Bounds of $q-$factorial $[n]_q!$

M. Mansour[1] and M. A. Obaid[2]
King AbdulAziz University, Faculty of Science, Mathematics Department,
P. O. Box 80203, Jeddah 21589 , Saudi Arabia.
[1] mansour@mans.edu.eg
[2]drmobaid@yahoo.com

Abstract

In this paper, we get the following upper and lower bounds for $q-$factorial $[n]_q!$

$$(q;q)_\infty(1-q)^{-n}e^{f_q(n+1)} < [n]_q! < (q;q)_\infty(1-q)^{-n}e^{g_q(n+1)},$$

where $n \geq 1$, $0 < q < 1$ and the two sequences $f_q(n)$ and $g_q(n)$ tends to zero through positive values. Also, we present two examples of the two sequences $f_q(n)$ and $g_q(n)$.

Keywords: Stirling's formula, $q-$gamma function, $q-$factorial.

1 Introduction.

Stirling's formula

$$n! \sim \sqrt{2n\pi}\left(\frac{n}{e}\right)^n \tag{1}$$

is used in many applications, especially in statistics and in the theory of probability to help estimate the value of $n!$, where $\sim$ is used to indicate that the ratio of the two sides goes to 1 as n goes to ∞. In other words, we have

$$\lim_{n\to\infty}\frac{n!}{n^{n+1/2}e^{-n}} = \sqrt{2\pi}. \tag{2}$$

Stirling's formula was actually discovered by De Moivre (1667-1754) but James Stirling (1692-1770) improved it by finding the value of the constant

2000 Mathematics Subject Classification: 33D05 and 11B65.

[1]Permanent address: M. Mansour, Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt.

$\sqrt{2\pi}$.

A number of upper and lower bounds for $n!$ have been obtained by various authors. Most bounds are of the form

$$\sqrt{2n\pi}\left(\frac{n}{e}\right)^n e^{\alpha_n} < n! < \sqrt{2n\pi}\left(\frac{n}{e}\right)^n e^{\beta_n}, \tag{3}$$

where α_n and β_n tends to zero through positive values. Here some examples of α_n and β_n:

$\alpha_n = \frac{1}{12n+1/4}$ $\beta_n = \frac{1}{12n}$ (E. Cesàro [2])

$\alpha_n = \frac{1}{12n+6}$ $\beta_n = \frac{1}{12n}$ (J. V. Uspensky [10])

$\alpha_n = \frac{1}{12n+1}$ $\beta_n = \frac{1}{12n}$ (H. Robbins [9])

$\alpha_n = \frac{1}{12n} - \frac{1}{360n^3}$ $\beta_n = \frac{1}{12n}$ (T. S. Nanjundiah [8])

$\alpha_n = \frac{1}{12n+\frac{3}{2(2n+1)}}$ (A. J. Maria [6])

$\alpha_n = \frac{1}{12n} - \frac{1}{360n^3}$ $\beta_n = \frac{1}{12n} - \frac{1}{(360+\gamma_n)n^3}$ (P. R. Beesack [1])

,where $\gamma_n = 30\frac{7n(n+1)+1}{n^2(n+1)^2}$

$\alpha_n = \frac{1}{12n} - \frac{1}{360n^3} - \frac{1}{120n^4}$ $\beta_n = \frac{1}{12n}$ (R. Michel [7])

The $q-$gamma function $\Gamma_q(x)$ is defined by the infinite product [4]

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x}; \qquad x \neq 0,-1,-2,..., \tag{4}$$

where q is a fixed real number $0 < q < 1$ and the $q-$shifted factorials are defined by

$(a;q)_0 = 1,$

$(a;q)_k = \prod_{j=0}^{k-1}(1-aq^j);$ $\qquad k = 1,2,\dots,$

$(a;q)_\infty = \prod_{i=0}^{\infty}(1-aq^i).$

This function is a $q-$analogue of the gamma function since we have

$$\lim_{q\to 1}\Gamma_q(x) = \Gamma(x).$$

Also, it satisfies the functional equation

$$\Gamma_q(x+1) = [x]_q\Gamma_q(x), \quad \Gamma_q(1) = 1, \tag{5}$$

which is a $q-$extension of the well-known functional equation

$$\Gamma(x+1) = x\Gamma(x), \quad \Gamma(1) = 1,$$

where $[x]_q = \frac{1-q^x}{1-q}$ is the $q-$number of x and $\lim_{q\to 1}[x]_q = x$, see [5] for details and related facts.

In this paper, we will get a $q-$analogue of the inequality (3) for the $q-$factorial which is defined by [5]

$$[n]_q! = [n]_q[n-1]_q...[2]_q[1]_q = \Gamma_q(n+1),$$

which is the $q-$analog of the relation $n! = \Gamma(n+1)$ where $\lim_{q\to 1}[n]_q! = n!$.

