Spanning Trees with Bounded Total Excess

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Abstract

Let $c(H)$ denote the number of components of a graph H. Win proved in 1989 that if a connected graph G satisfies

$$c(G \setminus S) \leq (k-2)|S| + 2, \text{ for every subset } S \text{ of } V(G),$$

then G has a spanning tree with maximum degree at most k.

For a spanning tree T of a connected graph, the k-excess of a vertex v is defined to be $\max\{0, \deg_T(v) - k\}$. The total k-excess $te(T,k)$ is the summation of the k-excesses of all vertices, namely,

$$te(T,k) = \textstyle\sum_{v \in V(T)} \max\{0, \deg_T(v) - k\}.$$

This paper gives a sufficient condition for a graph to have a spanning tree with bounded total k-excess. Our main result is as follows.

Suppose $k \geq 2$, $b \geq 0$, and G is a connected graph satisfying the following condition:

For every subset S of $V(G)$, $c(G \setminus S) \leq (k-2)|S| + 2 + b$.

Then, G has a spanning tree with total k-excess at most b.

1 Introduction

Let G be a graph, and let S be a subset of $V(G)$. The number of components in $G \setminus S$ is denoted by $c(G \setminus S)$. For a real number t, if $|S| \geq t \cdot c(G \setminus S)$

holds for every $S \subseteq V(G)$ with $c(G \setminus S) \geq 2$, then G is called t-tough. The maximum number t for which G is t-tough is the toughness of G. If G is a complete graph, its toughness is defined to be ∞.

The notion of toughness was introduced by Chvátal [6], related to the hamiltoncity of a graph. It is easy to see that every hamiltonian graph is 1-tough. Chvátal [6] conjectured that there exists a constant t_0 such that every t_0-tough graph is hamiltonian. Bauer, Broersma and Veldman [2] showed that if such a number t_0 exists, then $t_0 \geq \frac{9}{4}$.

A k-tree is a spanning tree whose maximum degree is less than or equal to k. Win[5] gave a sufficient condition for a graph G to contain a k-tree, in terms of $|S|$ and $c(G \setminus S)$.

Theorem 1 (Win, 1989 [5]) *Let k be an integer with $k \geq 2$. If G is a connected graph satisfying the following condition:*

$$\textit{For every subset } S \textit{ of } V(G),\ c(G \setminus S) \leq (k-2)|S| + 2.$$

Then, G has a k-tree.

For $k \geq 3$, this theorem implies that every $\frac{1}{k-2}$-tough graph has a k-tree.

In this paper, we consider what kind of spanning trees we can get if we replace the constant term in the inequality of the condition in Theorem 1. We give one answer to this problem, based on another proof of Theorem 1 by Ellingham and Zha [3]. We introduce the following notion.

Definition 1 *For a spanning tree T of a connected graph, we define the k-excess of a vertex v as* $\max\{0, \deg_T(v) - k\}$. *We define the total k-excess $te(T, k)$ as follows.*

$$te(T, k) = \sum_{v \in V(T)} \max\{0, \deg_T(v) - k\}$$

The main result in this paper is the following.

Theorem 2 *Suppose $k \geq 2$, $b \geq 0$, and G is a connected graph satisfying the following condition.*

$$\textit{For every subset } S \textit{ of } V(G),\ c(G \setminus S) \leq (k-2)|S| + b + 2.$$

Then, G has a spanning tree with total k-excess at most b.

2 Proof of Theorem 2

At first, we introduce a notion called bridge.

Definition 2 *For $S \subseteq V(G)$, an S-bridge of G is*

- *an edge both of whose ends are contained in S, or*
- *a subgraph consisting of a component C of $G \setminus S$ together with the edges joining S and C.*

If we assume the condition weaker than the one in Theorem 1, then we cannot avoid getting vertices with degree greater than k in a spanning tree.

A k-forest of G is a spanning subgraph of G which is a forest with maximum degree at most k. Take a k-forest F of G with the smallest number of components. Let r be the number of components in F.

Let $\mathcal{F}$ be the set of k-forests in G such that the vertex sets of the components coincide with the ones of F. For $S \subseteq V(G)$, let $\mathcal{F}(S)$ be the set of k-forests $F' \in \mathcal{F}$ such that the vertex sets of the S-bridges of F' coincide with those of the S-bridges of F. Let A_0 be the set of vertices which have degree k in all k-forests in $\mathcal{F}$. Let A_1 be the set of vertices which have degree k in all k-forests in $\mathcal{F}(A_0)$. In every tree in $\mathcal{F}(A_0)$, the degree of vertices in A_0 is k, therefore $A_0 \subseteq A_1$.

Claim 1 *Each edge of G which connects different components of $F \setminus A_0$ has an end vertex in A_1.*

Proof of Claim 1.

Let $uv \in E(G)$ be an edge which connects different components of $F \setminus A_0$. Then, for every $F' \in \mathcal{F}(A_0)$, u and v are contained in different components of $F' \setminus A_0$. Suppose $u \notin A_1$ and $v \notin A_1$. Then, there exist $F_1, F_2 \in \mathcal{F}(A_0)$ satisfying $\deg_{F_1}(u) < k$ and $\deg_{F_2}(v) < k$. By replacing the A_0-bridge in F_1 that contains v with the A_0-bridge in F_2 that contains v, we get another k-forest $F_3 \in \mathcal{F}(A_0)$ such that the degrees of u and v are less than k.

