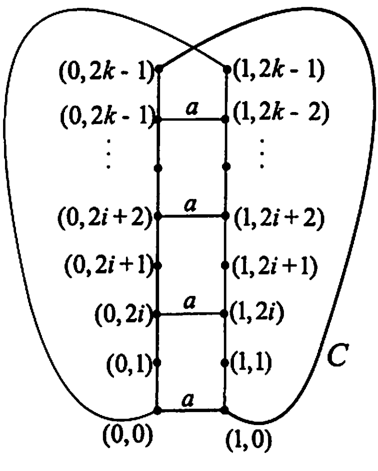


Figure 5. $H_{2,2k}$, $\alpha = l((0,0)(1,0))$.

3. $k \geq 3$. See Figure 5.

   Firstly, we show that the cycle $C = (1,0)(0,0)(0,1)(0,2)\ldots(0,2k-1)$ $(1,0)$ is an isometric cycle in $H_{2,2k}$. If not, there exist two vertices x, y such that $d_C(x,y) > d_{H_{2,2k}}(x,y)$. Take such a pair of vertices x, y that $d_{H_{2,2k}}(x,y)$ is as small as possible. Then any shortest x,y-path P in $H_{2,2k}$ intersects C only at x and y. Hence $d_{H_{2,2k}}(x) = d_{H_{2,2k}}(y) = 3$.

   If $x = (1,0)$ and $y = (0,2i)$ with $1 \leq i \leq k-1$. According to the choice of P in $H_{2,2k}$, $P = (1,0)(1,1)(1,2)\ldots(1,2i)(0,2i)$. The length of P is $2i+1$. The path $(1,0)(0,0)(0,1)(0,2)\ldots(0,2i)$ is a path joining x to y in C. So $d_C(x,y) \leq 2i+1$. This contradicts that $d_C(x,y) > d_{H_{2,2k}}(x,y)$.

If $x = (0,0)$ and $y = (0,2i)$, with $1 \le i \le k-1$. According to the choice of P in $H_{2,2k}$, $P = (0,0)(1,2k-1)(1,2k-2)\dots(1,2i)(0,2i)$. The length of P is $2k-2i+1$. The path $(0,0)(1,0)(0,2k-1)(0,2k-2)\dots(0,2i)$ is a path joining x and y in C. So $d_C(x,y) \le 2k-2i+1$. This contradicts that $d_C(x,y) > d_{H_{2,2k}}(x,y)$.

If both x and y are different from $(1,0)$ or $(0,0)$, without loss of generality, suppose that $x = (0,2i)$ and $y = (0,2j)$ with $i < j$. Then $(1,2i)$ and $(1,2j)$ must lie on P. According to the choice of P in $H_{2,2k}$, $P = (0,2i)(1,2i)(1,2i+1)(1,2i+2)\dots(1,2j)(0,2j)$. The length of P is $2j-2i+2$. The path $(0,2i)(0,2i+1)(0,2i+2)\dots(0,2j)$ is a path joining x to y in C. So $d_C(x,y) \le 2j-2i$. This contradicts that $d_C(x,y) > d_{H_{2,2k}}(x,y)$.

Secondly, we show that each face cycle of $H_{2,2k}$ is isometric in $H_{2,2k}$. Taking any face cycle $C_i = (0,2i)(1,2i)(1,2i+1)(1,2i+2)(0,2i+2)(0,2i+1)(0,2i)$ with $0 \le i \le k-2$, for any two vertices whose distance is 2 in C_i, we see that they are at distance 2 in $H_{2,2k}$. Then we are sufficient to prove that the distance between two vertices whose distance is 3 in C_i is also 3 in $H_{2,2k}$. Without loss of generality, suppose that $x = (0,2i)$ and $y = (1,2i+2)$, $d_C(x,y) = 3$. Since $N_{H_{2,2k}}(x) \cap N_{H_{2,2k}}(y) = \emptyset$, $d_{H_{2,2k}}(x,y) \ge 3$. Hence $d_{H_{2,2k}}(x,y) = d_C(x,y) = 3$. A similar discussion to the face cycle $C_{k-1} = (0,2k-2)(0,2k-1)(1,0)(0,0)(1,2k-1)(1,2k-2)(0,2k-2)$, we know that C_{k-1} is isometric in $H_{2,2k}$.

