NEIGHBOR RUPTURE DEGREE AND THE RELATIONS BETWEEN OTHER PARAMETERS

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ABSTRACT. The vulnerability shows the resistance of the network until communication breakdown after the disruption of certain stations or communication links. This study introduces a new vulnerability parameter, neighbor rupture degree. The neighbor rupture degree of a noncomplete connected graph G is defined to be

$$Nr(G) = \max\{w(G/S) - |S| - c(G/S) : S \subset V(G), w(G/S) \geq 1\}$$

where S is any vertex subversion strategy of G, $w(G/S)$ is the number of connected components in G/S, and $c(G/S)$ is the maximum order of the components of G/S. In this paper, the neighbor rupture degree of some classes of graphs are obtained and the relations between neighbor rupture degree and other parameters are determined.

1. INTRODUCTION

In a communication network, the vulnerability parameters measure the resistance of the network to disruption of operation after the failure of certain stations or communication links. When a network begins losing stations or communication links there is a loss in its effectiveness. Thus, a communication network must be constructed to be as stable as possible, not only with respect to the initial disruption, but also with respect to the possible reconstruction of the network. Connectivity, integrity [3], tenacity [7], rupture degree [17] and some other vulnerability parameters have been defined to measure the vulnerability of a graph. However most of these parameters do not consider the neighborhoods of the affected vertices. On the other hand, in spy networks, if a spy or a station is captured, then adjacent stations are unreliable. Therefore neighborhoods should be taken into consideration in spy networks. Nevertheless there are very few parameters concerning neighborhoods such as neighbor connectivity [11], neighbor integrity [8] and neighbor scattering [18].

Terminology and notation not defined in this paper can be found in [4]. Let G be a simple graph and let u be any vertex of G. The set $N(u) = \{v \in V(G) | v \neq u,$ v and u are adjacent$\}$ is the open neighborhood of u, and

2000 *Mathematics Subject Classification.* 05C40, 68M10, 68R10.
Key words and phrases. Vulnerability, Neighbor Integrity, Rupture Degree, Neighbor Rupture Degree.

$N[u] = \{u\} \cup N(u)$ is the closed neighborhood of u. A vertex u in G is said to be subverted if the closed neighborhood of u is removed from G. A set of vertices $S = \{u_1, u_2, \ldots, u_m\}$ is called a vertex subversion strategy of G if each of the vertices in S has been subverted from G. If S has been subverted from the graph G, then the survival subgraph is disconnected, a clique, or the empty graph. The survival subgraph is denoted by G/S. The most common vulnerability parameters concerning to spy networks are as follows.

The neighbor connectivity of a graph G is

$$K(G) = \min_{S \subseteq V(G)} \{|S|\}$$

where S is a subversion strategy of G [11].

The neighbor integrity of a graph G is defined to be

$$NI(G) = \min_{S \subseteq V(G)} \{|S| + c(G/S)\}$$

where S is any vertex subversion strategy of G and $c(G/S)$ is the order of the largest connected component of G/S [8].

The neighbor scattering number of a graph G is defined as

$$S(G) = \max_{S \subseteq V(G)} \{w(G/S) - |S| : w(G/S) \geq 1\}$$

where S is any vertex subversion strategy of G and $w(G/S)$ is the number of connected components in the graph G/S [18].

The known parameters concerning about the neighborhoods do not deal with the number of the removing vertices, the number of the components and the number of the vertices in the largest component of the remaining graph in a disrupted network simultaneously. In order to fill this void in the literature, the current study proposes a definition of neighbor rupture degree which is a new parameter concerning to these three values. Additionally, this study also analyzes the relations between neighbor rupture degree and some other graph parameters, and obtains neighbor rupture degree of some classes of graphs.

The neighbor rupture degree of a noncomplete connected graph G is defined to be

$$Nr(G) = \max\{w(G/S) - |S| - c(G/S) : S \subset V(G), w(G/S) \geq 1\}$$

where S is any vertex subversion strategy of G, $w(G/S)$ is the number of connected components in G/S, and $c(G/S)$ is the maximum order of the components of G/S [1]. In particular, the neighbor rupture degree of a complete graph K_n is defined to be $(1-n)$. A set $S \subset V(G)$ is said to be the Nr-set of G, if

$$Nr(G) = w(G/S) - |S| - c(G/S).$$

The neighbor rupture degree differs from neighbor integrity and rupture degree in showing the vulnerability of networks. Consider the two graphs of order $3st$ containing the corona operation

$$G_1 = K_s \circ [(t-2)K_2 \cup (t+3)K_1]$$
$$G_2 = K_t \circ [(s-2)K_2 \cup (s+3)K_1]$$

where s and t are integers and $s > t+2 > 4$. It can be easily seen that the neighbor integrity and the rupture degree of these graphs are equal.

$$NI(G_1) = NI(G_2) = 3$$
$$r(G_1) = r(G_2) = 2ts - 2$$

On the other hand, the neighbor rupture degree of G_1 and G_2 are different.

$$Nr(G_1) = 2ts - 2t + s - 4$$
$$Nr(G_2) = 2ts - t - 2$$

Thus, the neighbor rupture degree is a better parameter then the neighbor integrity and the rupture degree to distinguish these two graphs.

2. Relations Between the Neighbor Rupture Degree and Other Parameters

In this section some lower and upper bounds are given for the neighbor rupture degree of a graph using different graph parameters.

Theorem 1. *Let G be a graph of order n. Then,*

$$Nr(G) \geq 1 - n.$$

Proof. Let S be a subversion strategy of G. We have $|S| \leq |N[S]|$ and $w(G/S) \geq 1$ for any graph G. Then,

$$w(G/S) \leq n - |N[S]| - c(G/S) + 1 \leq n - |S| - c(G/S) + 1.$$

If we add $w(G/S)$ to both sides, we get

$$2w(G/S) - n - 1 \leq w(G/S) - |S| - c(G/S).$$

Then, by the definition of neighbor rupture degree, $Nr(G) \geq 1 - n$. □

Theorem 2. *Let G be a graph of order n and $K(G)$ be the neighbor connectivity of G. Then,*

$$Nr(G) \leq n - 2K(G) - 1.$$

Proof. Let S be an Nr-set of G. For any set S we have $w(G/S) \leq n - |N[S]|$, $c(G/S) \geq 1$ and $K(G) \leq |S| \leq |N[S]|$. Hence,

$$w(G/S) - |S| - c(G/S) \leq n - |N[S]| - |S| - c(G/S).$$

Thus, when we take the maximum of both sides, the proof is completed. □

Theorem 3. *Let G be a graph of order n. Then,*

$$Nr(G) \leq \alpha(G) - NI(G).$$

Proof. Let S be a subversion strategy of G. We have $NI(G) \leq |S| + c(G/S)$ for any S by the definition of neighbor integrity. If we subtract both sides from $w(G/S)$, we get

$$w(G/S) - NI(G) \geq w(G/S) - |S| - c(G/S).$$

Therefore, we have $Nr(G) \leq \alpha(G) - NI(G)$ since $w(G/S) \leq \alpha(G)$. □

Theorem 4. *For any graph G, we have*

$$Nr(G) \leq 2\alpha(G) - 2NI(G) - r(G).$$

Proof. Let S be an r-set and S' be an Nr-set of G. Hence

$$w(G - S) - |S| - m(G - S) = r(G) \text{ and}$$

$$w(G/S') - |S'| - c(G/S') = Nr(G).$$

If we add this two equalities, we get

$$r(G) + Nr(G) = w(G - S) - |S| - m(G - S) + w(G/S') - |S'| - c(G/S').$$

It follows from the definition of integrity and neighbor integrity that $|S| + m(G-S) \geq I(G)$ and $|S'| + c(G/S') \geq NI(G)$ for any S and S'. Since we have

$$w(G/S) \leq w(G - S) \leq \alpha(G)$$

for any S [17], $r(G) + Nr(G) \leq 2\alpha(G) - I(G) - NI(G)$.
The proof is completed using the inequality $NI(G) \leq I(G)$ [8]. □

Theorem 5. *For any graph G,*

$$Nr(G) \geq 3 - I(G) - NI(G) - r(G).$$

Proof. Let S be an I-set and S' be an NI-set of G. Then we have the equalities $|S| + m(G - S) = I(G)$ and $|S'| + c(G/S') = NI(G)$. It follows from the definition of rupture degree and neighbor rupture degree that

$$r(G) + Nr(G) \geq w(G - S) - |S| - m(G - S) + w(G/S') - |S'| - c(G/S').$$

Thus,

$$r(G) + Nr(G) \geq w(G - S) + w(G/S') - I(G) - NI(G).$$

Since we have $w(G - S) \geq 2$ and $w(G/S') \geq 1$ for any set S and S', $r(G) + Nr(G) \geq 3 - I(G) - NI(G)$ is obtained and the proof is completed. □

The following corollary is easily obtained from Theorem 4 and Theorem 5.

