# Binding numbers and $(a, b, k)$-critical graphs

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**Abstract**

Let $G$ be a graph of order $n$, and let $a, b, k$ be nonnegative integers with $1 \leq a < b$. A spanning subgraph $F$ of $G$ is called an $[a, b]$-factor if $a \leq d_F(x) \leq b$ for each $x \in V(G)$. Then a graph $G$ is called an $(a, b, k)$-critical graph if $G - N$ has an $[a, b]$-factor for each $N \subseteq V(G)$ with $|N| = k$. In this paper, it is proved that $G$ is an $(a, b, k)$-critical graph if $n \geq \frac{(a+b-1)(a+b-2)}{b} + \frac{bk}{b-1}$, $bind(G) \geq \frac{(a+b-1)(n-1)}{b(n-1-k)}$ and $\delta(G) \neq \lfloor \frac{(a-1)n+a+b+bk-2}{a+b-1} \rfloor$.

**Keywords:** operations research, graph, $[a, b]$-factor, binding number, $(a, b, k)$-critical graph

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## 1 Introduction

Many physical structures can conveniently be modelled by networks. A wide variety of systems can be described by complex networks. Factors and factorizations in networks are very useful in combinatorial design, network design, circuit layout and so on [1]. It is well-known that a network can be represented by a graph. Vertices and edges of the graph correspond to nodes and links between the nodes, respectively. Henceforth, we use the term *graph* instead of *network*.

All graphs considered in this paper will be finite and undirected simple graphs. Let $G$ be a graph. We use $V(G)$ and $E(G)$ to denote its vertex set and edge set, respectively. For any $x \in V(G)$, the degree and the neighborhood of $x$ in $G$ are denoted by $d_G(x)$ and $N_G(x)$, respectively.

For $S \subseteq V(G)$, we write $N_G(S) = \bigcup_{x \in S} N_G(x)$, and denote by $G[S]$ the subgraph of $G$ induced by $S$, and $G - S = G[V(G) \setminus S]$. We say that $S$ is independent if $N_G(S) \cap S = \emptyset$. We denote by $\delta(G)$ the minimum degree of $G$. The binding number of $G$ is defined as

$$bind(G) = min\{\frac{|N_G(X)|}{|X|} : \emptyset \neq X \subseteq V(G), N_G(X) \neq V(G)\}.$$

For a real number $r$, we use $\lfloor r \rfloor$ to denote the floor of $r$, which is the largest integer smaller than or equal to $r$, and also use $\lceil r \rceil$ to denote the ceiling of $r$, which is the least integer greater than or equal to $r$.

Let $a$ and $b$ be two integers with $0 \leq a \leq b$. Then a spanning subgraph $F$ of $G$ is called an $[a, b]$-factor if $a \leq d_F(x) \leq b$ for each $x \in V(G)$. (where of course $d_F$ denotes the degree in $F$). And if $a = b = k$, then an $[a, b]$-factor is called a $k$-factor. A graph $G$ is called an $(a, b, k)$-critical graph if $G - N$ has an $[a, b]$-factor for each $N \subseteq V(G)$ with $|N| = k$. If $G$ is an $(a, b, k)$-critical graph, then we also say that $G$ is $(a, b, k)$-critical. If $a = b = n$, then an $(a, b, k)$-critical graph is simply called an $(n, k)$-critical graph. In particular, a $(1, k)$-critical graph is simply called a $k$-critical graph. Some other terminologies and notations can be found in [2].

Many authors have investigated graph factors [3-13]. The following results on $(a, b, k)$-critical graphs are known.

**Theorem 1** [14] *Let $G$ be a graph of order $n$, and let $a, b$ and $k$ be non-negative integers such that $1 \leq a < b$. If the binding number $bind(G) > \frac{(a+b-1)(n-1)}{bn-(a+b)-bk+2}$ and $n \geq \frac{(a+b-1)(a+b-2)}{b} + \frac{bk}{b-1}$, then $G$ is an $(a, b, k)$-critical graph.*

**Theorem 2** [15] *Let $a$, $b$ and $k$ be nonnegative integers with $1 \leq a < b$, and let $G$ be a graph of order $n$ with $n \geq \frac{(a+b-1)(a+b-2)}{b} + k$. Suppose that*

$$|N_G(X)| > \frac{(a-1)n + |X| + bk - 1}{a + b - 1}$$

*for every non-empty independent subset $X$ of $V(G)$, and*

$$\delta(G) > \frac{(a-1)n + a + b + bk - 2}{a + b - 1}.$$

*Then $G$ is an $(a, b, k)$-critical graph.*

Motivated by the above theorems, we prove the following result, which is a binding number condition for graphs to be $(a, b, k)$-critical graphs. Our result is an improvement of Theorem 1.

**Theorem 3** *Let $G$ be a graph of order $n$, and let $a, b$ and $k$ be nonnegative integers such that $1 \le a < b$. If $n \ge \frac{(a+b-1)(a+b-2)}{b} + \frac{bk}{b-1}$, $bind(G) \ge \frac{(a+b-1)(n-1)}{b(n-1-k)}$ and $\delta(G) \neq \lfloor \frac{(a-1)n+a+b+bk-2}{a+b-1} \rfloor$, then $G$ is an $(a, b, k)$-critical graph.*

In Theorem 3, if $k = 0$, then we get the following corollary.

