An upper bound for total domination subdivision numbers

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Abstract

A set S of vertices of a graph $G = (V, E)$ without isolated vertex is a *total dominating set* if every vertex of $V(G)$ is adjacent to some vertex in S. *The total domination number* $\gamma_t(G)$ is the minimum cardinality of a total dominating set of G. *The total domination subdivision number* $sd_{\gamma_t}(G)$ is the minimum number of edges that must be subdivided (each edge in G can be subdivided at most once) in order to increase the total domination number. In this paper we first prove that $sd_{\gamma_t}(G) \leq n - \delta + 2$ for every simple connected graph G of order $n \geq 3$. We also classify all simple connected graphs G with $sd_{\gamma_t}(G) = n - \delta + 2, n - \delta + 1,$ and $n - \delta$.

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1 Introduction

Let $G = (V(G), E(G))$ be a graph of order n with no isolated vertices. The neighborhood of a vertex u is denoted by $N_G(u)$ and its degree $|N_G(u)|$ by $\deg_G(u)$ (briefly $N(u)$ and $\deg(u)$ when no ambiguity on the graph is possible). A *matching* in a graph is a set of non-loop edges with no shared end-vertices. A perfect matching M in a graph G is a matching with $V(M) = V(G)$. The maximum number of edges of a matching in G is denoted by $\alpha'(G)$. In order to work on the total dominating sets of G, we must suppose that the minimum degree δ of G is positive. We use [10] for terminology and notation which are not defined here.

A set S of vertices of G is a *dominating set* if $(V(G) \setminus S) \subseteq N(S)$. The *domination number* of G, denoted $\gamma(G)$, is the minimum cardinality of a dominating set of G. A $\gamma(G)$-set is a dominating set of G of cardinality $\gamma(G)$. When an edge uv of G is subdivided by inserting a new vertex x between u and v, the domination number does not decrease. The *domination subdivision number* $sd_\gamma(G)$ is the minimum number of edges of G that must be subdivided in order to increase the domination number. This concept was first introduced by Velammal in his Ph.D. thesis [9].

A set S of vertices of G is a *total (connected)* dominating set if it is a dominating set of G such that $G[S]$ has no isolated vertex ($G[S]$ is connected). The minimum cardinality of a total (connected) dominating set, denoted by $\gamma_t(G)$ ($\gamma_c(G)$), is called the *total (connected)* domination number of G. A *$\gamma_t(G)$-set* (*$\gamma_c(G)$-set*) is a total (connected) dominating set of G of cardinality $\gamma_t(G)$ ($\gamma_c(G)$). The *total domination subdivision number* $sd_{\gamma_t}(G)$ is the minimum number of edges of G that must be subdivided in order to increase the total domination number. Similarly, the *connected domination subdivision number* $sd_{\gamma_c}(G)$ is the minimum number of edges of a connected graph G that must be subdivided in order to increase the connected domination number. Since the total domination number of the graph K_2 dose not change when its only edge is subdivided, in the study of total (connected) domination subdivision number we must assume that the graph is of order $n \geq 3$.

It is rather difficult to construct graphs with large values of $sd_\gamma(G)$, $sd_{\gamma_t}(G)$ or $sd_{\gamma_c}(G)$ and the first conjecture on the subject (see [9]) was that $sd_\gamma(G) \leq 3$ for every G. However, it is now known that the three parameters can be arbitrary large (see [1] for $sd_\gamma(G)$, [7] for $sd_{\gamma_t}(G)$ and [3] for $sd_{\gamma_c}(G)$). It is also difficult to find general and good upper bounds for these parameters. Bhattacharya and Vijayakumar [1] prove that if $n = |V(G)|$ is large enough, then $sd_\gamma(G) \le 4\sqrt{n}\ln n + 5$ and the authors of [6] ask whether $sd_{\gamma_t}(G) \le n$. Some bounds are given in terms of the corresponding domination parameters. For instance, $sd_\gamma(G) \le \gamma(G) + 1$ [1, 5], $sd_{\gamma_t}(G) \le n - \gamma_t(G) + 1$ [4] and $sd_{\gamma_c}(G) \le n - \gamma_c(G) - 1$ with equality if and only if G is a path or a cycle [3].