Corollary 1. $3 - I(G) - NI(G) - r(G) \leq Nr(G) \leq 2\alpha(G) - 2NI(G) - r(G)$.

3. Neighbor Rupture Degree of Some Classes of Graphs

In this section, we consider the neighbor rupture degree of path graphs, cycle graphs, complete k-partite graphs, wheel graphs and complete k-ary trees.

Theorem 6. *Let P_n be a path graph of order $n \geq 12$. Then the neighbor rupture degree of P_n is*

$$Nr(P_n) = \begin{cases} 0, & n \equiv 1 \ (Mod\ 4); \\ -1, & n \equiv 0,2,3 \ (Mod\ 4). \end{cases}$$

Proof. Let S be a subversion strategy of P_n and $|S| = x$ be the number of removing vertices from P_n. There are two cases according to the number of the elements in S.

Case 1: If $x \leq \lfloor \frac{n-1}{4} \rfloor$, then $w(P_n/S) \leq x+1$ and $c(P_n/S) \geq \lceil \frac{n-3x}{x+1} \rceil$. Hence

$$w(P_n/S) - |S| - c(P_n/S) \leq x + 1 - x - \lceil \frac{n-3x}{x+1} \rceil \leq 1 - \frac{n-3x}{x+1}$$

and we have

$$Nr(P_n) \leq \max_x \{1 - \frac{n-3x}{x+1}\}.$$

The function $f(x) = 1 - \frac{n-3x}{x+1}$ is an increasing function and takes its maximum value at $x = \lfloor \frac{n-1}{4} \rfloor$. Thus, we have $f(\lfloor \frac{n-1}{4} \rfloor) = 4 - \frac{n+3}{\lfloor \frac{n-1}{4} \rfloor + 1}$ and since the neighbor rupture degree is integer valued, we get

$$Nr(P_n) \leq \lfloor 4 - \frac{n+3}{\lfloor \frac{n-1}{4} \rfloor + 1} \rfloor.$$

$Nr(P_n) \leq -1$ where $n \equiv 0,2,3$ $(Mod\ 4)$ and $Nr(P_n) \leq 0$ where $n \equiv 1$ $(Mod\ 4)$. Therefore, we have

$$Nr(P_n) \leq \begin{cases} 0, & n \equiv 1 \ (Mod\ 4); \\ -1, & n \equiv 0,2,3 \ (Mod\ 4). \end{cases} \tag{1}$$

Case 2: If $x \geq \lceil \frac{n}{4} \rceil$, then $w(P_n/S) \leq x$ and $c(P_n/S) \geq 1$. Therefore

$$w(P_n/S) - |S| - c(P_n/S) \leq x - x - 1$$

and when we take the maximum of both sides we have

$$Nr(P_n) \leq -1. \tag{2}$$

It can be easily seen that there is a subversion strategy S^* of P_n such that $|S^*| = \lfloor \frac{n-1}{4} \rfloor, w(P_n/S^*) = \lfloor \frac{n-1}{4} \rfloor + 1, c(P_n/S^*) = 2$ where $n \equiv 0,2,3$ $(Mod\ 4)$ and $c(P_n/S^*) = 1$ where $n \equiv 1$ $(Mod\ 4)$. Thus,

$$Nr(P_n) \geq \begin{cases} 0, & n \equiv 1 \ (Mod\ 4); \\ -1, & n \equiv 0,2,3 \ (Mod\ 4). \end{cases} \tag{3}$$

The proof is completed from (1), (2) and (3). □

The neighbor rupture degrees of the path graph P_n for $n < 12$ are given in Table 1 below.

TABLE 1. Neighbor rupture degrees of P_n for $n < 12$

n	2	3	4	5	6	7	8	9	10	11
$Nr(P_n)$	-1	-1	-1	0	-1	-1	-1	0	-1	-1

Theorem 7. *Let C_n be a cycle graph of order $n \geq 15$. Then the neighbor rupture degree of C_n is*

$$Nr(C_n) = \begin{cases} -1, & n \equiv 0 \ (Mod\ 4); \\ -2, & n \equiv 1,2,3 \ (Mod\ 4). \end{cases}$$

Proof. Let S be a subversion strategy of C_n and $|S| = x$ be the number of removing vertices of C_n. There are two cases according to the number of the vertices in S.

Case 1: If $x \leq \lfloor \frac{n}{4} \rfloor$, then we have $w(C_n/S) \leq x$ and $c(C_n/S) \geq \lceil \frac{n-3x}{x} \rceil$. Hence

$$w(C_n/S) - |S| - c(C_n/S) \leq x - x - \lceil \frac{n-3x}{x} \rceil \leq 3 - \frac{n}{x} \text{ and}$$

$$Nr(C_n) \leq \max_x \{3 - \frac{n}{x}\}.$$

The function $f(x) = 3 - \frac{n}{x}$ is an increasing function and takes its maximum value at $x = \lfloor \frac{n}{4} \rfloor$. The neighbor rupture degree is integer valued. Thus we get,

$$Nr(C_n) \leq \lfloor 3 - \frac{n}{\lfloor \frac{n}{4} \rfloor} \rfloor.$$

$Nr(C_n) \leq -1$ where $n \equiv 0 \ (Mod\ 4)$ and $Nr(C_n) \leq -2$ where $n \equiv 1,2,3 \ (Mod\ 4)$. Therefore,

$$Nr(C_n) \leq \begin{cases} -1, & n \equiv 0 \ (Mod\ 4); \\ -2, & n \equiv 1,2,3 \ (Mod\ 4). \end{cases} \tag{4}$$

Case 2: If $x \geq \frac{n}{4}$, then $w(C_n/S) \leq x$, $c(C_n/S) \geq 1$ where $n \equiv 0 \ (Mod\ 4)$ and $w(C_n/S) \leq x + 1$, $c(C_n/S) \geq 1$ where $n \equiv 1,2,3 \ (Mod\ 4)$. Therefore, we have

$$Nr(C_n) \leq \begin{cases} -1, & n \equiv 0 \ (Mod\ 4); \\ -2, & n \equiv 1,2,3 \ (Mod\ 4). \end{cases} \tag{5}$$

It can be easily seen that there is a subversion strategy S^* of C_n such that $|S^*| = \lfloor \frac{n}{4} \rfloor$, $w(C_n/S^*) = \lfloor \frac{n}{4} \rfloor$, $c(C_n/S^*) = 1$ where $n \equiv 0 \ (Mod\ 4)$ and $c(P_n/S^*) = 2$ where $n \equiv 1,2,3 \ (Mod\ 4)$. Thus, we have

$$Nr(C_n) \geq \begin{cases} -1, & n \equiv 0 \ (Mod\ 4); \\ -2, & n \equiv 1,2,3 \ (Mod\ 4). \end{cases} \tag{6}$$

The proof is completed from (4), (5) and (6). □

TABLE 2. Neighbor rupture degrees of C_n for $n < 15$

n	3	4	5	6	7	8	9	10	11	12	13	14
$Nr(C_n)$	-2	-1	-2	-2	-2	-1	-2	-2	-2	-1	-2	-2

The neighbor rupture degrees of the cycle graph C_n for $n < 15$ are given in Table 2 above.