Suppose that $H_{2,2k}$ is l_1-embeddable. Let $\alpha := l((0,0)(1,0))$. Then by Lemma 2.2, $l((0,2i)(1,2i)) = l((0,0)(1,0)) = \alpha$ with $1 \le i \le k-1$ and $l((0,0)(1,0)) \subset l((0,k-1)(0,k)) \cup l((0,k)(0,k+1))$ (in the cycle C).

If k is odd, by Lemma 2.2, $l((0,k-1)(1,k-1)) = l((0,0)(1,0)) \subset l((0,k-1)(0,k)) \cup l((0,k)(0,k+1))$ and by Lemma 2.1, we have that $d_{H_{2,2k}}((1,k-1),(0,k+1)) < 3$, but $d_{H_{2,2k}}((1,k-1),(0,k+1)) = 3$, a contradiction.

If k is even, by Lemma 2.2, $l((0,k)(1,k)) = l((0,0)(1,0)) \subset l((0,k-1)(0,k)) \cup l((0,k)(0,k+1))$ and $l((0,k)(1,k)) \cap l((0,k-1)(0,k)) \ne \emptyset$. The path $(0,k+1)(0,k)(1,k)(1,k-1)$ is a path connecting $(0,k+1)$ with $(1,k-1)$. Then by Lemma 2.1, $d_{H_{2,2k}}((0,k+1),(1,k-1)) < 3$, but $d_{H_{2,2k}}((0,k+1),(1,k-1)) = 3$, a contradiction.

Case 2. $m = 2$.

1. $k = 1$. see Figure 6.

Let $x = (1,0)$, $y = (3,0)$, $a = (0,1)$, $b = (2,1)$, and $c = (3,1)$. Then we obtain that: $d(a,b) = 3$, $d(a,c) = 2$, $d(b,c) = 3$, and $d(x,y) = 3$;

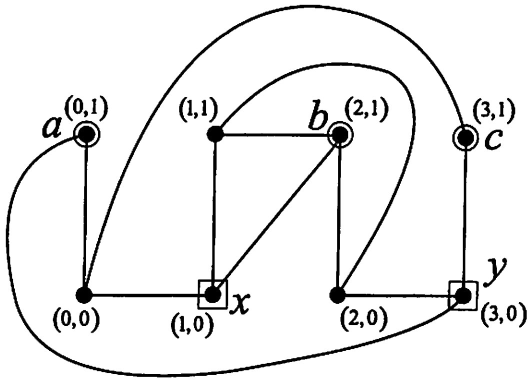


Figure 6. $H_{4,2}$

$d(x,a) = 2$, $d(x,b) = 1$, $d(x,c) = 2$, $d(y,a) = 1$, $d(y,b) = 2$ and $d(y,c) = 1$.
So

$$\begin{aligned} &d(x,y) + (d(a,b) + d(a,c) + d(b,c)) \\ &\quad = 3 + (3 + 2 + 3) \\ &\quad = 11, \end{aligned}$$

and

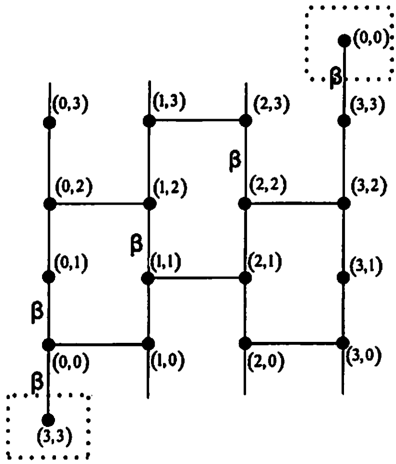
$$\begin{aligned} &(d(x,a) + d(x,b) + d(x,c)) + (d(y,a) + d(y,b) + d(y,c)) \\ &\quad = (2 + 1 + 2) + (1 + 2 + 1) \\ &\quad = 9. \end{aligned}$$

Then these five vertices violate the 5-gonal inequality in Lemma 3.1. So $H_{4,2}$ is not an l_1-graph.