Our purpose in this paper is that of establishing an upper bound for $sd_{\gamma_t}(G)$ in terms of the order and the minimum degree of G. We prove that $sd_{\gamma_t}(G) \le n - \delta + 2$ for every simple connected graph G of order $n \ge 3$. We also characterize all simple connected graphs G with $sd_{\gamma_t}(G) = n - \delta + 2, n - \delta + 1$, and $n - \delta$. We will use the following results on α', $\gamma_c(G)$, $\gamma_t(G)$ and $sd_{\gamma_t}(G)$.

Theorem A. [2] Let G be a simple graph of order n such that $\delta \ge k$ and $n \ge 2k$ for some $k \in \mathbb{N}$. Then $\alpha'(G) \ge k$.

Theorem B. [8] For every connected graph G, $\gamma_c(G) \le n - \Delta$.

Theorem C. [8] For every tree T, $\gamma_c(T) = n - \Delta$ if and only if T has at most one vertex of degree three or more.

Theorem D. [6] If G is a graph of order $n \ge 3$ and $\gamma_t(G) = 2$ or 3, then $1 \le sd_{\gamma_t}(G) \le 3$.

Theorem E. [4] For every simple connected graph G of order $n \ge 3$, $sd_{\gamma_t}(G) \le n - \gamma_t(G) + 1$.

Theorem F. [4] If G is a simple connected graph of order $n \ge 3$ different from P_3, C_3, K_4, P_6, C_6 then $sd_{\gamma_t}(G) \le n - \gamma_t(G)$.

Theorem G. [3] Let G be a graph of order $n \ge 3$. If there exists a γ_c-set S of G such that each vertex of S has an S-private neighbor, then $\mathrm{sd}_{\gamma_c}(G) \le \gamma_c(G)$.

The proof of the following lemma is straightforward and therefore omitted.

Lemma 1. (1) $sd_{\gamma_t}(K_3) = 2$ and $sd_{\gamma_t}(K_n) = 3$ for $n \ge 4$.
(2) For every matching M in K_n of size at least 2, $sd_{\gamma_t}(K_n - M) = 2$.

2 An upper bound

In this section we prove that for every simple connected graph G of order $n \ge 3$, $sd_{\gamma_t}(G) \le n - \delta + 2$. We make use of the following three lemmas in the proof of Theorem 5.

Lemma 2. If G contains a matching M such that $\gamma_t(G) < |M|$, then $sd_{\gamma_t}(G) \le \gamma_t(G) + 1$.

Proof. Let G' be obtained by subdividing $\gamma_t(G)+1$ edges of M. Each total dominating set of G' has order at least $\gamma_t(G)+1$. Hence, $\gamma_t(G') > \gamma_t(G)$ which implies $sd_{\gamma_t}(G) \leq \gamma_t(G)+1$. □

Lemma 3. Let G be a simple connected graph. If $v \in V(G)$ is contained in a 3-cycle and $\deg(v) \geq \gamma_t(G)$, then $sd_{\gamma_t}(G) \leq \deg(v)+1$.

Proof. Let $N(v) = \{v_1, \ldots, v_{\deg(v)}\}$ and $v_1v_2 \in E(G)$. Let G' be obtained from G by subdividing the edges v_1v_2 and vv_i, $1 \leq i \leq \deg(v)$, with $\deg(v)+1$ new vertices $a, b_1, \ldots, b_{\deg(v)}$, respectively. Let S be a $\gamma_t(G')$-set. If $v \in S$, then $b_i \in S$ for some i and v_1 or $v_2 \in S$. Now $S \setminus \{b_i \mid 1 \leq i \leq \deg(v)\}$ is a total dominating set for G. If $v \notin S$, then $b_i, v_i \in S$ for some i and $v_j \in S$ for each $j \neq i$. Both cases imply that $\gamma_t(G') \geq \gamma_t(G)+1$ and hence $sd_{\gamma_t}(G) \leq \deg(v)+1$. □

Lemma 4. If G is a simple connected graph of order $n \geq 3$ with $\delta \geq \gamma_t(G)$, then $sd_{\gamma_t}(G) \leq \gamma_t(G)+1$.