**Corollary 1** *Let $G$ be a graph of order $n$, and let $a, b$ be two integers such that $1 \le a < b$. If $n \ge \frac{(a+b-1)(a+b-2)}{b}$, $bind(G) \ge \frac{a+b-1}{b}$ and $\delta(G) \neq \lfloor \frac{(a-1)n+a+b-2}{a+b-1} \rfloor$, then $G$ has an $[a, b]$-factor.*

Unfortunately, the authors do not know whether the binding number condition in Theorem 3 are best possible. But, the result of Theorem 3 is stronger than one of Theorem 1 if $\delta(G) \neq \lfloor \frac{(a-1)n+a+b+bk-2}{a+b-1} \rfloor$.

## 2 The Proof of Theorem 3

**Proof of Theorem 3.** For any $X \subseteq V(G)$ with $X \neq \emptyset$ and $N_G(X) \neq V(G)$. Let $Y = V(G) \setminus N_G(X)$. Obviously, $\emptyset \neq Y \subseteq V(G)$. Now, we prove the following claims.

**Claim 1.** $X \cap N_G(Y) = \emptyset$.

**Proof.** Suppose that $X \cap N_G(Y) \neq \emptyset$. Then there exists $x \in X \cap N_G(Y)$. Since $x \in N_G(Y)$, we have $y \in Y$ such that $xy \in E(G)$. Thus, we obtain $y \in N_G(x) \subseteq N_G(X)$. Which contradicts $y \in Y = V(G) \setminus N_G(X)$. This completes the proof of Claim 1.

**Claim 2.** $(b-1)(n-1) \ge bk$.

**Proof.** Using $1 \le a < b$ and $n \ge \frac{(a+b-1)(a+b-2)}{b} + \frac{bk}{b-1}$, we have

$$\begin{aligned}(b-1)(n-1) &= (b-1)n - b + 1 \\ &\ge \frac{(b-1)(a+b-1)(a+b-2)}{b} + bk - b + 1 \\ &\ge (a+b-2) + bk - b + 1 \\ &= bk + a - 1 \ge bk.\end{aligned}$$

This completes the proof of Claim 2.

**Claim 3.** $|N_G(X)| > \frac{(a-1)n+|X|+bk-1}{a+b-1}$.

**Proof.** By Claim 1, we have

$$|X| + |N_G(Y)| \le n \tag{1}$$

and

$$N_G(Y) \neq V(G). \tag{2}$$

From (1), (2), $bind(G) \geq \frac{(a+b-1)(n-1)}{b(n-1-k)}$ and the definition of $bind(G)$, we get

$$\begin{aligned}\frac{(a+b-1)(n-1)}{b(n-1-k)} &\leq bind(G) \leq \frac{|N_G(Y)|}{|Y|}\\ &\leq \frac{n-|X|}{|V(G)\setminus N_G(X)|} = \frac{n-|X|}{n-|N_G(X)|},\end{aligned}$$

which implies

$$|N_G(X)| \geq n - \frac{b(n-|X|)(n-1-k)}{(a+b-1)(n-1)}. \tag{3}$$

According to (3), $1 \leq a < b$, Claim 2 and $X \neq \emptyset$, we obtain

$$\begin{aligned}&|N_G(X)| \geq n - \frac{b(n-|X|)(n-1-k)}{(a+b-1)(n-1)}\\ =\ & \frac{(a+b-1)(n-1)n - b(n-1-k)n + b(n-1-k)|X|}{(a+b-1)(n-1)}\\ =\ & \frac{(a-1)(n-1)n + (n-1)|X| + ((b-1)(n-1) - bk)|X| + bkn}{(a+b-1)(n-1)}\\ \geq\ & \frac{(a-1)(n-1)n + (n-1)|X| + ((b-1)(n-1) - bk) + bkn}{(a+b-1)(n-1)}\\ =\ & \frac{(a-1)(n-1)n + (n-1)|X| + bk(n-1) + (b-1)(n-1)}{(a+b-1)(n-1)}\\ =\ & \frac{(a-1)n + |X| + bk + (b-1)}{a+b-1}\\ >\ & \frac{(a-1)n + |X| + bk - 1}{a+b-1}.\end{aligned}$$

This completes the proof of Claim 3.

In terms of $\emptyset \neq X \subseteq V(G)$ and $|N_G(X)| \geq \frac{(a-1)n+|X|+bk+(b-1)}{a+b-1}$, we have

$$\delta(G) \geq \frac{(a-1)n + bk + b}{a+b-1}. \tag{4}$$

**Claim 4.** $\delta(G) > \frac{(a-1)n+a+b+bk-2}{a+b-1}$.

**Proof.** Assume that $\delta(G) \leq \frac{(a-1)n+a+b+bk-2}{a+b-1}$. Using (4), we obtain

$$\left\lceil\frac{(a-1)n+b+bk}{a+b-1}\right\rceil \leq \delta(G) \leq \left\lfloor\frac{(a-1)n+a+b+bk-2}{a+b-1}\right\rfloor,$$

which implies,

$$\delta(G) = \lceil\frac{(a-1)n+b+bk}{a+b-1}\rceil = \lfloor\frac{(a-1)n+a+b+bk-2}{a+b-1}\rfloor.$$

That contradicts the condition of Theorem 3. The proof of Claim 4 is complete.

According to Claim3, Claim 4 and Theorem 2, $G$ is an $(a, b, k)$-critical graph. This completes the proof of Theorem 3.

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