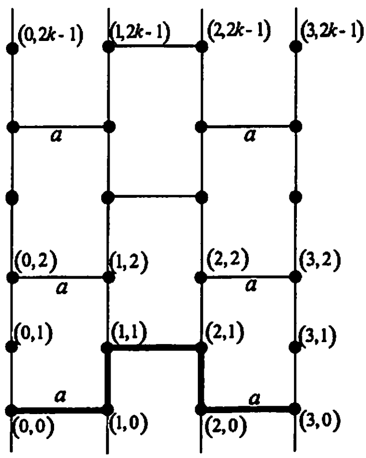
2. $k = 2$. see Figure 7(a). Suppose that $H_{4,4}$ is an l_1-graph. Let $\beta := l((0,0)(0,1))$. It's known that each face cycle is isometric in $H_{4,4}$. Then by Lemma 2.2, $l((3,3)(0,0)) = l((2,2)(2,3)) = l((1,1)(1,2)) = l((0,0)(0,1)) = \beta$. But the edges $(3,3)(0,0)$ and $(0,0)(0,1)$ are adjacent, a contradiction to Lemma 2.3.

3. $k \geq 3$. see Figure 7(b).

   The path $P = (0,0)(1,0)(1,1)(2,1)(2,0)(3,0)$ is a shortest path between $(0,0)$ and $(3,0)$. Suppose that $H_{4,2k}$ is an l_1-graph and let $\alpha =$



(a) $H_{4,4}$, $\beta = l((0,0)(1,0))$.



(b) $H_{4,2k}$, $\alpha = l((0,0)(1,0))$.

Figure 7.

$l((0,0)(1,0))$. Then by Lemma 2.2, we have that $l((0,2i)(1,2i)) = l((0,0)(1,0)) = \alpha$ $(1 \le i \le k-1)$ and $l((0,2k-2)(1,2k-2)) = l((2,0)(3,0))$. Hence $l((0,0)(1,0)) = l((2,0)(3,0)) = \alpha$, a contradiction to Lemma 2.1(2).

Case 3. $m \ge 3$.

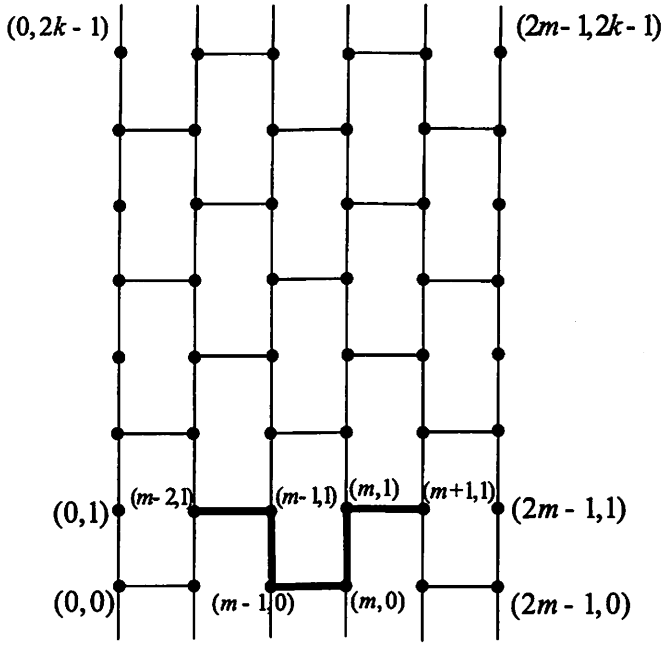


Figure 8. $H_{2m,2k}$, $m \ge 3$, m is odd.

1. m is odd. See Figure 8.

   The path $P = (m-2,1)(m-1,1)(m-1,0)(m,0)(m,1)(m+1,1)$ is a shortest path between $(m-2,1)$ and $(m+1,1)$. If $H_{2m,2k}$ is an l_1-graph, then by Lemma 2.2, $l((m-2,1)(m-1,1)) = l((m-2,2i+1)(m-1,2i+1))$ $(1 \le i \le k-1)$ and $l((m-2,2k-1)(m-1,2k-1)) = l((m,1)(m+1,1))$. Hence $l((m-2,1)(m-1,1)) = l((m,1)(m+1,1))$, a contradiction to Lemma 2.1(2).

2. m is even. See Figure 9.