Proof. If $\gamma_t(G) = 2$ or 3, then $sd_{\gamma_t}(G) = 3 \leq \gamma_t(G)+1$ by Theorem D. Let $\gamma_t(G) \geq 4$. If $\alpha'(G) \geq \gamma_t(G)+1$, then $sd_{\gamma_t}(G) \leq \gamma_t(G)+1$ by Lemma 2. Now assume $\alpha' = \alpha'(G) \leq \gamma_t(G)$ and $M = \{e_1 = u_1v_1, \ldots, e_{\alpha'} = u_{\alpha'}v_{\alpha'}\}$ is a maximum matching of G. Clearly, $X = V(G) \setminus \{u_i, v_i \mid 1 \leq i \leq \alpha'\}$ is an independent set. We consider two cases.

Case 1. $\delta \geq \lfloor \frac{n}{2} \rfloor$. By Theorem F, we see that $sd_\gamma(G) \leq n - \gamma_t(G)$. If $\gamma_t(G)+1 \leq \lfloor \frac{n}{2} \rfloor$, then $\delta \geq \gamma_t(G)+1$ and $n \geq 2(\gamma_t(G)+1)$. This implies that $\alpha'(G) \geq \gamma_t(G)+1$ by Theorem A, which is a contradiction. Therefore we have $\gamma_t(G) \geq \lfloor \frac{n}{2} \rfloor$. This implies that $sd_\gamma(G) \leq n - \lfloor \frac{n}{2} \rfloor \leq \lfloor \frac{n}{2} \rfloor + 1 \leq \gamma_t(G)+1$.

Case 2. $\delta \leq \lfloor \frac{n}{2} \rfloor - 1$. If $\delta \geq \gamma_t(G)+1$, then $\alpha'(G) \geq \gamma_t(G)+1$ by Theorem A, which is a contradiction. Therefore we have $\delta = \gamma_t(G)$, which implies $\alpha'(G) \geq \gamma_t(G)$ by Theorem A. Hence, $\alpha'(G) = \gamma_t(G) = \delta$ by assumption. Since $2\alpha'(G) \leq n-2$, it follows that $|X| \geq 2$. Let $x, y \in X$. If $xu_i, yv_i \in E(G)$ for some i (the case $xv_i, yu_i \in E(G)$ is similar), then $M' = (M \setminus \{u_iv_i\}) \cup \{xu_i, yv_i\}$ is a matching of G larger than M, a contradiction. Therefore $2\delta \leq \deg(x) + \deg(y) \leq 2\alpha' = 2\delta$, which implies $\deg(x) = \gamma_t(G) = \alpha'(G)$ for each $x \in X$.
If $xu_i, xv_i \in E(G)$ for some $x \in X$ and some $1 \leq i \leq \alpha'(G)$, then $sd_{\gamma_t}(G) \leq \gamma_t(G)+1$ by Lemma 3. Now, without loss of generality, we may assume $N(x) = \{u_i \mid 1 \leq i \leq \alpha'(G)\}$ for each $x \in X$. Let G' be obtained from G by subdividing the edges xu_i, $1 \leq i \leq \alpha'(G)$, with new vertices $b_1, \ldots, b_{\gamma_t(G)}$, respectively. Let S be a $\gamma_t(G')$-set. If $x \in S$, then $b_i \in S$ for some i and

since $|X| \geq 2$ we see that $u_j \in S$ for some j. Now $S \setminus \{b_i \mid 1 \leq i \leq \gamma_t(G)\}$ is a total dominating set for G. Hence, $\gamma_t(G') \geq \gamma_t(G) + 1$. If $x \notin S$, then $b_i \in S$ for some i and $u_j \in S$ for each j. Therefore $\gamma_t(G') \geq \gamma_t(G) + 1$. This implies $sd_{\gamma_t}(G) \leq \gamma_t(G)$. □

Now we are ready to prove the main theorem of this section.

Theorem 5. If G is a simple connected graph of order $n \geq 3$, then $sd_{\gamma_t}(G) \leq n - \delta + 2$.

Proof. If $\delta \leq \gamma_t(G) - 1$, then

$$sd_{\gamma_t}(G) \leq n - \gamma_t(G) + 1 \leq n - \delta, \tag{1}$$

by Theorem E. Let $\delta \geq \gamma_t(G)$. If $\gamma_c(G) = 1$, then obviously $\gamma_t(G) = 2$ and hence

$$sd_{\gamma_t}(G) \leq 3 \leq n - \delta + 2, \tag{2}$$

by Theorem D. If $\gamma_c(G) \geq 2$, then $\gamma_t(G) \leq \gamma_c(G)$. Now by Lemma 4 and Theorem B,

$$sd_{\gamma_t}(G) \leq \gamma_t(G) + 1 \leq \gamma_c(G) + 1 \leq n - \Delta + 1 \leq n - \delta + 1. \tag{3}$$

This completes the proof. □

The next theorem characterizes the graphs whose total domination subdivision number is $n - \delta + 2$.