Theorem 8. *The neighbor rupture degree of the complete k-partite graph is*

$$Nr(K_{n_1,n_2,\ldots,n_k}) = \max\{n_1, n_2, \ldots, n_k\} - 3.$$

Proof. Let S be a subversion strategy of $K_{n_1,n_2,\ldots,n_k}$ and let S contain the elements which belongs to only one of the sets V_i. Otherwise $w(K_{n_1,n_2,\ldots,n_k}/S) = 0$ and this contradicts to the definition. Therefore let $S \subseteq V_i$ and $|S| = x$. $V(K_{n_1,n_2,\ldots,n_k}) = V_1 \cup V_2 \cup \ldots \cup V_k$ is a partition where $|V_1| = n_1$, $|V_2| = n_2$, $\ldots$, $|V_k| = n_k$. Then, $w(K_{n_1,n_2,\ldots,n_k}/S) = n_i - x$ and $c(K_{n_1,n_2,\ldots,n_k}/S) = 1$. By the definition of the neighbor rupture degree, we get

$$Nr(K_{n_1,n_2,\ldots,n_k}) = \max_x\{n_i - 2x - 1\}.$$

The function $f(x) = n_i - 2x - 1$ is a decreasing function and since $1 \le x \le n_i$ we have $Nr(K_{n_1,n_2,\ldots,n_k}) = n_i - 3$. The proof is completed by taking $n_i = \max\{n_1, n_2, \ldots, n_k\}$. □

Corollary 2. *The neighbor rupture degree of the complete bipartite graph is*

$$Nr(K_{m,n}) = \max\{m, n\} - 3.$$

Corollary 3. *The neighbor rupture degree of the star graph $K_{1,n}$ is*

$$Nr(K_{1,n}) = n - 3.$$

Theorem 9. *Let W_n be a wheel graph of order $n \ge 6$. Then,*

$$Nr(W_n) = \begin{cases} -1, & n \equiv 1 \ (Mod\ 4); \\ -2, & n \equiv 0,2,3 \ (Mod\ 4). \end{cases}$$

Proof. A wheel graph W_n is the join of a cycle C_{n-1} and the complete graph K_1, $W_n = K_1 + C_{n-1}$. Let S be a subversion strategy of W_n. Then, the set S contains only some of the vertices belonging to the cycle, otherwise it contradicts to $w(W_n/S) \ge 1$. Let $S = \{v\}$ where $v \in V(C_{n-1})$, then we get $W_n/S = P_{n-4}$ and $Nr(W_n) = Nr(P_{n-4}) - 1$. If we substitute

$$Nr(P_{n-4}) = \begin{cases} 0, & n \equiv 1 \ (Mod\ 4); \\ -1, & n \equiv 0,2,3 \ (Mod\ 4) \end{cases}$$

we complete the proof. □

Theorem 10. *Let $T_{k,d}$ be a complete k-ary tree of depth d where $k \geq 2$. Then*

$$Nr(T_{k,d}) = \begin{cases} \dfrac{k^{d+2} - k^2}{k^2+1}, & d \equiv 0 \ (Mod\ 4); \\ \dfrac{k^{d+2} - 3k^2 + k - 3}{k^2+1}, & d \equiv 1 \ (Mod\ 4); \\ \dfrac{k^{d+2} - k^2 - 2}{k^2+1}, & d \equiv 2 \ (Mod\ 4); \\ \dfrac{k^{d+2} - k^2 - k - 1}{k^2+1}, & d \equiv 3 \ (Mod\ 4). \end{cases}$$

Proof. Let S be a subversion strategy of $T_{k,d}$ and let $|S| = x$ be the number of removing vertices. There are four cases according to the depth of $T_{k,d}$.

Case 1: Let $d \equiv 0 \ (Mod\ 4)$.

(i) If $0 \leq x \leq \frac{k(k^d-1)}{k^4-1}$, then $w(T_{k,d}/S) \leq (k^2+k-1)x + 1$ and $c(T_{k,d}/S) \geq 1$. Thus,

$$Nr(T_{k,d}) \leq \max_x\{(k^2+k-1)x + 1 - x - 1\} = \max_x\{(k^2+k-2)x\}.$$

The function $f(x) = (k^2+k-2)x$ is an increasing function and it takes its maximum value at $x = \frac{k(k^d-1)}{k^4-1}$. Then,

$$Nr(T_{k,d}) \leq \frac{k^{d+2} + 2k^{d+1} - k^2 - 2k}{(k^2+1)(k+1)}. \tag{7}$$

(ii) If $\frac{k(k^d-1)}{k^4-1} + 1 \leq x \leq \frac{k^2(k^d-1)}{k^4-1}$, then $w(T_{k,d}/S) \leq (k^2+k-1)(\frac{k(k^d-1)}{k^4-1}) + (k^2-1)(x - \frac{k(k^d-1)}{k^4-1}) + 1 = \frac{k^2(k^d-1)}{k^4-1} + x(k^2-1) + 1$ and $c(T_{k,d}/S) \geq 1$. Then,

$$Nr(T_{k,d}) \leq \max_x\{\frac{k^2(k^d-1)}{k^4-1} + x(k^2-1) + 1 - x - 1\} = \max_x\{\frac{k^2(k^d-1)}{k^4-1} + x(k^2-2)\}.$$

The function $f(x) = \frac{k^2(k^d-1)}{k^4-1} + x(k^2-2)$ is an increasing function and it takes its maximum value at $x = \frac{k^2(k^d-1)}{k^4-1}$. Therefore,

$$Nr(T_{k,d}) \leq \frac{k^2(k^d-1)}{k^2+1}. \tag{8}$$

(iii) If $\frac{k^2(k^d-1)}{k^4-1} + 1 \leq x \leq \frac{k^{d+1}-1}{k-1} - 2$, then we have $w(T_{k,d}/S) \leq \frac{k^{d+4}-1}{k^4-1}$ and $c(T_{k,d}/S) \geq 1$. Thus,

$$Nr(T_{k,d}) \leq \max_x\{\frac{k^{d+4}-1}{k^4-1} - x - 1\}.$$

The function $f(x) = \frac{k^{d+4}-1}{k^4-1} - x - 1$ is a decreasing function and it takes its maximum value at $x = \frac{k^2(k^d-1)}{k^4-1} + 1$. Then,

$$Nr(T_{k,d}) \leq \frac{k^{d+2} - 2k^2 - 1}{k^2+1}. \tag{9}$$

It can be easily seen that there is a subversion strategy S^* of $T_{k,d}$ such that $|S^*| = \frac{k^2(k^d-1)}{k^4-1}$ where S contains all the vertices on the $\{2nd, 6th, 10th, 12th, ..., (d-2)th\}$ levels. Then $w(T_{k,d}/S^*) = \frac{k^{d+4}-1}{k^4-1}$ and $c(T_{k,d}/S^*) = 1$. Thus

$$Nr(T_{k,d}) \geq \frac{k^2(k^d-1)}{(k^2+1)}. \tag{10}$$

The proof is completed by (7), (8), (9) and (10).

Case 2: Let $d \equiv 1$ $(Mod\ 4)$.

(i) If $0 \leq x \leq \frac{k(k^d-k)}{k^4-1}$, then $w(T_{k,d}/S) \leq (k^2+k-1)x+1$ and $c(T_{k,d}/S) \geq 2$. Thus,

$$Nr(T_{k,d}) \leq \max_x\{(k^2+k-1)x+1-x-1\} = \max_x\{(k^2+k-2)x\}.$$

The function $f(x) = (k^2+k-2)x$ is an increasing function and it takes its maximum value at $x = \frac{k(k^d-k)}{k^4-1}$. Then,

$$Nr(T_{k,d}) \leq \frac{k^{d+2}+2k^{d+1}-k^3-2k^2}{(k^2+1)(k+1)}. \tag{11}$$

(ii) If $\frac{k(k^d-k)}{k^4-1} + 1 \leq x \leq \frac{k^2(k^d-k)}{k^4-1}$, then $w(T_{k,d}/S) \leq (k^2+k-1)\frac{k(k^d-k)}{k^4-1} + (k^2-1)(x - \frac{k(k^d-k)}{k^4-1}) = \frac{k^2(k^d-k)}{k^4-1} + (k^2-1)(x-1)$ and $c(T_{k,d}/S) \geq 1$. Then,

$$\begin{aligned} Nr(T_{k,d}) &\leq \max_x\{\frac{k^2(k^d-k)}{k^4-1} + (k^2-1)(x-1) - x - 1\} \\ &= \max_x\{\frac{k^2(k^d-k)}{k^4-1} - k^2 + (k^2-2)x\} \end{aligned}$$