   The path $P = (m-2,0)(m-1,0)(m-1,1)(m,1)(m,0)(m+1,0)$ is a shortest path between $(m-2,0)$ and $(m+1,0)$. If $H_{2m,2k}$ is an l_1-graph, then by Lemma 2.2, $l((m-2,0)(m-1,0)) = l((m-2,2i)(m-1,2i))$ with $1 \le i \le k-1$ and in the face cycle $(m-2,2k-2)(m-2,2k-1)(m+1,0)(m,0)(m-1,2k-1)(m-1,2k-2)(m-2,2k-2)$, $l((m-2,2k-2)(m-1,2k-2)) = l((m,0)(m+1,0))$. So $l((m-2,0)(m-1,0)) = l((m,0)(m+1,0))$, a contradiction to Lemma 2.1(2).

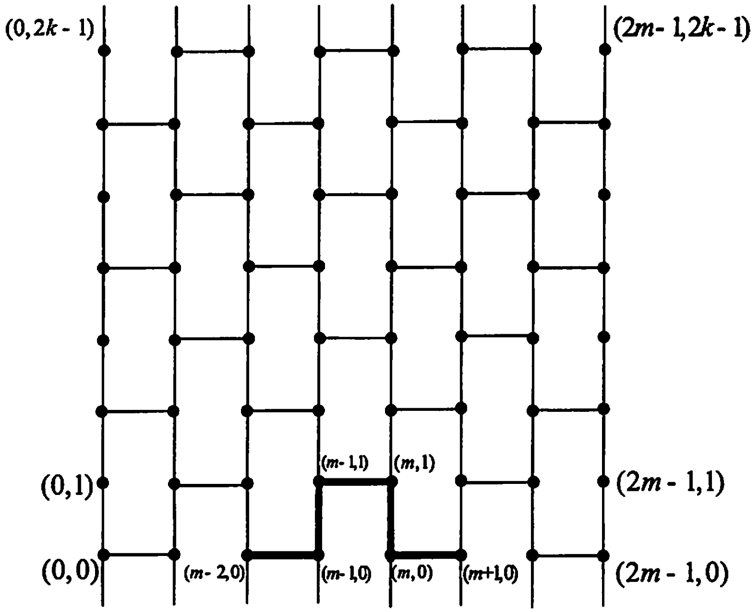


Figure 9. $H_{2m,2k}$, $m \ge 3$, m is even.

So far all cases have been studied and the proof is complete. □

Theorem 3.4. *$H_{2m+1,2k+1}$ is an l_1-graph if and only if $m = k = 1$.*

Proof. For the graph $H_{3,3}$, we use the recognition algorithm of l_1-graphs to determine that it is an l_1-graph. For any of the other graphs $H_{2m+1,2k+1}$, we use reductio ad absurdum to show that its edges do not possess l_1-labels.

To prove our results, like in Theorem 3.3, we classify the graphs $H_{2m+1,2k+1}$ into several cases according to the values of m and k.

Case 1. $m = 1$.

1. $k = 1$. See Figure 10(a). In fact, it is isomorphic to the graph shown in Figure 10(b). By the recognition algorithm of l_1-graphs described

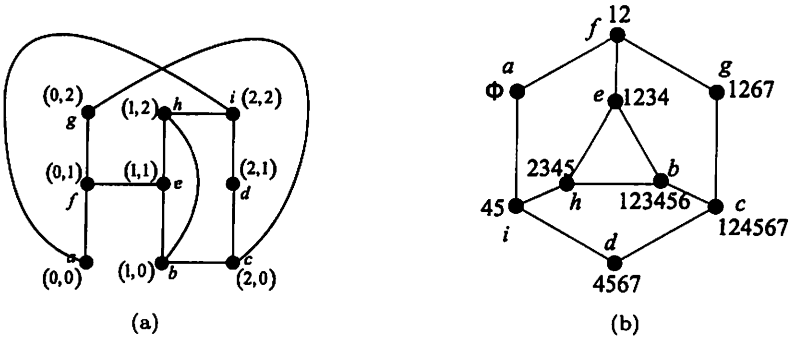


Figure 10. $H_{3,3}$.

in [9], we obtain that $H_{3,3}$ can be scale 2 embedded into Q_7 and the labels of vertices are also labeled out in Figure 10(b).

2. $k > 1$. See Figure 11.