Theorem 6. Let G be a simple connected graph of order $n \geq 3$. Then $sd_{\gamma_t}(G) = n - \delta + 2$ if and only if G is isomorphic to K_n $(n \geq 4)$.

Proof. If $G \simeq K_n$ $(n \geq 4)$, then $sd_{\gamma_t}(G) = 3 = n - \delta + 2$ by Lemma 1. Conversely, let $sd_{\gamma_t}(G) = n - \delta + 2$. By (1) and (3) in Theorem 5 we have $\delta \geq \gamma_t(G)$ and $\gamma_t(G) = 2$. By Theorem D, $3 \leq n - \delta + 2 = sd_{\gamma_t}(G) \leq 3$, which implies $sd_{\gamma_t}(G) = 3$ and $\delta = n - 1$. Therefore, $G \simeq K_n$ $(n \geq 3)$. Since $sd_{\gamma_t}(K_3) = 2$ by Lemma 1, it follows that $n \geq 4$. This completes the proof. □

3 Graphs with $sd_{\gamma_t}(G) = n - \delta + 1$

In this section first we characterize all simple connected graphs G with $\gamma_t(G) = 2$ and $sd_{\gamma_t}(G) = 3$. Then we characterize all simple connected graphs whose total domination subdivision number is $n - \delta + 1$.

We make use of the graph $K_m^C \vee K_{n-m}$, $1 \leq m \leq n-1$, in the remaining sections. These graphs are known in the literature as *split graphs with the maximum number of edges*. We note that $P_3 \simeq K_2^C \vee K_1$, $K_3 \simeq K_1^C \vee K_2$ and $K_4 - e \simeq K_2^C \vee K_2$.

Lemma 7. *Let $n \geq 3$ and $1 \leq m \leq n-1$. Then $\gamma_t(K_m^C \vee K_{n-m}) = 2$ and*

$$sd_{\gamma_t}(K_m^C \vee K_{n-m}) = \begin{cases} 2 & \text{when} \quad m \geq n-2 \geq 1 \\ 3 & \text{otherwise.} \end{cases}$$

Proof. If $m = n-1$, then $K_{n-1}^C \vee K_1$ is a star $K_{1,n-1}$. Obviously, we have $\gamma_t(K_{1,n-1}) = 2$ and since $n \geq 3$, we have $sd_{\gamma_t}(K_{1,n-1}) = 2$. If $m = n-2$, then every pair of adjacent vertices forms a $\gamma_t(K_{n-2}^C \vee K_2)$. Now it is easy to see that $sd_{\gamma_t}(K_{n-2}^C \vee K_2) = 2$. Finally, let $1 \leq m \leq n-3$ and $G = K_m^C \vee K_{n-m}$. Then there is a subset X of $V(G)$ such that $G \simeq K_{|X|}^c \vee K_{n-|X|}$ and $1 \leq |X| \leq n-3$. Obviously, every pair of distinct vertices of $V(G) \setminus X$ forms a total dominating set for G, hence $\gamma_t(G) = 2$. Let $e_1 = u_1v_1, e_2 = u_2v_2 \in E(G)$ and let G' be obtained from G by subdividing the edges e_1 and e_2. At least one endpoint of e_1 and e_2, say v_1 and v_2, must be in $V(G) \setminus X$. Moreover, a vertex $z \in V(G) \setminus X$ is adjacent to every vertex $x \in V(G) - \{z\}$. If $v_1 = v_2$ and $V(G) - X \neq \{u_1, u_2, v_1\}$, then $\{v_1, z\}$ is a $\gamma_t(G')$-set, where $z \in V(G) \setminus X$ and $z \notin \{u_1, u_2, v_1\}$. If $v_1 = v_2$ and $V(G) - X = \{u_1, u_2, v_1\}$, then $\{v_1, z\}$ is a $\gamma_t(G')$-set, where $z \in X$. If $v_1 \neq v_2$ and $u_1 = u_2$, then $\{u_1, z\}$ is a $\gamma_t(G')$-set, where $z \in N_G(u_1)$, $z \neq v_1, v_2$. If $v_1 \neq v_2$ and $u_1 \neq u_2$, then $\{v_1, v_2\}$ is a $\gamma_t(G')$-set. Therefore, we must bisect at least three edges. Hence, by Theorem D, $sd_{\gamma(t)}(G) = 3$. □

Theorem 8. Let G be a simple connected graph of order n. The following statements are equivalent.

