Unimodality of sequences associated to Pell numbers

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Abstract

In this paper we show that the sequences $p(n,k) := 2^{n-2k}\binom{n-k}{k}$ and $q(n,k) := 2^{n-2k}\frac{n}{n-k}\binom{n-k}{k}$, $k = 0,\dots,\lfloor\frac{n}{2}\rfloor$, are strictly log-concave and then unimodal with at most two consecutive modes. We localize the modes and the integers where there is a plateau. We also give a combinatorial interpretation of $p(n,k)$ and $q(n,k)$. These sequences are associated respectively to the Pell numbers and the Pell-Lucas numbers for which we give some trigonometric relations.

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1 Introduction

A finite positive sequence v_k is said to be unimodal if there exists an integer l such that $v_0 \le v_1 \le \cdots \le v_{l-1} \le v_l \ge v_{l+1} \ge \cdots \ge v_n$. The integer l is called a mode of the sequence (v_k). The sequence is logarithmically concave (log-concave for short) if $v_k^2 \ge v_{k-1}v_{k+1}$ for $1 \le k \le n-1$. Obviously, log-concavity is stronger than unimodality. Also if the sequence is strictly log-concave (SLC for short), i.e. if the previous inequalities are strict, then the sequences have at most two consecutive modes (a peak or a plateau).

The following Newton's Theorem, see [3], is usually used for the study of unimodality.

Theorem 1 *Let $(v_j)_{j=0}^n$ be a real sequence. If the polynomial $\sum_{j=0}^n v_j x^j$ has only real zeros, then*

$$v_j^2 \geq \frac{n-j+1}{n-j}\frac{j+1}{j} v_{j-1} v_{j+1}, \text{ for } 1 \leq j \leq n-1.$$

As a consequence, if $(v_j)_{j=0}^n$ is positive and satisfies the hypothesis of the previous Theorem then it is SLC. In this case, its smallest mode r satisfies $v_{r+1} - v_r \leq 0$ and $v_r - v_{r-1} > 0$ and the sequence $(v_j)_{j=0}^n$ admits a plateau of two elements $\{r, r+1\}$ if and only if we have $v_{r+1} - v_r = 0$.

Let (F_n), (L_n), (P_n) and (Q_n) be the sequences defined, for $n \in \mathbb{N}$, by

$$F_n = F_{n-1} + F_{n-2},\ F_0 = 0,\ F_1 = 1, \quad P_n = 2P_{n-1} + P_{n-2},\ P_0 = 0,\ P_1 = 1,$$
$$L_n = L_{n-1} + L_{n-2},\ L_0 = 2,\ L_1 = 1, \quad Q_n = 2Q_{n-1} + Q_{n-2},\ Q_0 = 2,\ Q_1 = 2.$$

(F_n) is the sequence of Fibonacci numbers (SLOANE A000045), (L_n) the sequence of Lucas numbers (SLOANE A000032), (P_n) the sequence of Pell numbers (SLOANE A000129), and (Q_n) the sequence of Pell-Lucas numbers (SLOANE A002203). We have

$(F_n) = (0, 1, 1, 2, 3, 5, 8, 13, \ldots)$, $(L_n) = (2, 1, 3, 4, 7, 11, 18, 29, \ldots)$,
$(P_n) = (0, 1, 2, 5, 12, 29, 70, \ldots)$, $(Q_n) = (2, 2, 6, 14, 34, 82, 198, \ldots)$.

We can prove easily the following Binet forms:

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right) \text{ and } L_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n,$$
$$P_n = \frac{1}{2\sqrt{2}}\left((1+\sqrt{2})^n - (1-\sqrt{2})^n\right) \text{ and } Q_n = (1+\sqrt{2})^n + (1-\sqrt{2})^n.$$

Tanny and Zuker [10] and Benoumhani [1] establish, respectively, the unimodality of the sequences $\binom{n-k}{k}$ and $\frac{n}{n-k}\binom{n-k}{k}$. They determined explicitly there modes and the integers where there is a plateau. These two sequences are related to the Fibonacci and Lucas numbers by the relations

$$F_{n+1} = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} \ (n \geq 0) \text{ and } L_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{n-k}\binom{n-k}{k} \ (n \geq 1).$$

The combinatorial significations of $\binom{n-k}{k}$ and $\frac{n}{n-k}\binom{n-k}{k}$ are known, see Graham and al [2, p. 310], Riordan [6, p. 198], Stanley [9, p. 73], and Sloane [8, A034807].