The function $f(x) = \frac{k^2(k^d-k)}{k^4-1} - k^2 + (k^2-2)x$ is an increasing function and it takes its maximum value at $x = \frac{k^2(k^d-k)}{k^4-1}$. Therefore,

$$Nr(T_{k,d}) \leq \frac{k^{d+2}-k^3-k^2-1}{k^2+1}. \tag{12}$$

(iii) If $x = \frac{k^2(k^d-k)}{k^4-1} + 1$, then $w(T_{k,d}/S) \leq (k^2+k-1)\frac{k(k^d-k)}{k^4-1} + (k^2-1)(\frac{k^2(k^d-k)}{k^4-1} - \frac{k(k^d-k)}{k^4-1}) + (k-1)$ and $c(T_{k,d}/S) \geq 1$. Then,

$$\begin{aligned} Nr(T_{k,d}) &\leq \max_x\{(2k^2+k-2)\frac{k(k^d-k)}{k^4-1} + k - 1 - x - 1\} \\ &= \frac{k^{d+2} - 3k^2 + k - 3}{k^2+1} \end{aligned} \quad (13)$$

(iv) If $\frac{k^2(k^d-k)}{k^4-1} + 2 \leq x \leq \frac{k^{d+1}-1}{k-1} - 2$, then $w(T_{k,d}/S) \leq \frac{k^{d+4}-k^4-k+1}{k^4-1}$ and $c(T_{k,d}/S) \geq 1$. Then,

$$Nr(T_{k,d}) \leq \max_x\{\frac{k^{d+4}-k^4-k+1}{k^4-1} - x - 1\}.$$

The function $f(x) = \frac{k^{d+4}-k^4-k+1}{k^4-1} - x - 1$ is a decreasing function and it takes its maximum value at $x = \frac{k^2(k^d-1)}{k^4-1} + 2$. Therefore,

$$Nr(T_{k,d}) \leq \frac{k^{d+2} - 4k^2 + k - 4}{k^2+1}. \quad (14)$$

It is obvious that there is a subversion strategy S^* of $T_{k,d}$ such that $|S^*| = \frac{k^2(k^d-k)}{k^4-1} + 1$ where S contains all the vertices on the $\{3rd, 7th, 11th, ..., (d-2)th\}$ levels and one of the vertices on the first level. Then $w(T_{k,d}/S^*) = \frac{k^{d+4}-k^4-k+1}{k^4-1}$ and $c(T_{k,d}/S^*) = 1$. Hence we get

$$Nr(T_{k,d}) \geq \frac{k^{d+2} - 3k^2 + k - 3}{k^2+1}. \quad (15)$$

The proof is completed by (11), (12), (13), (14) and (15).

Case 3: Let $d \equiv 2$ $(Mod\ 4)$.

(i) If $0 \leq x \leq \frac{k(k^d-k^2)}{k^4-1}$, then $w(T_{k,d}/S) \leq (k^2+k-1)x + 1$ and $c(T_{k,d}/S) \geq 1$. Thus,

$$Nr(T_{k,d}) \leq \max_x\{(k^2+k-1)x + 1 - x - 1\} = \max_x\{(k^2+k-2)x\}.$$

The function $f(x) = (k^2+k-2)x$ is an increasing function and it takes its maximum value at $x = \frac{k(k^d-k^2)}{k^4-1}$. Then,

$$Nr(T_{k,d}) \leq \frac{k^{d+2} + 2k^{d+1} - k^4 - 2k^3}{(k^2+1)(k+1)}. \quad (16)$$

(ii) If $x = \frac{k(k^d-k^2)}{k^4-1} + 1$, then $w(T_{k,d}/S) \leq (k^2+k-1)\frac{k(k^d-k^2)}{k^4-1} + k^2$ and $c(T_{k,d}/S) \geq 1$. Then,

$$\begin{aligned} Nr(T_{k,d}) &\leq (k^2+k-1)\frac{k(k^d-k^2)}{k^4-1} + k^2 - x - 1 \\ &= \frac{k^{d+2}+2k^{d+1}+k^5-3k^3-k^2-2k-2}{(k^2+1)(k+1)}. \end{aligned} \tag{17}$$

(iii) If $\frac{k(k^d-k^2)}{k^4-1} + 1 \leq x \leq \frac{k^{d+2}-1}{k^4-1}$, then $w(T_{k,d}/S) \leq (k^2+k-1)\frac{k(k^d-k^2)}{k^4-1} + k^2 + (k^2-1)(x - \frac{k(k^d-k^2)}{k^4-1} - 1) = \frac{k^{d+2}-1}{k^4-1} + (k^2-1)x$ and $c(T_{k,d}/S) \geq 1$. Thus,

$$Nr(T_{k,d}) \leq \max_x\{\frac{k^{d+2}-1}{k^4-1} + (k^2-1)x - x - 1\}.$$

The function $f(x) = \frac{k^{d+2}-1}{k^4-1} + (k^2-2)x - 1$ is an increasing function and it takes its maximum value at $x = \frac{k^{d+2}-1}{k^4-1}$. Then,

$$Nr(T_{k,d}) \leq \frac{k^{d+2}-k^2-2}{k^2+1}. \tag{18}$$

(iv) If $\frac{k^{d+2}-1}{k^4-1} + 1 \leq x \leq \frac{k^{d+1}-1}{k-1} - 2$, then $w(T_{k,d}/S) \leq \frac{k^2(k^{d+2}-1)}{k^4-1}$ and $c(T_{k,d}/S) \geq 1$. Then,

$$Nr(T_{k,d}) \leq \max_x\{\frac{k^2(k^{d+2}-1)}{k^4-1} - x - 1\}.$$

The function $f(x) = \frac{k^2(k^{d+2}-1)}{k^4-1} - x - 1$ is a decreasing function and it takes its maximum value at $x = \frac{k^{d+2}-1}{k^4-1} + 1$. Therefore,

$$Nr(T_{k,d}) \leq \frac{k^{d+2}-2k^2-3}{k^2+1}. \tag{19}$$

It is obvious that there is a subversion strategy S^* of $T_{k,d}$ such that $|S^*| = \frac{k^{d+2}-1}{k^4-1}$ where S contains all the vertices on the $\{0th, 4th, 8th, ..., (d-2)th\}$ levels. Then $w(T_{k,d}/S^*) = \frac{k^2(k^{d+2}-1)}{k^4-1}$ and $c(T_{k,d}/S^*) = 1$. Hence we get

$$Nr(T_{k,d}) \geq \frac{k^{d+2}-k^2-2}{k^2+1}. \tag{20}$$

The proof is completed by (16), (17), (18), (19) and (20).

Case 4: Let $d \equiv 3\ (Mod\ 4)$.

(i) If $0 \leq x \leq \frac{k^{d+1}-1}{k^4-1} - 1$, then $w(T_{k,d}/S) \leq (k^2+k-1)x + 1$ and $c(T_{k,d}/S) \geq 1$. Thus,

$$Nr(T_{k,d}) \leq \max_x\{(k^2+k-1)x + 1 - x - 1\} = \max_x\{(k^2+k-2)x\}.$$

The function $f(x) = (k^2 + k - 2)x$ is an increasing function and it takes its maximum value at $x = \frac{k^{d+1}-1}{k^4-1} - 1$. Then,

$$Nr(T_{k,d}) \leq \frac{k^{d+2} + 2k^{d+1} - k^5 - 2k^4}{(k^2+1)(k+1)}. \tag{21}$$