1. $\gamma_t(G) = 2$ and $sd_{\gamma_t}(G) = 3$.

2. $\delta \geq 3$ and each two edges of G are contained in a $K_4 - e$.

3. G is isomorphic to $K_m^c \vee K_{n-m}$ for some $1 \leq m \leq n-3$.

Proof. ($1 \implies 2$) First assume $\delta = 1$, $x \in V(G)$ with $\deg(x) = 1$ and xyz is a path in G. Let G' be obtained from G by subdividing the edges xy, yz. Then $\gamma_t(G') \geq 3$, a contradiction. Now assume $\delta = 2$, $x \in V(G)$ with $\deg(x) = 2$ and $xy, xz \in E(G)$. Let G' be obtained from G by subdividing the edges xy, xz. Then $\gamma_t(G') \geq 3$, a contradiction. Hence, $\delta \geq 3$ and $n \geq 4$. If $n = 4$, then $G \simeq K_4$ by Theorem 6. Let $n \geq 5$ and $e_1 = u_1v_1, e_2 = u_2v_2 \in E(G)$. Let G' be obtained from G by subdividing the edges e_1, e_2. First suppose that e_1 and e_2 do not have a common end-vertex. Since $sd_{\gamma_t}(G) = 3$ and $\gamma_t(G) = 2$, we may assume $\{u_1, u_2\}$ is a $\gamma_t(G')$-set. This implies that $u_1u_2, u_1v_2, u_2v_1 \in E(G)$. Therefore, e_1, e_2 are contained in a $K_4 - e$. Now suppose that $u_1 = u_2$. Let S be a total dominating set of G'. Since $\gamma_t(G) = 2$ and $sd_{\gamma_t}(G) = 3$, we have $u_1 \in S$.

Let $S = \{u_1, z\}$. Then z must be adjacent to u_1, v_1, v_2, which implies that e_1, e_2 are contained in a $K_4 - e$.

($2 \Longrightarrow 3$) Let X be a maximum independent set of G. If $|X| = 1$, then $G \simeq K_n = K_1 \vee K_{n-1}$. Let $|X| \geq 2$. Since $\delta \geq 3$, $|V(G) \setminus X| \geq 3$.

Claim 1 Each vertex of X is adjacent to every vertex of $V(G) \setminus X$.

Proof of Claim 1: Let, to the contrary, $x \in X, y \in V(G)\setminus X$ and $xy \notin E(G)$. Since G is connected, $xw \in E(G)$ for some $w \in V(G) \setminus X$. On the other hand, since X is a maximum independent set, y is adjacent to a vertex z of X. By assumption xw, yz are contained in a $K_4 - e$, which implies $xy \in E(G)$, a contradiction. This proves Claim 1.

Claim 2 The induced subgraph $G[V(G) \setminus X]$ is a complete graph.

Proof of Claim 2: Let $x, y \in V(G)\setminus X$ and $z_1, z_2 \in X$. By Claim 1, we have $xz_1, yz_2 \in E(G)$. Now by assumption, xz_1, yz_2 are contained in a $K_4 - e$, which implies $xy \in E(G)$. This proves Claim 2.

Therefore $G \simeq K^c_{|X|} \vee K_{n-|X|}$ and $|X| \leq n - 3$.

($3 \Longrightarrow 1$) By Lemma 7. □

Lemma 9. If G is a simple connected graph of order n with $\gamma_t(G) \geq 3$ and at least two vertices of degree three or more, then $\gamma_t(G) \leq n - \Delta - 1$.

Proof. Since $\gamma_t(G) \geq 3$, $\gamma_t(G) \leq \gamma_c(G) \leq n - \Delta$, the latter inequality by Theorem B. Equality holds in the latter inequality if and only if G has at most one vertex of degree three or more by Theorem C. Now the result follows. □

The next theorem characterizes the graphs whose total domination subdivision number is $n - \delta + 1$.