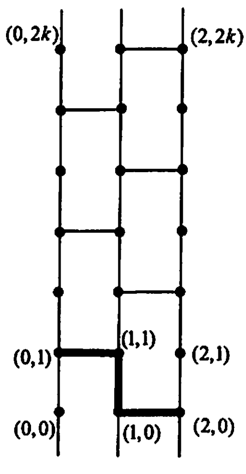


Figure 11. $H_{3,2k+1}$.

The path $P = (0,1)(1,1)(1,0)(2,0)$ is a shortest path between $(0,1)$ and $(2,0)$. If $H_{3,2k+1}$ is an l_1-graph, then by Lemma 2.2, $l((0,1)(1,1)) = l((0,2i+1)(1,2i+1))$ with $1 \leq i \leq k-1$. At the same time, in the face cycle $(0,2k-1)$ $(0,2k)$ $(2,0)$ $(1,0)$ $(1,2k)$ $(1,2k-1)$ $(0,2k-1)$, $l((0,2k-1)(1,2k-1)) = l((1,0)(2,0))$. So $l((0,1)(1,1)) = l((1,0)(2,0))$, a contradiction.

Case 2. $m \geq 2$.

1. m is odd. See Figure 12.

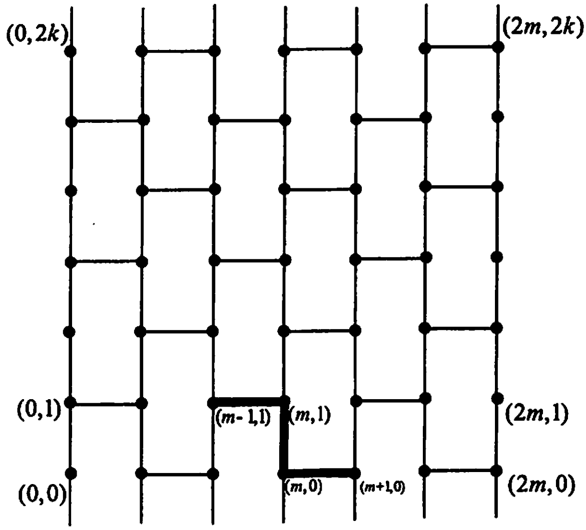


Figure 12. $H_{2m+1,2k+1}$, $m \geq 2$, m is odd.

The path $P = (m-1,1)(m,1)(m,0)(m+1,0)$ is a shortest path between $(m-1,1)$ and $(m+1,0)$. Suppose that $H_{2m+1,2k+1}$ is an l_1-graph. Then by Lemma 2.2, $l((m-1,1)(m,1)) = l((m-1,2i+1)(m,2i+1))$ with $1 \leq i \leq k-1$ and in the face cycle $(m-1,2k-1)(m-1,2k)(m+1,0)(m,0)(m,2k)(m,2k-1)(m-1,2k-1)$, $l((m-1,2k-1)(m,2k-1)) = l((m,0)(m+1,0))$. Hence $l((m-1,1)(m,1)) = l((m,0)(m+1,0))$, a contradiction to Lemma 2.1(2).

2. m is even. See Figure 13.

   The path $P = (m-1,0)(m,0)(m,1)(m+1,1)$ is a shortest path between $(m-1,0)$ and $(m+1,1)$. Suppose $H_{2m+1,2k+1}$ is an l_1-graph. Then $l((m-1,0)(m,0)) = l((m-1,2i)(m,2i))$ with $1 \leq i \leq k$ and $l((m-1,2k)(m,2k)) = l((m,1)(m+1,1))$. Hence $l((m-1,0)(m,0)) = l((m,1)(m+1,1))$, a contradiction to Lemma 2.1(2).

Up to now we have completed our proof and we can see that in $H_{2m+1,2k+1}$ only $H_{3,3}$ is l_1-embeddable. □

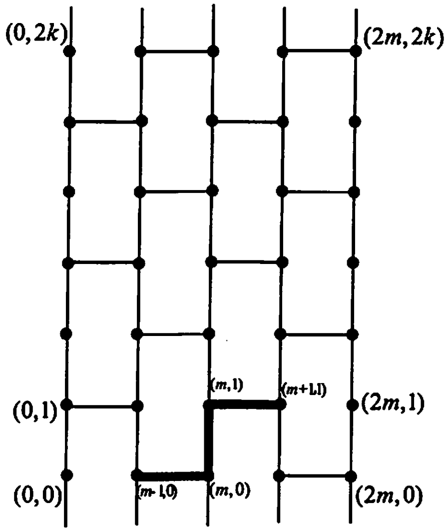


Figure 13. $H_{2m+1,2k+1}$, $m \geq 2$, m is even.