2 Main result.

We begin with the sequence $\{K_n\}$ defined by

$$K_n = [n-1]_q!(1-q)^{n-1/2}; \qquad 0 < q < 1; \quad n \geq 1, \tag{6}$$

for which

$$\frac{K_n}{K_{n+1}} = \frac{1}{1-q^n}. \tag{7}$$

By using relation (7), we can show that $K_n \geq K_{n+1}$ for all $n \geq 1$. Now, the idea of the proof is to find positive sequences $\{f_q(n)\}$, $\{g_q(n)\}$ both of which tend to zero, such that

$$f_q(n) - f_q(n+1) < -\log(1-q^n) < g_q(n) - g_q(n+1), \qquad n \geq 1.$$

Then

$$\exp(f_q(n) - f_q(n+1)) < \frac{K_n}{K_{n+1}} < \exp(g_q(n) - g_q(n+1)) \tag{8}$$

and hence

$$K_{n+1}\ \exp(-f_q(n+1)) < K_n\ \exp(-f_q(n)), \tag{9}$$

$$K_{n+1}\ \exp(-g_q(n+1)) > K_n\ \exp(-g_q(n)). \tag{10}$$

Now define the following two sequences

$$x_n = K_n\ \exp(-f_q(n)), \tag{11}$$

$$y_n = K_n\ \exp(-g_q(n)). \tag{12}$$

By using relation (9), the sequence $\{x_n\}$ is monotone decreasing and bounded below by zero. Then $\lim_{n\to\infty} x_n = a_q$ exist and $a_q \geq 0$. But $f_q(n)$ tends to zero, then

$$\lim_{n\to\infty} K_n = \lim_{n\to\infty} x_n\ \exp(f_q(n)) = a_q \tag{13}$$

also exists. Similarly, By using relation (10), the sequence $\{y_n\}$ is monotone increasing with $y_{n+1} < K_{n+1} < K_n < \dots < K_1$, so that $\lim_{n\to\infty} y_n = b_q$ exists with $b_q > 0$. Then

$$a_q = \lim_{n\to\infty} K_n = \lim_{n\to\infty} y_n\ \exp(g_q(n)) = b_q. \tag{14}$$

But

$$[n]_q! = \frac{(q;q)_n}{(1-q)^n}$$

then

$$a_q = \lim_{n\to\infty} K_n = \lim_{n\to\infty} [n-1]_q!(1-q)^{n-1/2}$$

$$= \lim_{n\to\infty} (q;q)_{n-1}(1-q)^{1/2} = (q;q)_\infty (1-q)^{1/2}. \qquad (15)$$

Then the relation

$$y_n < a_q < x_n, \qquad n \geq 1;\ 0 < q < 1 \qquad (16)$$

gives us the following theorem:

Theorem 1. *The $q-$factorial $[n]_q!$ satisfies the double inequality*

$$(q;q)_\infty (1-q)^{-n} e^{f_q(n+1)} < [n]_q! < (q;q)_\infty (1-q)^{-n} e^{g_q(n+1)}, \quad n \geq 1;\ 0 < q < 1 \qquad (17)$$

where $f_q(n)$ and $g_q(n)$ are two sequences tend to zero through positive values and satisfy

$$f_q(n) - f_q(n+1) < -\log(1-q^n) < g_q(n) - g_q(n+1), \qquad n \geq 1. \qquad (18)$$

The double inequality (17) is a $q-$analogue of the inequality (3).

3 Some special cases of the two sequences $f_q(n)$ and $g_q(n)$.

3.1 Case 1

By applying the Mean value theorem to the natural log, we get

$$\log(1+x) - \log(1) = \frac{(1+x)-1}{\xi} \qquad (19)$$

for some $\xi \in (1, 1+x)$. Then

$$\frac{x}{x+1} < \log(1+x) < x, \qquad x > 0. \qquad (20)$$

This is the logarithmic inequality. Also, the range of validity can be extended to include $-1 < x < 0$ as well [3]. So, if we put $x = -q^n$; $n \geq 1$, we get

$$\frac{-q^n}{1-q^n} < \log(1-q^n) < -q^n, \qquad n \geq 1. \qquad (21)$$

Then

$$q^n < -\log(1-q^n) < \frac{q^n}{1-q^n}; \qquad n \geq 1, \qquad (22)$$

but

$$\frac{q^n}{1-q^n} < \frac{q^n}{1-q}, \qquad n \geq 1;\ 0 < q < 1. \qquad (23)$$