Theorem 10. Let G be a simple connected graph of order $n \geq 3$. Then $sd_{\gamma_t}(G) = n - \delta + 1$ if and only if $G \simeq K_3$ or $G \simeq K_2^c \vee K_{n-2}$ ($n \geq 5$).

Proof. If $G \simeq K_3$ or $G \simeq K_2^c \vee K_{n-2}$ ($n \geq 5$), then $sd_{\gamma_t}(G) = n - \delta + 1$ by Lemma 1 and Theorem 8.

Conversely, let $sd_{\gamma_t}(G) = n - \delta + 1$. By (1) in Theorem 5, $\delta \geq \gamma_t(G)$. If $\gamma_c(G) = 1$, then $\gamma_t(G) = 2$ and hence $n - \delta + 1 = sd_{\gamma_t}(G) \leq 3$ by Theorem D. This implies $\delta \geq n - 2$. If $\delta = n - 1$, then $G \simeq K_3$ by Lemma 1. Let $\delta = n - 2$. Then $\gamma_t(G) = 2$ and $sd_{\gamma_t}(G) = 3$, hence $G \simeq K_2^c \vee K_{n-2}$ ($n \geq 5$) by Theorem 8. Let $\gamma_c(G) \geq 2$. Then by (3) in Theorem 5, $\gamma_t(G) = n - \delta$ and $\delta = \Delta$. First suppose that $n \geq 6$. We have $\delta \geq \gamma_t(G) = n - \delta$, which implies $\delta \geq 3$. Now by Lemma 9 we obtain $\gamma_t(G) \leq 3$. Moreover, $n - \delta + 1 = \gamma_t(G) + 1 = sd_{\gamma_t}(G) \leq 3$ by (3) in Theorem 5 and Theorem D, which implies $\gamma_t(G) = 2$, $sd_{\gamma_t}(G) = 3$ and $\delta = \Delta = n - 2$. Therefore

$G \simeq K_n - M$, where M is a perfect matching of G. This contradicts Lemma 1 (2). Let $n \leq 5$. If $n = 3$, then obviously $G \simeq K_3$. If $n = 4$, since G is connected, it follows that $\gamma_t(G) = 2$. Now, by Theorem D, we have $n - \delta + 1 = sd_{\gamma_t}(G) \leq 3$. Therefore $\delta \geq 2$. By Lemma 1, $\delta = 2$ and we have $sd_{\gamma_t}(G) = 4 - 2 + 1 = 3$. Since $sd_{\gamma_t}(C_4) = 1$, we must have $\Delta = 3$. This leads to $G \simeq K_4 - e$, a contradiction by Theorem 8. Hence, there is no graph of order 4 with $sd_{\gamma_t}(G) = 4 - \delta + 1$. Finally, if $n = 5$, then $\gamma_t(G) \leq 3$ and we have $5 - \delta + 1 = sd_{\gamma_t}(G) \leq 3$. This leads to $\delta \geq 3$. By Lemma 1, $\delta = 3$ and we obtain $\gamma_t(G) = 2$ and $sd_{\gamma_t}(G) = 3$. This implies $G \simeq K_2^c \vee K_3$ by Theorem 8. □

4 Graphs with $sd_{\gamma_t}(G) = n - \delta$

In this section we characterize all simple connected graphs G whose total domination subdivision number is $n - \delta$. We make use of the following three lemmas in this section.

Lemma 11. For every simple connected graph G of order $n \geq 6$ and $\delta \geq n - 3$, $\gamma_t(G) = 2$.

Proof. Let $x \in V(G)$ be a vertex of degree δ and $X = V(G) \setminus N[x]$. Clearly $|X| \leq 2$. If $X = \emptyset$, then G is a complete graph and $\gamma_t(G) = 2$. If $|X| = 1$ and $x_1 \in X$, then $N(x) = N(x_1)$, which implies that $\{x, y\}$ is a $\gamma_t(G)$-set for each $y \in N(x)$. Let $|X| = 2$ and $x_1, x_2 \in X$. Clearly $N(x_1) \cap N(x_2) \neq \emptyset$. Now $\{x, y\}$ is a $\gamma_t(G)$-set for each $y \in N(x_1) \cap N(x_2)$. □

Lemma 12. For every simple connected graph G of order n, $2 \leq n \leq 9$, and $\delta \geq \lfloor \frac{n}{2} \rfloor$, $\gamma_t(G) \leq 3$.