4 l_1-embeddability of quadrilateral Möbius graphs

In this section, we make a study of l_1-embeddability of the quadrilateral Möbius graphs. The methods of our proof are similar to the methods in Theorems 3.3 and 3.4.

Theorem 4.1. *$Q_{p,q}$ is an l_1-graph if and only if $p = q = 2$ or $p = 1$.*

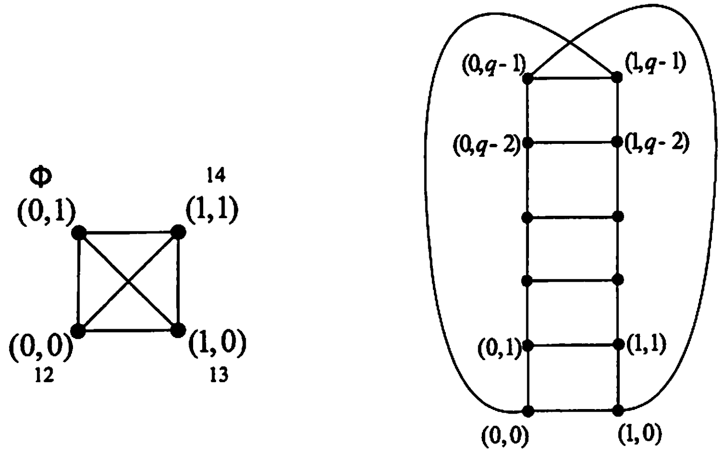
Proof. If $p = 1$, $Q_{p,q}$ is a cycle and it is evidently l_1-embeddable. The graphs $Q_{p,q}$ ($p \geq 2$) fall into two general types $p = 2$ and $p \geq 3$. For the first type we fix p and let q range. For the second type we further classify $Q_{p,q}$ into two classes with respect to the parity of p.
Case 1. $p = 2$.

1. $q = 2$. See Figure 14(a).

   Then $Q_{2,2}$ is isomorphic to K_4. It's known that K_4 is an l_1-graph [7] and a scale 2 embedding of $Q_{2,2}$ into Q_4 is given in Figure 14(a).

2. $q \geq 3$. See Figure 14(b).

   Firstly, we have the following claim: the face cycles $(0,i)(1,i)(1,i+1)(0,i+1)(0,i)$ with $0 \leq i \leq q-2$ and $C_{q-1} = (0,0)(1,0)(0,q-1)(1,q-1)(0,0)$ are isometric in $Q_{2,q}$. In fact, it is necessary to



(a) $Q_{2,2}$. (b) $Q_{2,q}$.

Figure 14.

consider the vertices whose distance is 2 in C_i and C_{q-1}. These vertices are not adjacent in $Q_{2,q}$. The claim is clear.

Similarly as shown in Theorem 3.3, the cycle $C = (1,0)\ (0,0)(0,1)\ (0,2)\dots(0,q-1)\ (1,0)$ is isometric in $Q_{2,q}$. The length of C is equal to $q+1$.

Suppose that $Q_{2,q}$ is an l_1-graph, then $l((0,0)(1,0)) = l((0,i)(1,i))$ for all $1 \le i \le q-1$.

If q is odd, then $l((0,\frac{q-1}{2})(0,\frac{q+1}{2})) = l((0,0)(1,0))$. Since $l((0,\frac{q-1}{2})\ (1,\frac{q-1}{2})) = l((0,0)(1,0))$, so $l((0,\frac{q-1}{2})(1,\frac{q-1}{2})) = l((0,\frac{q-1}{2})(0,\frac{q+1}{2}))$. But the two edges $(0,\frac{q-1}{2})(0,\frac{q+1}{2})$ and $(0,\frac{q-1}{2})(1,\frac{q-1}{2})$ are adjacent. This contradicts to Lemma 2.3.