In a dual way, we prove the unimodality of sequences $p(n,k)$ and $q(n,k)$. These two sequences are related respectively to the Pell and Pell-Lucas numbers by the relations

$$P_{n+1} = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} 2^{n-2k}\binom{n-k}{k} \text{ and } Q_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} 2^{n-2k}\frac{n}{n-k}\binom{n-k}{k}.$$

Our main results are the following two Theorems and the combinatorial interpretation of $p(n,k)$ and $q(n,k)$ given in section two;

Theorem 2 *The sequences* $(p(n,k))$ *and* $(q(n,k))$, $k = 0,\dots,\lfloor \frac{n}{2} \rfloor$, *are unimodal with at most two consecutive modes. Their respective smallest modes* r_n *and* s_n *are*

$$r_n = \left\lceil \frac{4n-3-\sqrt{8n^2+16n+9}}{8} \right\rceil \quad \textit{and } s_n = \left\lceil \frac{4n-5-\sqrt{8n^2-7}}{8} \right\rceil .$$

They satisfy $r_n, s_n \in \left\{ \left\lfloor \frac{n}{2}\left(1-\frac{\sqrt{2}}{2}\right) \right\rfloor, \left\lceil \frac{n}{2}\left(1-\frac{\sqrt{2}}{2}\right) \right\rceil \right\}$.

By setting, $\alpha_m := \frac{1}{2}P_{4m} - 1$, $\beta_m := \frac{1}{16}(Q_{4m-1} - 14)$, $\gamma_m := \frac{1}{2}(Q_{2m} - (-1)^m P_{2m})$, *and* $\delta_m := \frac{1}{16}(8P_{2m-1} - (-1)^m Q_{2m-1} - 10)$, *the integers* n *for which these sequences admit a plateau* $\{r_n, r_n+1\}$ *and* $\{s_n, s_n+1\}$ *are respectively* $n = \alpha_m$ *and* $n = \gamma_m$ *with* $r_n = \beta_m$ *and* $s_n = \delta_m$, *for* $m \geq 1$.

The sequences (α_m), (β_m), (γ_m), (δ_m) satisfy the following recurrent relations $\alpha_m = 34\alpha_{m-1} - \alpha_{m-2} + 32$, $\beta_m = 34\beta_{m-1} - \beta_{m-2} + 28$, $\gamma_m = 34\gamma_{m-2} - \gamma_{m-4}$, and $\delta_m = 34\delta_{m-2} - \delta_{m-4} + 20$. These recurrent relations and the computation of the first terms show that these sequences are sequences of integers.

The first values of n for which the sequences admit a plateau are respectively

m	$n = \alpha_m$	$r_n = \beta_m$
1	5	0
2	203	29
3	6 929	1 014
4	235 415	34 475
5	7 997 213	1 171 164
6	271 669 859	39 785 129
7	9 228 778 025	1 351 523 250

and

m	$n = \gamma_m$	$s_n = \delta_m$
1	11	1
2	134	19
3	373	54
4	4 552	666
5	12 671	1 855
6	154 634	22 645
7	430 441	63 036

Melham [5] gives sums involving Fibonacci, Lucas, Pell and Pell-Lucas numbers. We establish complementary relations in the same vein:

Theorem 3 *For* $m > n \geq 0$, *we have*

1. $\tan^{-1}\left(\frac{P_n}{P_m}\right) + \tan^{-1}\left(\frac{Q_n}{Q_m}\right) = \tan^{-1}\left(\frac{2}{Q_{m-n}}\right)$, *for* $m - n \equiv 1 \bmod 2$.

2. $\tan^{-1}(P_{2n+2}) - \tan^{-1}(P_{2n}) = \tan^{-1}\left(\frac{2}{P_{2n+1}}\right)$ *and* $\sum_{j\geq 0} \tan^{-1}\left(\frac{2}{P_{2j+1}}\right) = \frac{\pi}{2}$.

3. $\frac{1}{2}\tan^{-1}\left(\frac{Q_{2n+1}}{2}\right) - \frac{1}{2}\tan^{-1}\left(\frac{Q_{2n-1}}{2}\right) = \tan^{-1}\left(\frac{2}{Q_{2n}}\right)$, $n \geq 1$ *and* $\sum_{j\geq 0}\tan^{-1}\left(\frac{2}{Q_{2j}}\right) = \frac{3\pi}{8}$.