(ii) If $\frac{k^{d+1}-1}{k^4-1} \leq x \leq \frac{k(k^{d+1}-1)}{k^4-1}$, then $w(T_{k,d}/S) \leq (k^2 + k - 1)\frac{k^{d+1}-1}{k^4-1} + (k^2-1)(x - \frac{k^{d+1}-1}{k^4-1}) = \frac{k(k^{d+1}-1)}{k^4-1} + (k^2-1)x$ and $c(T_{k,d}/S) \geq 1$. Then,

$$Nr(T_{k,d}) \leq \max_x\{\frac{k(k^{d+1}-1)}{k^4-1} + (k^2-1)x - x - 1\}.$$

The function $f(x) = \frac{k(k^{d+1}-1)}{k^4-1} + (k^2-2)x - 1$ is an increasing function and it takes its maximum value at $x = \frac{k(k^{d+1}-1)}{k^4-1}$. Therefore,

$$Nr(T_{k,d}) \leq \frac{k^{d+2} - k^2 - k - 1}{k^2+1}. \tag{22}$$

(iii) If $\frac{k(k^{d+1}-1)}{k^4-1} + 1 \leq x \leq \frac{k^{d+1}-1}{k-1} - 2$, then $w(T_{k,d}/S) \leq \frac{k^3(k^{d+1}-1)}{k^4-1}$ and $c(T_{k,d}/S) \geq 1$. Then,

$$Nr(T_{k,d}) \quad \leq \quad \max_x\{\frac{k^3(k^{d+1}-1)}{k^4-1} - x - 1\}$$

The function $f(x) = \frac{k^3(k^{d+1}-1)}{k^4-1} - x - 1$ is a decreasing function and it takes its maximum value at $x = \frac{k(k^{d+1}-1)}{k^4-1} + 1$. Therefore,

$$Nr(T_{k,d}) \leq \frac{k^{d+2} - 2k^2 - k - 2}{k^2+1}. \tag{23}$$

It is clear that there is a subversion strategy S^* of $T_{k,d}$ such that $|S^*| = \frac{k(k^{d+1}-1)}{k^4-1}$ where S contains all the vertices on the $\{1st, 5th, 9th, ..., (d-2)th\}$ levels. Then $w(T_{k,d}/S^*) = \frac{k^3(k^{d+1}-1)}{k^4-1}$ and $c(T_{k,d}/S^*) = 1$. Hence we get

$$Nr(T_{k,d}) \geq \frac{k^{d+2} - k^2 - k - 1}{k^2+1}. \tag{24}$$

The proof is completed by (21), (22), (23) and (24). □

In the following theorems, we obtain the neighbor rupture degree of the cartesian product of a complete graph and a complete k-ary tree.

Theorem 11. *Let $T_{k,d}$ be a complete k-ary tree of depth d and K_p be a complete graph with p vertices where $2 \leq k < p - 1$. Then,*

$$Nr(K_p \times T_{k,d}) = \begin{cases} \frac{k^{d+1} + k - kp - p}{k+1}, & \text{if } d \text{ is odd;} \\ \frac{k^{d+1} + k - kp - p + 2}{k+1}, & \text{if } d \text{ is even.} \end{cases}$$

Proof. A complete k-ary tree of depth d has $\frac{k^{d+1}-1}{k-1}$ vertices and the covering number of $T_{k,d}$ is

$$\beta(T_{k,d}) = \begin{cases} \dfrac{k^{d+1}-1}{k^2-1}, & \text{d is odd;} \\ \dfrac{k(k^d-1)}{k^2-1}, & \text{d is even.} \end{cases}$$

Let S be a subversion strategy of $K_p \times T_{k,d}$ and let $|S| = x$ be the number of the removing vertices. There are two cases according to the cardinality of S:

Case 1: If $1 \leq x \leq \beta(T_{k,d}) - 1$, then $w((K_p \times T_{k,d})/S) \leq kx + 1$ and $c((K_p \times T_{k,d})/S) > p - 1$. Thus we have

$$Nr(K_p \times T_{k,d}) < \max_x\{kx + 1 - x - p + 1\} = \max_x\{(k-1)x + 2 - p\}.$$

The function $f(x) = (k-1)x + 2 - p$ is an increasing function and it takes its maximum value at $x = \beta(T_{k,d}) - 1$. Then,

$$Nr(K_p \times T_{k,d}) < \begin{cases} \dfrac{k^{d+1} - k^2 + 2k - kp - p + 2}{k+1}, & \text{if } d \text{ is odd;} \\ \dfrac{k^{d+1} - k^2 + k - kp - p + 3}{k+1}, & \text{if } d \text{ is even.} \end{cases} \quad (25)$$

Case 2: If $\beta(T_{k,d}) \leq x < \frac{k^{d+1}-1}{k-1}$, then $w((K_p \times T_{k,d})/S) \leq \frac{k^{d+1}-1}{k-1} - x$ and $c((K_p \times T_{k,d})/S) \geq p - 1$. Then, we get

$$Nr(K_p \times T_{k,d}) \leq \max_x\{\frac{k^{d+1}-1}{k-1} - 2x - p + 1\}$$

The function $f(x) = \frac{k^{d+1}-1}{k-1} - 2x - p + 1$ is a decreasing function and takes its minimum value at $x = \beta(T_{k,d})$. Thus

$$Nr(K_p \times T_{k,d}) \leq \begin{cases} \dfrac{k^{d+1} + k - kp - p}{k+1}, & \text{if } d \text{ is odd;} \\ \dfrac{k^{d+1} + k - kp - p + 2}{k+1}, & \text{if } d \text{ is even.} \end{cases} \quad (26)$$

It is obvious that there is a subversion strategy S^* of $K_p \times T_{k,d}$ such that $|S^*| = \beta(T_{k,d})$, $c((K_p \times T_{k,d})/S) = p - 1$ and $w((K_p \times T_{k,d})/S) = \frac{k(k^{d+1}-1)}{k^2-1}$ where d is odd and $w((K_p \times T_{k,d})/S) = \frac{k^{d+2}-1}{k^2-1}$ where d is even. Hence we get

$$Nr(K_p \times T_{k,d}) \geq \begin{cases} \dfrac{k^{d+1} + k - kp - p}{k+1}, & \text{if } d \text{ is odd;} \\ \dfrac{k^{d+1} + k - kp - p + 2}{k+1}, & \text{if } d \text{ is even.} \end{cases} \quad (27)$$

The proof is completed by (25), (26) ve (27). □

Theorem 12. *Let $T_{k,d}$ be a complete k-ary tree of depth d and K_p be a complete graph with p vertices where $k \geq p-1 \geq 1$. Then,*

$$Nr(K_p \times T_{k,d}) = \begin{cases} \frac{k^{d+5}-k^{d+4}+k^{d+3}+k^{d+2}-k^{d+1}+k^4-k^3-k^2+k-2-k^5(p-1)+p}{k^5-1}, & d \equiv 0 \ (Mod\ 5); \\ \frac{k^{d+5}-k^{d+4}+k^{d+3}+k^{d+2}-k^{d+1}-k^4-k^3+k^2-k-k^5(p-1)+p}{k^5-1}, & d \equiv 1 \ (Mod\ 5); \\ \frac{k^{d+5}-k^{d+4}+k^{d+3}+k^{d+2}-k^{d+1}-k^4+k^3-k^2+k-2-k^5(p-1)+p}{k^5-1}, & d \equiv 2 \ (Mod\ 5); \\ \frac{k^{d+5}-k^{d+4}+k^{d+3}+k^{d+2}-k^{d+1}+k^4-k^3+k^2-k-k^5(p+1)+p}{k^5-1}, & d \equiv 3 \ (Mod\ 5); \\ \frac{k^{d+5}-k^{d+4}+k^{d+3}+k^{d+2}-k^{d+1}-k^4+k^3-k^2-k-k^5(p-1)+p}{k^5-1}, & d \equiv 4 \ (Mod\ 5). \end{cases}$$

Proof. Let S be a subversion strategy of $K_p \times T_{k,d}$ and let $|S| = x$ be the number of removing vertices. There are five cases according to the depth of the complete k-ary tree $T_{k,d}$.

Case 1: Let $d \equiv 0\ (Mod\ 5)$.