Then

$$q^n < -\log(1-q^n) < \frac{q^n}{1-q}; \qquad n \geq 1, \tag{24}$$

and hence

$$q^n - q^{n+1} < -\log(1-q^n) < \frac{q^n}{(1-q)^2} - \frac{q^{n+1}}{(1-q)^2}; \qquad n \geq 1, \tag{25}$$

where $q^n - q^{n+1} > 0$ for $0 < q < 1$.
It will then follow that (18) holds with

$$M_q(n) = q^n \tag{26}$$

and

$$N_q(n) = \frac{q^n}{(1-q)^2}, \tag{27}$$

where, the two sequences $M_q(n)$ and $N_q(n)$ tends to zero through positive values. By using (17), we have

Lemma 3.1. *The $q-$factorial $[n]_q!$ satisfies the double inequality*

$$(q;q)_\infty (1-q)^{-n} e^{q^{n+1}} < [n]_q! < (q;q)_\infty (1-q)^{-n} e^{\frac{q^{n+1}}{(1-q)^2}}, \quad n \geq 1;\ 0 < q < 1. \tag{28}$$

3.2 Case 2

In view of the two sequences $M_q(n)$ and $N_q(n)$, we can improve the double inequality (28). Consider the sequence

$$T_q(n) = \frac{q^n}{(1-q)(1-q^n)},$$

which tends two zero through positive values. Let

$$\psi_q(n) = T_q(n) - T_q(n+1) + \log(1-q^n) = \frac{q^n}{(1-q^n)(1-q^{n+1})} + \log(1-q^n),$$

which tends to zero as n tends to infinity. Also,

$$\frac{d}{dn}\psi_q(n) = \frac{q^{2n}\left((1-q^{n+1}) + q(2-q^{n+1})(1-q^n)\right)}{(1-q^n)^2(1-q^{n+1})^2} \log q \ < 0,$$

where $0 < q < 1$ and $n \geq 1$. Then

$$\psi_q(n) > 0$$

and hence

$$-\log(1-q^n) < T_q(n) - T_q(n+1).$$

Also, consider the sequence

$$S_q(n) = \frac{q^n}{1-q},$$

which tends two zero through positive values. Let

$$\varphi_q(n) = S_q(n) - S_q(n+1) + \log(1-q^n) = q^n + \log(1-q^n),$$

which tends to zero as n tends to infinity. Also,

$$\frac{d}{dn}\varphi_q(n) = -\frac{q^{2n}\log q}{(1-q^n)} > 0,$$

where $0 < q < 1$ and $n \geq 1$. Then

$$\varphi_q(n) < 0$$

and hence

$$S_q(n) - S_q(n+1) < -\log(1-q^n).$$

It will then follow that (18) holds with the two sequences $S_q(n)$ and $T_q(n)$. Hence by using (17), we get

Lemma 3.2. *The q–factorial $[n]_q!$ satisfies the double inequality*

$$(q;q)_\infty(1-q)^{-n}e^{\frac{q^{n+1}}{1-q}} < [n]_q! < (q;q)_\infty(1-q)^{-n}e^{\frac{q^{n+1}}{(1-q)(1-q^{n+1})}}, \quad n \geq 1;\ 0 < q < 1. \tag{29}$$

Of course the double inequality (29) is better than the double inequality (28) for $n > 1$.

References

[1] P. R. Beesack, Improvement of Stirling's formula by elementary methods, Univ. Beograd, Publ. Elektrotenhn, Fak. Ser. Mat. Fiz. No. 274-301, (1969) 17-21.

[2] E. Cesàro, Elements Lehrbuch der algebraischen analysis und der Infinitesimalrechnung, Leipzig, 1922.

[3] M. J. Cloud and B. C. Drachman , Inequalities with applications to engineering, Springer-Verlag New York, 1998.

[4] G. Gasper and M. Rahman, Basic hypergeometric series, Cambridge Univ. Press, 1990.

[5] T. Kim, Lee-C. Jang and Heungsu Yi, A note on the modified $q-$Bernstein polynomials, Discrete Dynamics in Nature and Society Vol. 2010, Article ID 706483, 12 pages.

[6] A. J. Maria, A remark on Striling's formula, Amer. Math. Monthly, 72, (1965) 1096-1098.

[7] R. Michel, On Stirling's formula, Amer. Math. Monthly, 109(4): (2002) 388-390.

[8] T. S. Nanjundiah, Note on Stirling's formula, Amer. Math. Monthly, 66, (1965) 701-703.

[9] H. Robbins, A remark on Stirling's formula, Amer. Math. Monthly, 62, (1955) 26-29.

[10] J. V. Uspensky, Introduction to mathematical probability, McGraw Hill, New York, 1937.