As a consequence of the first relation of Theorem 3, we obtain

$$\tan^{-1}\left(\frac{P_n}{P_{n+1}}\right) + \tan^{-1}\left(\frac{Q_n}{Q_{n+1}}\right) = \frac{\pi}{4}.$$

This relation can also be deduced by adding the relations (2.24) and (2.25) given in Melham [5].

The infinite sums given by 2. and 3. of Theorem 3 are similar to the following result of Lehmer [4]: $\sum_{j\geq 0}\tan^{-1}\left(\frac{1}{F_{2j+1}}\right) = \frac{\pi}{2}$.

2 Combinatorial interpretation

Among the main results, we give a combinatorial interpretation of $p(n,k)$ and $q(n,k)$.

Theorem 4

1. *The number of words formed with the letters A, B, C and D, of length n, beginning with k consecutive A's and containing exactly k times the letter B is equal to* $p(n,k)$.
2. *The number of ways to arrange k adults and* $n-k$ *children, among whom at least k are boys, around a table in such a way the left neighbor of each adult is a boy, is equal to* $q(n,k)$.

Proof. The first interpretation is obtained by a simple combinatorial computation.

For the second one, label the places in the table $1, 2, \ldots, n$ in the clockwise order. We wish to place k adults no two consecutive. Place the $n-k$ children. When the place labeled 1 is occupied by a child, we insert k adults into the $n-k$ spaces between the children in $\binom{n-k}{k}$ ways. If not, place the $k-1$ adults into the $n-k-1$ spaces between the children (we omit the places labeled 2 and n adjacent to the labeled place 1) in $\binom{n-k}{k}$ ways. Hence, we obtain $\binom{n-k}{k} + \binom{n-k}{k} = \frac{n}{n-k}\binom{n-k}{k}$ possibilities. For the places occupied by the children, we have at least k boys, then $(n-k)-k$ can be occupied by a boy or a girl, which gives 2^{n-2k} possibilities, which clearly establishes the result. ■

For n even, the sequence $(-1)^{n-k}p(n,k)$ is considered in Sloane [8, A117438].

3 Proof of Theorem 2 and Theorem 3

We need the following lemmas.

Lemma 5 *We have*

1. $S_1 := \{(x,y) \in \mathbb{N}^2 \mid x^2 - 2y^2 = 1\} = \{(\frac{1}{2}Q_{2m}, P_{2m}) \mid m \in \mathbb{N}\}$,
2. $S_2 := \{(x,y) \in \mathbb{N}^2 \mid x^2 - 8y^2 = 1\} = \{(\frac{1}{2}Q_{2m}, \frac{1}{2}P_{2m}) \mid m \in \mathbb{N}\}$,
3. $S_3 := \{(x,y) \in \mathbb{N}^2 \mid x^2 - 8y^2 = 1 \text{ and } x \equiv 1\,(mod\,4)\}$
   $= \{(\frac{1}{2}Q_{4m}, \frac{1}{2}P_{4m}) \mid m \in \mathbb{N}\}$,
4. $S_4 := \{(x,y) \in \mathbb{N}^2 \mid x^2 - 8y^2 = -7 \text{ and } x \equiv -1\,(mod\,4)\}$
   $= \left\{\left(4P_{2m} - \frac{(-1)^m}{2}Q_{2m}, \frac{1}{2}Q_{2m} - \frac{(-1)^m}{2}P_{2m}\right) \mid m \in \mathbb{N}^*\right\}$