If there does not exist a (u, v)-path in F_3, $F_3 + uv$ is a k-forest of G with less number of components than F. This contradicts the minimality of F.

If there exists a (u, v)-path $F_3(u, v)$ in F_3, the path contains a vertex w of A_0. By adding uv, and removing one of the edges in $F_3(u, v)$ incident with w, we obtain a k-forest in $\mathcal{F}$ such that the degree of w is less than k. This contradicts the fact that $w \in A_0$. Therefore, we establish $u \in A_1$ or $v \in A_1$. Thus the proof of Claim 1 is completed.

To continue this inductively, we define A_{j+1} as the set of vertices which have degree k in all forests in $\mathcal{F}(A_j)$. Then we can show the following claim by the same argument as Claim 1.

Claim 2 *Each edge connecting different components of $F \setminus A_j$ has an end vertex in A_{j+1}.*

In $\mathcal{F}(A_j)$, degree of any vertex in A_j is constant, therefore $A_j \subseteq A_{j+1}$. Therefore, we get the following progression, where $D_k(F)$ is the set of all vertices whose degree is k in F.

$$A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots \subseteq A_j \subseteq \cdots \subseteq D_k(F)$$

Because $D_k(F)$ is a finite set, we get $A_m = A_{m+1}$ at some integer m. Then, by Claim 2, A_m has the property that any edge connecting different components of $F \setminus A_m$ has an end vertex in A_m. In other words, there is no edge of G connecting different components of $F \setminus A_m$. This implies that for $S = A_m$, we have $c(G \setminus S) = c(F \setminus S)$.

Let $r = c(F)$, and let s be the number of components in F which does not contain a vertex of S. If $r = 1$, then F is a desired k-tree. Assume $r \geq 2$. Then, since G is connected, we have $S \neq \phi$. Thus, we have $s + 1 \leq r$.

We shall construct a spanning tree of G by adding edges to F. At first, we add edges connecting a component C containing no vertices of S with another component C'. Note that C' must contain a vertex of S.

For otherwise, we first replace one or both of C and C' by a component on the same vertex set in which the end of the edge has degree less than k. Then by adding the edge, we would obtain a k-forest with the number of components fewer than F. Thus $C' \cap S \neq \phi$. Similarly in this case, we replace C if necessary, and add the edge connecting C and C'. At this point, the total k-excess increases by at most 1 for adding one edge. We repeat this produce until there is no component containing no vertices of S. Then the total k-excess increases by at most s. Next, we add edges between the components until only one component remains. The total k-excess increases by at most 2 for adding one edge. So, this operation increases the total k-excess by at most $2(r-s-1)$. Therefore, the total k-excess of the resulting spanning tree T is at most $2(r-1)-s$.

On the other hand, we can evaluate $c(F \setminus S)$ as follows. At first, the number of components in F is r. For each component of F containing a vertex of S, when we remove the first vertex in S, the number of components increases $k-1$, since the degree of this vertex is k. Then we remove vertices of S according to the distance from the first vertex. If the removing vertex is adjacent to the vertex already removed, then the number of components increases by $k-2$. Otherwise, the removal increases the number of components by $k-1$. Taking sum of them, we have $c(F \setminus S) \geq r + (k-2)|S| + r - s = (k-2)|S| + 2r - s$.

Therefore, by the condition of this theorem $c(G \setminus S) \leq (k-2)|S| + b + 2$, we obtain $(k-2)|S| + 2r - s \leq c(F \setminus S) = c(G \setminus S) \leq (k-2)|S| + b + 2$. So we have $2r - s \leq b + 2$. Thus the total k-excess of T is at most $2(r-1) - s \leq b$. □

3 Remarks

When the constant term b in the condition of Theorem 2 is negative, what kind of spanning trees does the graph contain? In [4], Ellingham, Nam and Voss proved the following result, which is a generalization of Win's theorem.

Theorem 3 ([4]) *Let G be a connected graph, and let h be a positive integer-valued function on $V(G)$. Then, G has a spanning tree T with $\deg_T(v) \leq h(v)$ for every $v \in V(G)$, if for every $S \subseteq V(G)$*

$$c(G \setminus S) \leq \sum_{v \in S} (h(v) - 2) + 2.$$

For a given subset $X \subseteq V(G)$ with $|X| = b$, define

$$h(v) = \begin{cases} k-1, & v \in X \\ k, & v \in V(G) \setminus X. \end{cases}$$

Suppose that G satisfies the following condition; for every nonempty subset $S \subseteq V(G)$,

$$c(G \setminus S) \leq (k-2)|S| + 2 - b.$$

Then,

$$c(G \setminus S) \leq (k-2)|S| + 2 - |S \cap X| \leq \sum_{v \in S} (h(v) - 2) + 2.$$

Thus, by Theorem 3, G has a k-tree in which the vertices in X have degree less than k.

Next, for a subset $X \subseteq V(G)$ with $|X| = b$, we consider the following function;

$$h(v) = \begin{cases} k+1, & v \in X \\ k, & v \in V(G) \setminus X. \end{cases}$$

By Theorem 3, if for every subset $S \subseteq V(G)$,

$$c(G \setminus S) \leq (k-2)|S| + 2 + |S \cap X|,$$

then G has a spanning $(k+1)$-tree T such that $\deg_T(x) \leq k$ for $x \in V(G) \setminus X$. In particular, G has a spanning tree T with $te(T,k) \leq b$. This condition is slightly stronger than the one in Theorem 2. Thus, Theorem 3 does not imply Theorem 2.

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