Proof. If $n = 2, 3, 4$, then clearly $\gamma_t(G) = 2$. Let $n = 5$. If $\Delta > 2$, then $\gamma_t(G) = 2$. If $\Delta = 2$, then G is a 5-cycle and $\gamma_t(G) = 3$. If $n = 6$, then the statement is true by Lemma 11. Let $n = 7$ (respectively, 8). If $\delta \geq 4$ (respectively, 5), then the statement holds by Lemma 11. Let $\delta = 3$ (respectively, 4) and $\deg(x) = \delta$. Assume $X = V(G) \setminus N[x] = \{x_1, x_2, x_3\}$. If $\Delta(G[X]) = 2$ and $x_1x_2, x_1x_3 \in E(G)$, then $\{x, y, x_1\}$ is a total dominating set of G for every vertex $y \in N(x_1) \cap N(x)$. If $\Delta(G[X]) = 0$, then obviously $\{x, y\}$ is a $\gamma_t(G)$-set for each $y \in N(x)$. Now assume $\Delta(G[X]) = 1$ and $x_1x_2 \in E(G)$. Then $\{x, z, x_1\}$ is a total dominating set of G for each vertex $z \in N(x_1) \cap N(x)$. Finally, let $n = 9$. If $\delta \geq 6$, then the statement holds by Lemma 11. If $\delta = 5$, then an argument similar to that described for $n = 7$ shows that the statement is true. Now assume $\delta = 4$, $\deg(x) = 4$ and $X = V(G) \setminus N[x] = \{x_1, x_2, x_3, x_4\}$. If $\Delta(G[X]) = 3$ and $x_1x_2, x_1x_3, x_1x_4 \in E(G)$, then $\{x, y, x_1\}$ is a total dominating set of G for each $y \in N(x_1) \cap N(x)$. Let $\Delta(G[X]) = 2$ and $x_1x_2, x_1x_3 \in E(G)$. If $\deg_{G[X]}(x_4) \leq 1$, then $N(x) \cap N(x_1) \cap N(x_4) \neq \emptyset$, and hence $\{x, x_1, y\}$ is a total dominating set of G for each $y \in N(x) \cap N(x_1) \cap N(x_4)$. If $\deg_{G[X]}(x_4) = 2$, then $G[X]$ is the 4-cycle (x_1, x_2, x_4, x_3). It is now clear that there exist vertices $y_1, y_2 \in N(x)$, which dominate x_i for $i = 1, 2, 3, 4$. Thus $\{x, y_1, y_2\}$ is a total dominating set of G. If $\Delta(G[X]) = 1$, then $\{x, y_1, y_2\}$ is a total dominating set of G for each $y_1, y_2 \in N(x)$. If $\Delta(G[X]) = 0$, then $\{x, y\}$ is a $\gamma_t(G)$-set for every $y \in N(x)$. This completes the proof. □

Lemma 13. For every simple connected graph G of order $n \geq 10$ and $\delta \geq n - 5$, $\gamma_t(G) \leq 3$.

Proof. If $\delta \geq n - 3$, then the statement is true by Lemma 11. If $\delta = n - 4$, then an argument similar to that described for $n = 7$ in Lemma 12 shows that $\gamma_t(G) \leq 3$. Let $\delta = n - 5$, $\deg(x) = \delta$ and $X = V(G) \setminus N[x] = \{x_1, x_2, x_3, x_4\}$. If $\Delta(G[X]) = 0, 1, 2$, then there are vertices, say x_1 and x_2, such that $x_1x_2 \notin E(G)$. Now $\{x, x_1, y\}$ is a total dominating set of G for each $y \in N(x) \cap N(x_1) \cap N(x_4)$. Assume $\Delta(G[X]) = 3$ and $x_1x_2, x_1x_3, x_1x_4 \in E(G)$. Then $\{x, x_1, y\}$ is a total dominating set for each $y \in N(x) \cap N(x_1)$. This completes the proof. □

Putting Lemmas 11, 12 and 13 together, we have:

Theorem 14. For every simple connected graph G of order $n \geq 2$ and $\delta \geq \max\{n - 5, \lfloor \frac{n}{2} \rfloor\}$, $\gamma_t(G) \leq 3$.