(i) If $0 \leq x \leq \frac{k(k^3+1)(k^d-1)}{k^5-1}$, then $w((K_p \times T_{k,d})/S) \leq kx + \frac{k^2}{k^3+1}x + 1$ and $c((K_p \times T_{k,d})/S) \geq p-1$. Thus,

$$Nr(K_p \times T_{k,d}) \leq \max_x\{kx + \frac{k^2}{k^3+1}x + 1 - x - p + 1\} = \max_x\{(k + \frac{k^2}{k^3+1} - 1)x - p + 2\}.$$

The function $f(x) = (k + \frac{k^2}{k^3+1} - 1)x - p + 2$ is an increasing function and it takes its maximum value at $x = \frac{k(k^3+1)(k^d-1)}{k^5-1}$. Then,

$$Nr(K_p \times T_{k,d}) \leq \frac{k^{d+5} - k^{d+4} + k^{d+3} + k^{d+2} - k^{d+1} + k^4 - k^3 - k^2 + k - 2 - k^5(p-1) + p}{k^5-1} \quad (28)$$

(ii) If $\frac{k(k^3+1)(k^d-1)}{k^5-1} + 1 \leq x \leq \frac{k^{d+1}-1}{k-1} - 1$, then $c((K_p \times T_{k,d})/S) \geq p-1$ and $w((K_p \times T_{k,d})/S) \leq \frac{k^{d+5}+k^{d+3}+k^{d+2}-k^3-k^2-1}{k^5-1}$. Then,

$$Nr((K_p \times T_{k,d})) \leq \max_x\{\frac{k^{d+5} + k^{d+3} + k^{d+2} - k^3 - k^2 - 1}{k^5-1} - x - p + 1\}.$$

The function $f(x) = \frac{k^{d+5}+k^{d+3}+k^{d+2}-k^3-k^2-1}{k^5-1} - x - p + 1$ is a decreasing function and it takes its maximum value at $x = \frac{k(k^3+1)(k^d-1)}{k^5-1} + 1$. Therefore,

$$Nr(K_p \times T_{k,d}) \leq \frac{k^{d+5} - k^{d+4} + k^{d+3} + k^{d+2} - k^{d+1} - k^5 p + k^4 - k^3 - k^2 + k - 1 + p}{k^5-1} \quad (29)$$

It can be easily seen that there is a subversion strategy S^* of $K_p \times T_{k,d}$ such that $|S^*| = \frac{k(k^3+1)(k^d-1)}{k^5-1}$ where S contains all the vertices on the $\{1st, 4th, 6th, 9th, 11th, ..., (d-4)th, (d-1)th\}$ levels of $T_{k,d}$. Then $c((K_p \times T_{k,d})/S^*) = p-1$ and $w((K_p \times T_{k,d})/S^*) = \frac{k^{d+5}+k^{d+3}+k^{d+2}-k^3-k^2-1}{k^5-1}$. Hence we get

$$Nr(K_p \times T_{k,d}) \geq \frac{k^{d+5} - k^{d+4} + k^{d+3} + k^{d+2} - k^{d+1} + k^4 - k^3 - k^2 + k - 2 - k^5(p-1) + p}{k^5-1} \quad (30)$$

The proof is completed by (28), (29), and (30).

Case 2: Let $d \equiv 1$ $(Mod\ 5)$.

(i) If $0 \leq x \leq \frac{k^{d+4}+k^{d+1}-k^2-1}{k^5-1} - 1$, then $w((K_p \times T_{k,d})/S) \leq kx + \frac{k^2}{k^3+1}x + 1$ and $c((K_p \times T_{k,d})/S) \geq p - 1$. Thus,

$$Nr(K_p \times T_{k,d}) \leq \max_x\{kx + \frac{k^2}{k^3+1}x + 1 - x - p + 1\} = \max_x\{(k + \frac{k^2}{k^3+1} - 1)x - p + 2\}.$$

The function $f(x) = (k + \frac{k^2}{k^3+1} - 1)x - p + 2$ is an increasing function and it takes its maximum value at $x = \frac{k^{d+4}+k^{d+1}-k^2-1}{k^5-1} - 1$. Then,

$$Nr(K_p \times T_{k,d}) \leq \frac{k^{d+5} - k^{d+4} + k^{d+3} + k^{d+2} - k^{d+1} - k^6 + 3k^5 - k^4 - k^3 + k^2 - 2 - k^5p + p}{k^5 - 1} \quad (31)$$

(ii) If $x = \frac{k^{d+4}+k^{d+1}-k^2-1}{k^5-1}$, then $w((K_p \times T_{k,d})/S) \leq (x-1)(k + \frac{k^2}{k^3+1}) + k$ and $c((K_p \times T_{k,d})/S) \geq p - 1$. Then,

$$\begin{aligned} Nr(K_p \times T_{k,d}) &\leq (x-1)(k + \frac{k^2}{k^3+1}) + k - x - p + 1 \\ &= \frac{k^{d+5} - k^{d+4} + k^{d+3} + k^{d+2} - k^{d+1} + k^5 - k^4 - k^3 + k^2 - k - k^5p + p}{k^5 - 1} \end{aligned} \quad (32)$$

(iii) If $\frac{k^{d+4}+k^{d+1}-k^2-1}{k^5-1} + 1 \leq x \leq \frac{k^{d+1}-1}{k-1} - 1$, then $c((K_p \times T_{k,d})/S) \geq p - 1$ and $w((K_p \times T_{k,d})/S) \leq \frac{k^{d+5}+k^{d+3}+k^{d+2}-k^4-k^3-k}{k^5-1}$. Thus,

$$Nr(K_p \times T_{k,d}) \leq \max_x\{\frac{k^{d+5} + k^{d+3} + k^{d+2} - k^4 - k^3 - k}{k^5 - 1} - x - p + 1\}.$$

The function $f(x) = \frac{k^{d+5}+k^{d+3}+k^{d+2}-k^4-k^3-k}{k^5-1} - x - p + 1$ is a decreasing function and it takes its maximum value at $x = \frac{k^{d+4}+k^{d+1}-k^2-1}{k^5-1} + 1$. Therefore,

$$Nr(K_p \times T_{k,d}) \leq \frac{k^{d+5} - k^{d+4} + k^{d+3} + k^{d+2} - k^{d+1} - k^4 - k^3 + k^2 - k - k^5p + p + 1}{k^5 - 1} \quad (33)$$

It can be easily seen that there is a subversion strategy S^* of $K_p \times T_{k,d}$ such that $|S^*| = \frac{k^{d+4}+k^{d+1}-k^2-1}{k^5-1}$ where S contains all the vertices on the $\{0th, 2nd, 5th, 7th, 10th, \ldots, (d-4)th, (d-1)th\}$ levels of $T_{k,d}$. Then $w((K_p \times T_{k,d})/S^*) = \frac{k^{d+5}+k^{d+3}+k^{d+2}-k^4-k^3-k}{k^5-1}$ and $c((K_p \times T_{k,d})/S^*) = p - 1$. Hence we get

$$Nr(K_p \times T_{k,d}) \geq \frac{k^{d+5} - k^{d+4} + k^{d+3} + k^{d+2} - k^{d+1} + k^5 - k^4 - k^3 + k^2 - k - k^5p + p}{k^5 - 1} \quad (34)$$

The proof is completed by (31), (32), (33) and (34).

Case 3: Let $d \equiv 2$ $(Mod\ 5)$.

(i) If $0 \le x \le \frac{k^{d+4}+k^{d+1}-k^6-k^3}{k^5-1}$, then $w((K_p \times T_{k,d})/S) \le kx + \frac{k^2}{k^3+1}x + 1$ and $c((K_p \times T_{k,d})/S) \ge p-1$. Thus we have

$$Nr(K_p \times T_{k,d}) \le \max_x\{kx + \frac{k^2}{k^3+1}x + 1 - x - p + 1\} = \max_x\{(k + \frac{k^2}{k^3+1} - 1)x - p + 2\}.$$

The function $f(x) = (k + \frac{k^2}{k^3+1} - 1)x - p + 2$ is an increasing function and it takes its maximum value at $x = \frac{k^{d+4}+k^{d+1}-k^6-k^3}{k^5-1}$. Then,

$$Nr(K_p \times T_{k,d}) \le \frac{k^{d+5} - k^{d+4} + k^{d+3} + k^{d+2} - k^{d+1} - k^7 + k^6 + k^5 - k^4 + k^3 - k^5p + p - 2}{k^5 - 1} \quad (35)$$