If q is even, then by Lemma 2.2, $l((0,0)(1,0)) \subset l((0,\frac{q}{2}-1)(0,\frac{q}{2})) \cup l((0,\frac{q}{2})(0,\frac{q}{2}+1))$ and $l((0,0)(1,0)) \cap l((0,\frac{q}{2})(0,\frac{q}{2}+1)) \ne \emptyset$. Since $l((0,\frac{q}{2})(1,\frac{q}{2})) = l((0,0)(1,0))$, $l((0,\frac{q}{2})(1,\frac{q}{2})) \cap l((0,\frac{q}{2})(0,\frac{q}{2}+1)) \ne \emptyset$. By Lemma 2.1, this shows that $d((0,\frac{q}{2}+1),(1,\frac{q}{2})) < 2$, but $(0,\frac{q}{2}+1)$ is not adjacent to $(1,\frac{q}{2})$, a contradiction.

Case 2. $p \ge 3$.

1. p is odd. See Figure 15.

   The path $(\frac{p-3}{2},0)(\frac{p-1}{2},0)(\frac{p+1}{2},0)$ is a shortest path between $(\frac{p-3}{2},0)$ and $(\frac{p+1}{2},0)$. Suppose that $Q_{p,q}$ is an l_1-graph. Then by Lemma

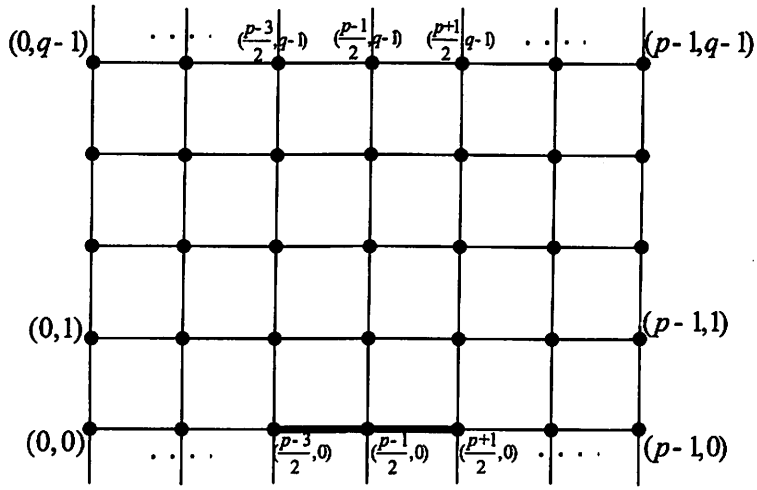


Figure 15. $Q_{p,q}$, p is odd.

2.2, $l((\frac{p-3}{2},0)(\frac{p-1}{2},0)) = l((\frac{p-3}{2},i)(\frac{p-1}{2},i))$ with $1 \le i \le q-1$ and $l((\frac{p-1}{2},q-1)(\frac{p+1}{2},q-1)) = l((\frac{p-1}{2},0)(\frac{p+1}{2},0))$. Therefore, $l((\frac{p-3}{2},0)(\frac{p-1}{2},0)) = l((\frac{p-1}{2},0)(\frac{p+1}{2},0))$ But the two edges $(\frac{p-3}{2},0)(\frac{p-1}{2},0)$ and $(\frac{p-1}{2},0)(\frac{p+1}{2},0)$ are adjacent, a contradiction to Lemma 2.3.

2. p is even. See Figure 16.
Let $x = (\frac{p}{2}-2,0)$, $z = (\frac{p}{2}-1,0)$, $w = (\frac{p}{2},0)$ and $y = (\frac{p}{2}+1,0)$. Suppose that $Q_{p,q}$ is scale λ embeddable into a hypercube. Let $\alpha := l(xz)$. Then by Lemma 2.2 $l((\frac{p}{2}-2,i)(\frac{p-1}{2},i)) = l(xz) = \alpha$ $(1 \le i \le q-1)$ and $l((\frac{p}{2}-2,q-1)(\frac{p-1}{2},q-1)) = l(wy)$. So $l(xz) = l(wy) = \alpha$. Then $|\phi(x)\Delta\phi(y)| = |l(xz)\Delta l(zw)\Delta l(wy)| = |l(zw)| = \lambda$. Therefore, x and y are adjacent. That is a contradiction and thus $Q_{p,q}$ is not an l_1-graph.