Proof. 1. is well known (see [11]). 2. is a consequence of 1. using the fact that $(x,y) \in S_2 \Leftrightarrow (x,2y) \in S_1$ with P_{2m} even. 3. use $\frac{1}{2}Q_{2m} \equiv 1\,(mod\,4) \Leftrightarrow m \equiv 0\,(mod\,2)$. 4. For $\varepsilon \in \{-1,1\}$ we have $x^2 - 8y^2 = -7 \Leftrightarrow \left(y + \varepsilon\frac{x+\varepsilon y}{7}\right)^2 - 8\left(\frac{x+\varepsilon y}{7}\right)^2 = 1$. In addition for $(x,y) \in S_4$, we have $x \geq y \geq \frac{x-y}{7}$ and $x \pm y \equiv 0\,(mod\,7)$, and then $\frac{x+\varepsilon y}{7} \geq 0$ and $y + \varepsilon\frac{x+\varepsilon y}{7} \geq 0$ for $\varepsilon = \pm 1$. Thus $(x,y) \in S_4$ if and only if we can find $\varepsilon \in \{-1,1\}$ such that $x \equiv -1\,(mod\,4)$ and $\left(y + \varepsilon\frac{x+\varepsilon y}{7}, \frac{x+\varepsilon y}{7}\right) \in S_2$, using 2., this is equivalent to $x = 4P_{2m} - \frac{\varepsilon}{2}Q_{2m}$, $y = \frac{1}{2}(Q_{2m} - \varepsilon P_{2m})$, $(x,y) \in \mathbb{N}$ and $x \equiv -1\,(mod\,4)$. Since Q_{2m} and P_{2m} are even, necessarily x and y are integers. The condition $x \equiv -1\,(mod\,4)$ is equivalent to $\varepsilon \equiv \frac{1}{2}Q_{2m}\,(mod\,4)$. Using Binet formula, we obtain $\frac{1}{2}Q_{2m} \equiv 1 - 2m \equiv (-1)^m\,(mod\,4)$, which gives $\varepsilon = (-1)^m$. We verify then that x and y belong to $\mathbb{N}$ only for $m \geq 1$. ■

Lemma 6 *For $m, n \in \mathbb{N}$, we have*

1. $4P_n = Q_n + Q_{n-1}$ *and* $Q_n = 2P_n + 2P_{n-1}$, *for* $n \geq 1$.
2. $P_mQ_n + P_nQ_m = 2P_{n+m}$ *and* $P_mQ_m + (-1)^{m-n}P_nQ_n = Q_{m-n}P_{m+n}$, *for* $m \geq n$.
3. $P_{2n+1}^2 = 1 + P_{2n}P_{2n+2}$ *and* $Q_{2n}^2 = 8 + Q_{2n-1}Q_{2n+1}$, *for* $n \geq 1$.

Proof. Use Binet forms. ■

Let $F_n(x)$, $L_n(x)$, $P_n(x)$ and $Q_n(x)$ be the generating polynomials of the sequences $\binom{n-k}{k}$, $\frac{n}{n-k}\binom{n-k}{k}$, $p(n,k)$ and $q(n,k)$, $k = 0, 1, \ldots, \lfloor\frac{n}{2}\rfloor$ respectively. The values of these polynomials at $x = 1$ are respectively F_{n+1}, L_n, P_{n+1} and Q_n.

Benoumhani [1], established the reality of zeros of polynomials $F_n(x)$ and $L_n(x)$. We obtain the reality of zeros of polynomials $P_n(x)$ and $Q_n(x)$, using the relations

$$P_n(x) := \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} 2^{n-2k} \binom{n-k}{k} x^k = 2^n F_n\left(\frac{x}{4}\right),$$
$$Q_n(x) := \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} 2^{n-2k} \frac{n}{n-k} \binom{n-k}{k} x^k = 2^n L_n\left(\frac{x}{4}\right).$$

Their zeros are respectively $z_k = -(1 + \tan^2 \frac{k\pi}{n+1})$ and $z'_k = -(1 + \tan^2 \frac{(2k-1)\pi}{2n})$, for $k = 1, \ldots, \lfloor \frac{n}{2} \rfloor$.

The sequences $p(n,k)$ and $q(n,k)$ are SLC by Theorem 1, then unimodal with a peak or a plateau with two elements. For all integer $k = 0, \ldots, \lfloor \frac{n}{2} \rfloor - 1$, we have $p(n,k+1) - p(n,k) = 2^{n-2k-2} \frac{(n-k-1)!}{(k+1)!(n-2k)!} A_n(k)$ and $q(n,k+1) - q(n,k) = 2^{n-2k-2} \frac{(n-k-2)!n}{(k+1)!(n-2k)!} B_n(k)$, where $A_n(x) = 8x^2 - 2(4n-3)x + n(n-5)$ and $B_n(x) = 8x^2 - 2(4n-5)x + n^2 - 5n + 4$. The signs of $(p(n,k+1) - p(n,k))$ and $(q(n,k+1) - q(n,k))$ are the same of $A_n(k)$ and $B_n(k)$ respectively. For $n > 5$, we have $A_n(0) = n(n-5) > 0$, $A_n\left(\frac{n}{2}\right) = -n(n+2) < 0$, $B_n(0) = n(n-5) + 4 > 0$, $B_n\left(\frac{n}{2}\right) = 4 - n^2 < 0$. We deduce that each of $A_n(x)$ and $B_n(x)$ have a unique root in the interval $\left[0, \frac{n}{2}\right]$. By setting them $h(n)$ and $l(n)$ respectively, we have $r_n = \lceil h(n) \rceil = \left\lceil \frac{4n-3-\sqrt{8n^2+16n+9}}{8} \right\rceil$ and $s_n = \lceil l(n) \rceil = \left\lceil \frac{4n-5-\sqrt{8n^2-7}}{8} \right\rceil$.