We are now ready to characterize the graphs G of order $n \geq 3$ whose total domination subdivision number is $n - \delta$.

Theorem 15. Let G be a simple connected graph of order $n \geq 3$. Then $sd_{\gamma_t}(G) = n - \delta$ if and only if $G \simeq K_4 - e$, P_3, $K_3^c \vee K_{n-3}$ $(n \geq 6)$ or $K_n - M$, where M is a matching of size at least 2.

Proof. If $G \simeq K_4 - e$, P_3, or $K_3^c \vee K_{n-3}$ $(n \geq 6)$, then obviously $sd_{\gamma_t}(G) = n - \delta$. If $G \simeq K_n - M$, where M is a matching of size at least 2, then $sd_{\gamma_t}(G) = n - \delta$ by Lemma 1.

Conversely, Let $sd_{\gamma_t}(G) = n - \delta$. First suppose $\delta \leq \gamma_t(G) - 1$. Then by Theorem E,

$$n - \delta = sd_{\gamma_t}(G) \leq n - \gamma_t(G) + 1 \leq n - \delta.$$

Hence, $sd_{\gamma_t}(G) = n - \gamma_t(G) + 1$ and $\gamma_t(G) = \delta + 1$. By Theorems E and F, $G \simeq P_3$. Now suppose $\delta \geq \gamma_t(G)$. By Lemma 4, we have $n - \delta = sd_{\gamma_t}(G) \leq \gamma_t(G) + 1$. This implies that $n \leq 2\delta + 1$ or $\delta \geq \lfloor \frac{n}{2} \rfloor$. If $\gamma_c(G) = 1$, then

$\Delta = n-1$ and $\gamma_t(G) = 2$. By Theorem D, $n-\delta = sd_{\gamma_t}(G) \leq 3$ and hence $\delta \geq n-3$. On the other hand, by Lemma 1, $\delta \leq n-2$. If $\delta = n-3$, then $sd_{\gamma_t}(G) = 3$ and $G \simeq K_3^c \vee K_{n-3}$ $(n \geq 5)$ by Theorem 8. If $\delta = n-2$, then $\gamma_t(G) = sd_{\gamma_t}(G) = 2$ and $G \simeq K_n - M$, where M is a matching of G of size at least two by Theorem 8.
If $\gamma_c(G) \geq 2$, then by Lemmas 9 and 12 we have $\gamma_t(G) \leq 3$. By Theorem D, $n-\delta = sd_{\gamma_t}(G) \leq 3$, which implies $\delta \geq n-3$. If $n \geq 6$, then $\gamma_t(G) = 2$ by Lemma 11. Now we have $\gamma_c(G) = 2$, which implies $n-\delta = sd_{\gamma_t}(G) = sd_{\gamma_c}(G) \leq 2$ by Theorem G, and hence $\delta = n-2$ by Lemma 1. Moreover, since $\gamma_c(G) = 2$, we have $\Delta = \delta = n-2$. Therefore $G \simeq K_n - M$, where M is a perfect matching of G.
If $n = 3$, then obviously $\delta = n-2 = 1$, $sd_{\gamma_t} = n-\delta = 2$ and $G \simeq P_3$. Finally, if $4 \leq n \leq 5$, then $\gamma_t(G) \leq 3$ by Lemma 12, and $sd_{\gamma_t}(G) \leq 3$ by Theorem D. This implies $\delta = n - sd_{\gamma_t}(G) \geq n-3$. On the other hand, by Lemma 1 we have $\delta \leq n-2$. If $n = 4$, then obviously $sd_{\gamma_t}(G) \leq 2$, which implies $\delta = 2$ and hence $G \simeq K_4 - e$. If $n = 5$ and $\delta = 3$, then since n is odd and $sd_{\gamma_t}(G) = 2$, it follows that $\Delta = 4$ and $G \simeq K_n - M$, where M is a matching of size 2. If $n = 5$ and $\delta = 2$, then since $sd_{\gamma_t}(C_5) = 1$, it follows that $\Delta \geq 3$ and $\gamma_t(G) = 2$. This implies $sd_{\gamma_t}(G) \leq 2$, a contradiction. This completes the proof. □

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