(ii) If $\frac{k^{d+4}+k^{d+1}-k^6-k^3}{k^5-1}+1 \le x \le \frac{k^{d+4}+k^{d+1}-k^3-k}{k^5-1}$, then $c((K_p \times T_{k,d})/S) \ge p-1$ and $w((K_p \times T_{k,d})/S) \le (\frac{k^{d+4}+k^{d+1}-k^6-k^3}{k^5-1})(k+\frac{k^2}{k^3+1})+k(x-\frac{k^{d+4}+k^{d+1}-k^6-k^3}{k^5-1})+1$. Then,

$$\begin{aligned} Nr(K_p \times T_{k,d}) &\le \max_x\{(\frac{k^{d+4}+k^{d+1}-k^6-k^3}{k^5-1})(k+\frac{k^2}{k^3+1}) \\ &+ k(x - \frac{k^{d+4}+k^{d+1}-k^6-k^3}{k^5-1}) - x - p + 2\} \\ &= \max_x\{\frac{k^{d+3}-k^5}{k^5-1} + (k-1)x - p + 2\}. \end{aligned}$$

The function $f(x) = \frac{k^{d+3}-k^5}{k^5-1} + (k-1)x - p + 2$ is an increasing function and it takes its maximum value at $x = \frac{k^{d+4}+k^{d+1}-k^3-k}{k^5-1}$. Therefore,

$$Nr(K_p \times T_{k,d}) \le \frac{k^{d+5} - k^{d+4} + k^{d+3} + k^{d+2} - k^{d+1} + k^5 - k^4 + k^3 - k^2 + k - k^5p + p - 2}{k^5 - 1} \quad (36)$$

(iii) If $\frac{k^{d+4}+k^{d+1}-k^3-k}{k^5-1} + 1 \le x \le \frac{k^{d+1}-1}{k-1} - 1$, then $c((K_p \times T_{k,d})/S) \ge p-1$ and $w((K_p \times T_{k,d})/S) \le \frac{k^{d+5}+k^{d+3}+k^{d+2}-k^4-k^2-1}{k^5-1}$. Then,

$$Nr(K_p \times T_{k,d}) \le \max_x\{\frac{k^{d+5}+k^{d+3}+k^{d+2}-k^4-k^2-1}{k^5-1} - x - p + 1\}.$$

The function $f(x) = \frac{k^{d+5}+k^{d+3}+k^{d+2}-k^4-k^2-1}{k^5-1} - x - p + 1$ is a decreasing function and it takes its maximum value at $x = \frac{k^{d+4}+k^{d+1}-k^3-k}{k^5-1} + 1$. Therefore,

$$Nr(K_p \times T_{k,d}) \le \frac{k^{d+5} - k^{d+4} + k^{d+3} + k^{d+2} - k^{d+1} - k^4 - k^3 + k^2 - k - k^5p + p + 1}{k^5 - 1} \quad (37)$$

It can be easily seen that there is a subversion strategy S^* of $K_p \times T_{k,d}$ such that $|S^*| = \frac{k^{d+4}+k^{d+1}-k^3-k}{k^5-1}$ where S contains all the vertices on the $\{1st, 3rd, 6th, 8th, 11th, \ldots, (d-4)th, (d-1)th\}$ levels of $T_{k,d}$. Then, we have $w((K_p \times T_{k,d})/S^*) = \frac{k^{d+5}+k^{d+3}+k^{d+2}-k^4-k^2-1}{k^5-1}$ and $c((K_p \times T_{k,d})/S^*) = p-1$. Hence we get

$$Nr(K_p \times T_{k,d}) \ge \frac{k^{d+5} - k^{d+4} + k^{d+3} + k^{d+2} - k^{d+1} + k^5 - k^4 + k^3 - k^2 + k - k^5p + p - 2}{k^5 - 1} \quad (38)$$

The proof is completed by (35), (36), (37) and (38).

Case 4: Let $d \equiv 3$ $(Mod\ 5)$.

(i) If $0 \le x \le \frac{k^{d+4}+k^{d+1}-k^7-k^4}{k^5-1}$, then $w((K_p \times T_{k,d})/S) \le kx + \frac{k^2}{k^3+1}x + 1$ and $c((K_p \times T_{k,d})/S) \ge p-1$. Thus,

$$Nr(K_p \times T_{k,d}) \le \max_x\{kx + \frac{k^2}{k^3+1}x + 1 - x - p + 1\} = \max_x\{(k + \frac{k^2}{k^3+1} - 1)x - p + 2\}.$$

The function $f(x) = (k + \frac{k^2}{k^3+1} - 1)x - p + 2$ is an increasing function and it takes its maximum value at $\frac{k^{d+4}+k^{d+1}-k^7-k^4}{k^5-1}$. Then,

$$Nr(K_p \times T_{k,d}) \le \frac{k^{d+5} - k^{d+4} + k^{d+3} + k^{d+2} - k^{d+1} - k^8 + k^7 - k^6 + k^5 + k^4 - k^5p + p - 2}{k^5-1} \quad (39)$$

(ii) If $\frac{k^{d+4}+k^{d+1}-k^7-k^4}{k^5-1} + 1 \le x \le \frac{k^{d+4}+k^{d+1}-k^4-k^2}{k^5-1}$, then we have $w((K_p \times T_{k,d})/S) \le (\frac{k^{d+4}+k^{d+1}-k^7-k^4}{k^5-1})(k + \frac{k^2}{k^3+1}) + k(x - \frac{k^{d+4}+k^{d+1}-k^7-k^4}{k^5-1}) + 1$ and $c((K_p \times T_{k,d})/S) \ge p-1$. Therefore,

$$\begin{aligned} Nr(K_p \times T_{k,d}) &\le \max_x\{(\frac{k^{d+4}+k^{d+1}-k^7-k^4}{k^5-1})(k + \frac{k^2}{k^3+1}) \\ &+ k(x - \frac{k^{d+4}+k^{d+1}-k^7-k^4}{k^5-1}) + 1 - x - p + 1\} \\ &= \max_x\{\frac{k^{d+3}-k^6}{k^5-1} + (k-1)x - p + 2\}. \end{aligned}$$

The function $f(x) = \frac{k^{d+3}-k^6}{k^5-1} + (k-1)x - p + 2$ is an increasing function and it takes its maximum value at $x = \frac{k^{d+4}+k^{d+1}-k^4-k^2}{k^5-1}$. Thus,

$$Nr(K_p \times T_{k,d}) \le \frac{k^{d+5} - k^{d+4} + k^{d+3} + k^{d+2} - k^{d+1} - k^6 + k^5 + k^4 - k^3 + k^2 - 2 - k^5p + p}{k^5-1} \quad (40)$$

(iii) If $x = \frac{k^{d+4}+k^{d+1}-k^4-k^2}{k^5-1} + 1$, then we have $c((K_p \times T_{k,d})/S) \ge p-1$ and $w((K_p \times T_{k,d})/S) \le (\frac{k^{d+4}+k^{d+1}-k^7-k^4}{k^5-1})(k + \frac{k^2}{k^3+1}) + k^3 + k$. Thus

$$\begin{aligned} Nr(K_p \times T_{k,d}) &\le \max_x\{(\frac{k^{d+4}+k^{d+1}-k^7-k^4}{k^5-1})(k + \frac{k^2}{k^3+1}) + k^3 + k - x - p + 1\} \\ &= \frac{k^{d+5} - k^{d+4} + k^{d+3} + k^{d+2} - k^{d+1} - k^5 + k^4 - k^3 + k^2 - k - k^5p + p}{k^5-1} \end{aligned} \quad (41)$$

(iv) If $\frac{k^{d+4}+k^{d+1}-k^4-k^2}{k^5-1} + 2 \le x \le \frac{k^{d+1}-1}{k-1} - 1$, then $w((K_p \times T_{k,d})/S) \le \frac{k^{d+5}+k^{d+3}+k^{d+2}-k^5-k^3-k}{k^5-1}$ and $c((K_p \times T_{k,d})/S) \ge p-1$. Thus,

$$Nr(K_p \times T_{k,d}) \le \max_x\{\frac{k^{d+5}+k^{d+3}+k^{d+2}-k^5-k^3-k}{k^5-1} - x - p + 1\}.$$

The function $f(x) = \frac{k^{d+5}+k^{d+3}+k^{d+2}-k^5-k^3-k}{k^5-1} - x - p + 1$ is a decreasing function and it takes its maximum value at $x = \frac{k^{d+4}+k^{d+1}+k^5-k^4-k^2-1}{k^5-1} + 1$. Therefore,

$$Nr(K_p \times T_{k,d}) \leq \frac{k^{d+5} - k^{d+4} + k^{d+3} + k^{d+2} - k^{d+1} - 2k^5 + k^4 - k^3 + k^2 - k - k^5p + p + 1}{k^5 - 1} \quad (42)$$

It can be easily seen that there is a subversion strategy S^* of $K_p \times T_{k,d}$ such that $|S^*| = \frac{k^{d+4}+k^{d+1}-k^4-k^2}{k^5-1} + 1$ where S contains all the vertices on the $\{0th, 2nd, 4th, 7th, 9th, 12th, \ldots, (d-4)th, (d-1)th\}$ levels of $T_{k,d}$. Then $w((K_p \times T_{k,d})/S^*) = \frac{k^{d+5}+k^{d+3}+k^{d+2}-k^5-k^3-k}{k^5-1}$ and $c((K_p \times T_{k,d})/S^*) = p - 1$. Hence we get

$$Nr(K_p \times T_{k,d}) \geq \frac{k^{d+5} - k^{d+4} + k^{d+3} + k^{d+2} - k^{d+1} - k^5 + k^4 - k^3 + k^2 - k - k^5p + p}{k^5 - 1} \quad (43)$$

The proof is completed by (39), (40), (41), (42) and (43).