The sequence $(p(n,k))$ admits a plateau if and only if $h(n) \in \mathbb{N}$, which is equivalent to the existence of an integer $x \geq 0$ satisfying $8n^2 + 16n + 9 = x^2$ and $4n - 3 - x \equiv 0\,(mod\,8)$. These conditions can be written as follows $(x, n+1) \in S_3$ and $4n - 3 - x \equiv 0\,(mod\,8)$. The Lemma 5 gives $x = \frac{1}{2}Q_{4m}$ and $n = \frac{1}{2}P_{4m} - 1$, with $m \geq 1$. By using the first relation of Lemma 6, we then have $4n - 3 - x = 8\beta_m \equiv 0\,(mod\,8)$. All integers $m \geq 1$, answer to the question and $r_n = \beta_m$. We can do the same for the sequence $(q(n,k))$.

For the proof of Theorem 3: The first relation follows from the well known relations $\tan^{-1}(x) + \tan^{-1}(y) = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$ if $xy < 1$ and $\tan^{-1}(x) - \tan^{-1}(y) = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$ if $xy > -1$, with $x = \frac{P_n}{P_m}$ and $y = \frac{Q_n}{Q_m}$, for $m > n$, and using the second relations of Lemma 6. The last relations of Lemma 6 give the second and the third relations of the Theorem. The infinite sum follows immediately.

Remark 7 *Let $(T_n(x))$ and $(U_n(x))$ be the sequences of Chebyshev polynomials of the first and second kind [7]. It is well known that*

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x) \text{ with } T_0(x) = 1 \text{ and } T_1(x) = x,$$
$$U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x) \text{ with } U_0(x) = 1 \text{ and } U_1(x) = 2x.$$

We have $T_n(x) = \frac{x^n}{2} Q_n\left(\frac{-1}{x^2}\right)$ and $U_n(x) = x^n P_n\left(\frac{-1}{x^2}\right)$, for $n \geq 1$, and thus

$T_n(x) = \frac{1}{2}\sum_k(-1)^k q(n,k)x^{n-2k}$ *and* $U_n(x) = \sum_k(-1)^k p(n,k)x^{n-2k}$.

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References

[1] M. Benoumhani: A sequence of binomial coefficients related to Lucas and Fibonacci numbers. Journal of Integer Sequences, Vol. 6, 2003, Article 03.2.1.

[2] R. L. Graham, D. E. Knuth, O. Patashnik: Concrete Mathematics. Addison Wesley Publ. Comp., Inc., 1994.

[3] G. H. Hardy, J. E. Littlewood, G. Pólya: Inequalities. Cambridge University Press, 1956.

[4] D. H.. Lehmer: Problem 3801. The American Mathematical Monthly 632, 1936.

[5] R. Melham: Sums involving Fibonacci and Pell numbers. Portugaliae Mathematica, Vol. 56, Fasc. 3, 1999.

[6] J. Riordan: Introduction to combinatorial analysis. Dover, 2002.

[7] T. J. Rivlin: Chebyshev Polynomials: From Approximation Theory to Algebra and Number Theory, second edition, Wiley Interscience, 1990.

[8] N. Sloane: Online Encyclopedia of Integer Sequences, www.research.att.com/ ~njas/ sequences/ index.html.

[9] N. Stanley: Enumerative combinatorics, Wadsworth and Brooks / Cole, Monterey, California 1986.

[10] S. Tanny, M. Zuker: On a unimodal sequence of binomial coefficients. Discrete Math. 9, 1974, 79-89.

[11] S. Y. Yan: Number theory for computing. Springer Verlag, 2002.