Summarizing the above two cases, we obtain that only $Q_{2,2}$ is l_1-embeddable in $Q_{p,q}$. □

5 Acknowledgements

The authors wish to thank Professor M. Deza for suggesting this study.

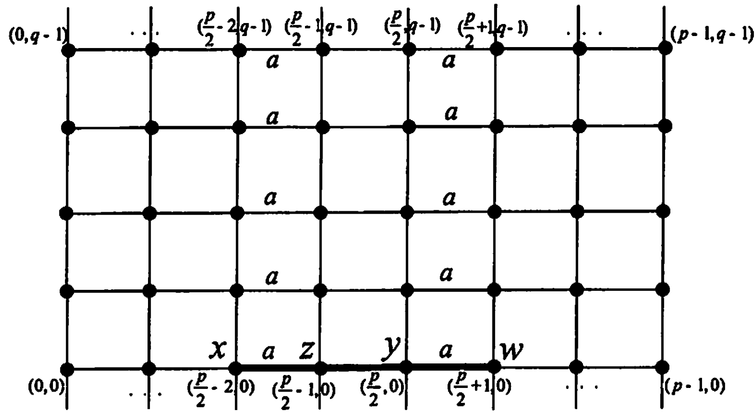


Figure 16. $Q_{p,q}$, p is even, $\alpha = l(xz)$.

References

[1] P. Assouad, M. Deza, Espaces metriques plongeables dans un hypercube: Aspects combinatoires, Ann. Discrete Math. 8 (1980) 197-210.

[2] V. Chepoi, M. Deza, V. Grishukhin, Clin d'oeil on L_1-embeddable planar graphs, Discrete Appl. Math. 80 (1997) 3-19.

[3] A. Deza, M. Deza and V. Grishukhin, Fullerenes and coordination polyhedra versus half-cube embeddings, Discrete Math. 192 (1998) 41-80.

[4] M. Deza (as M.E. Tylkin), On Hamming geometry of unitary cubes (in Russian), Dokl. Akad. Nauk SSSR 134 (1960) 1037-1040.

[5] M. Deza, V. Grishukhin, Hypermetric graphs, Quart. J. Math. Oxford (2) 44 (1993) 399-433.

[6] M. Deza, V. Grishukhin and M. Shtogrin, Scale-Isometric Polytopal Graphs in Hypercubes and Cubic Lattices, Imperial College Press, London (2004).

[7] M. Deza, M. Laurent, l_1-rigid graphs, J. Algebra Combin. 3 (1994) 153-175.

[8] M. Deza, M. Laurent, Geometry of Cuts and Metrics, Springer-Verlag, Berlin, 1997.

[9] M. Deza, S. Shpectorov, Recognition of the l_1-graphs with complexity $O(nm)$, or football in a hypercube, Europ. J. Combin. 17 (1996) 279-289.

[10] M. Deza, S. Shpectorov, Polyhexes that are l_1 graphs, European J. Combin. 30 (2009), 1090-1100.

[11] M. Deza, M.I. Shtogrin, Embeddings of chemical graphs in hypercubes, Mat. Zametki 68:3 (2000) 339-352; English tansl., Math. Notes. 68 (2000) 295-305.

[12] M. Deza, M.D. Sikiric, S. Shpectorov, Graphs 4_n that are isometrically embeddable in hypercubes, Southeast Asian Bull. Math. 29 (2005) 469-484.

[13] D. Garijo, I. Gitler, A. Márquez, M.P. Revuelta, Hexagonal tilings and locally C_6 graphs, arXiv:math/0512332v1, 14 Dec 2005.

[14] W. Imrich, S. Klavžar, Product Graphs: Structure and Recognition, John Wiley & Sons, New York, 2000.

[15] M. Marcušanu, The classification of l_1-embeddable fullerenes, Ph.D. Thesis, Bowling Green State University, 2007.

[16] Ch. Prisăcaru, P. Soltan, V. Chepoi, On embeddings of planar graphs into hypercubes, Proc. Moldavian Academy of Sci. Math. 1 (1990) 43-50 (in Russian).

[17] S.V. Shpectorov, On scale embeddings of graphs into hypercubes, European J. Combin. 14 (1993) 117-130.

[18] C. Thomassen, Tilings of the torus and the Klein bottle and vertex-transitive graphs on a fixed surface, Trans. Amer. Math. Soc. 323 (1991) 605-635.