Case 5: Let $d \equiv 4 \ (Mod\ 5)$.

(i) If $0 \leq x \leq \frac{k^{d+4}+k^{d+1}-k^8-k^5}{k^5-1}$, then $w((K_p \times T_{k,d})/S) \leq kx + \frac{k^2}{k^3+1}x + 1$ and $c((K_p \times T_{k,d})/S) \geq p - 1$. Thus,

$$Nr(K_p \times T_{k,d}) \leq \max_x\{kx + \frac{k^2}{k^3+1}x + 1 - x - p + 1\} = \max_x\{(k + \frac{k^2}{k^3+1} - 1)x - p + 2\}.$$

The function $f(x) = (k + \frac{k^2}{k^3+1} - 1)x - p + 2$ is an increasing function and it takes its maximum value at $x = \frac{k^{d+4}+k^{d+1}-k^8-k^5}{k^5-1}$. Then,

$$Nr(K_p \times T_{k,d}) \leq \frac{k^{d+5} - k^{d+4} + k^{d+3} + k^{d+2} - k^{d+1} - k^9 + k^8 - k^7 - k^6 + 3k^5 - k^5p + p - 2}{k^5 - 1} \quad (44)$$

(ii) If $\frac{k^{d+4}+k^{d+1}-k^8-k^5}{k^5-1} + 1 \leq x \leq \frac{k^{d+4}+k^{d+1}-k^8+k^6-k-1}{k^5-1}$, then we have $w((K_p \times T_{k,d})/S) \leq (\frac{k^{d+4}+k^{d+1}-k^8-k^5}{k^5-1})(k + \frac{k^2}{k^3+1}) + k(x - \frac{k^{d+4}+k^{d+1}-k^8-k^5}{k^5-1}) + 1$ and $c((K_p \times T_{k,d})/S) \geq p - 1$. Then,

$$\begin{aligned} Nr(K_p \times T_{k,d}) &\leq \max_x\{(\frac{k^{d+4} + k^{d+1} - k^8 - k^5}{k^5 - 1})(k + \frac{k^2}{k^3+1}) \\ &+ k(x - \frac{k^{d+4} + k^{d+1} - k^8 - k^5}{k^5 - 1}) + 1 - x - p + 1\} \\ &= \max_x\{\frac{k^{d+3} - k^7}{k^5 - 1} + (k-1)x - p + 2\}. \end{aligned}$$

The function $f(x) = \frac{k^{d+3}-k^7}{k^5-1} + (k-1)x - p + 2$ is an increasing function and it takes its maximum value at $x = \frac{k^{d+4}+k^{d+1}-k^8+k^6-k-1}{k^5-1}$. Therefore,

$$Nr(K_p \times T_{k,d}) \leq \frac{k^{d+5} - k^{d+4} + k^{d+3} + k^{d+2} - k^{d+1} - k^9 + k^8 - k^6 + 2k^5 - k^2 - 1 - k^5p + p}{k^5 - 1} \quad (45)$$

(iii) If $\frac{k^{d+4}+k^{d+1}-k^8+k^6-k-1}{k^5-1}+1 \le x \le \frac{k^{d+4}+k^{d+1}-k^3-1}{k^5-1}$, then $w((K_p \times T_{k,d})/S) \le (\frac{k^{d+4}+k^{d+1}-k^8-k^5}{k^5-1})(k+\frac{k^2}{k^3+1})+k(k+1)+(x-\frac{k^{d+4}+k^{d+1}-k^8+k^6-k-1}{k^5-1})(k+\frac{1}{k})+1$ and $c((K_p \times T_{k,d})/S) \ge p-1$. Therefore,

$$\begin{aligned} Nr(K_p \times T_{k,d}) &\le \max_x\{(\frac{k^{d+4}+k^{d+1}-k^8-k^5}{k^5-1})(k+\frac{k^2}{k^3+1}) \\ &+ (x-\frac{k^{d+4}+k^{d+1}-k^8+k^6-k-1}{k^5-1})(k+\frac{1}{k})+1-x-p+1\} \\ &= \max_x\{\frac{k^{d+1}+k^6-k-1}{k(k^5-1)}+(k+\frac{1}{k}-1)x-p+2\}. \end{aligned}$$

The function $f(x) = \frac{k^{d+1}+k^6-k-1}{k(k^5-1)}+(k+\frac{1}{k}-1)x-p+2$ is an increasing function and it takes its maximum value at $x = \frac{k^{d+4}+k^{d+1}-k^3-1}{k^5-1}$. Thus,

$$Nr(K_p \times T_{k,d}) \le \frac{k^{d+5}-k^{d+4}+k^{d+3}+k^{d+2}-k^{d+1}+k^5-k^4+k^3-k^2-k-k^5p+p}{k^5-1} \quad (46)$$

(iv) If $\frac{k^{d+4}+k^{d+1}-k^3-1}{k^5-1}+1 \le x \le \frac{k^{d+1}-1}{k-1}-1$, then $c((K_p \times T_{k,d})/S) \ge p-1$ and $w((K_p \times T_{k,d})/S) \le \frac{k^{d+5}+k^{d+3}+k^{d+2}-k^4-k^2-k}{k^5-1}$. Then we get

$$Nr(K_p \times T_{k,d}) \le \max_x\{\frac{k^{d+5}+k^{d+3}+k^{d+2}-k^4-k^2-k}{k^5-1}-x-p+1\}.$$

The function $f(x) = \frac{k^{d+5}+k^{d+3}+k^{d+2}-k^4-k^2-k}{k^5-1}-x-p+1$ is a decreasing function and it takes its maximum value at $x = \frac{k^{d+4}+k^{d+1}-k^3-1}{k^5-1}+1$. Therefore,

$$Nr(K_p \times T_{k,d}) \le \frac{k^{d+5}-k^{d+4}+k^{d+3}+k^{d+2}-k^{d+1}-k^4+k^3-k^2-k-k^5p+p+1}{k^5-1} \quad (47)$$

It is obvious that there is a subversion strategy S^* of $K_p \times T_{k,d}$ such that $|S^*| = \frac{k^{d+4}+k^{d+1}-k^3-1}{k^5-1}$ where S contains all the vertices on the $\{0th, 3rd, 5th, 8th, 10th, 13th, \ldots, (d-4)th, (d-1)th\}$ levels of $T_{k,d}$. Then $c((K_p \times T_{k,d})/S^*) = p-1$ and $w((K_p \times T_{k,d})/S^*) = \frac{k^{d+5}+k^{d+3}+k^{d+2}-k^4-k^2-k}{k^5-1}$. Hence we get

$$Nr(K_p \times T_{k,d}) \ge \frac{k^{d+5}-k^{d+4}+k^{d+3}+k^{d+2}-k^{d+1}+k^5-k^4+k^3-k^2-k-k^5p+p}{k^5-1} \quad (48)$$

The proof is completed by (44), (45), (46), (47) and (48).

□

Acknowledgements. Part of this paper was published in the proceedings of the 1st International Symposium on Computing in Science and Engineering, 3-5 June 2010, Kusadasi, Aydin, Turkey.

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