Real-graceful labellings: a generalisation of graceful labellings

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Abstract

Every graph can be associated to a characteristic exponential equation involving powers of (say) 2, whose unknowns represent vertex labels and whose general solution is equivalent to a graceful labelling of the graph. If we do not require that the solutions be integers, we obtain a generalisation of a graceful labelling that uses real numbers as labels. Some graphs that are well known to be non-graceful become graceful in this more general context. Among other things, "real-graceful" labellings provide some information on the rigidity to be non-graceful, also asymptotically.

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1 Introduction

A *graceful labelling* of a graph having e edges is a vertex labelling that uses distinct integers in $[0, e]$ and produces, as differences on the edges, all the integers whose absolute values range in $[1, e]$. Graceful labellings are an increasingly popular research field whose connections with real-life problems were first emphasised in [1, 2] and whose mathematical interest and beauty are more and more acknowledged (see for example [3]). As an example how a graceful labelling of a given graph G, with e edges, can be fruitful in a close research field, we recall that the complete graph K_{2e+1} can be decomposed into $2e+1$ copies of G once we regard its vertices as the integers $0, 1, ..., 2e$ and define any vertex of the i-th copy G_i $(0 \leq i \leq 2e)$ as the corresponding graceful label increased by i (see [6]).

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The effort of showing that some graphs or some classes of graphs are graceful (that is, they admit a graceful labelling) often gives rise – and has given, for decades - to a great number of nice constructions. In many cases we can speak of art pieces tailored to a particular context. Nobody has still succeeded in proving that all trees are graceful (see [5]). As to non-existence results, the one that towers above is Rosa's theorem (see [6]): all Eulerian graphs with $4t+1$ or $4t+2$ edges (with any t) are not graceful. Its proof is based on a double counting (mod 2) of the differences. For a different reason, namely the total adjacency, complete graphs of order larger than 4 are not graceful (see [4]). Other necessary conditions for gracefulness seem to be not enough explored yet (see [8]).

The present paper is devoted to a generalisation of a graceful labelling in a new direction. Instead of relaxing the interval of admissible integers for the labelling, and instead of allowing repetitions or gaps (see e.g. [7]), we postulate that the labels be real numbers ranging in the usual interval $[0, e]$. This choice would appear ridiculously trivial if we required that the differences on the edges be the classical ones. On the contrary, the extension to real numbers relies on a different way of defining gracefulness, by means of an equation involving powers of 2 and having all the labels as exponential unknowns. Using a simple arithmetical argument, we shall show that in the standard setting (where labels are assumed to be integers) such equation holds precisely when the unknowns are the labels of a graceful labelling. Then, since nothing prevents the interpretation of the equation in the field of real numbers, we make this step and look for possible real-graceful labellings for some graphs that are, classically, not graceful. In details, we provide "real-graceful" labellings for all non-graceful cycles and for the complete graphs K_i with $5 \leq i \leq 7$. To this end we assign all but one integer label, then we find the remaining label by solving the corresponding equation.

The response of the label to the request of the exponential equation seems an interesting information, which among other things expresses the rigidity, the extent to which a graph is far from being graceful.

2 An arithmetical property and the basic definition

The definition of real-graceful labelling, which we are about to give, heavily depends on a simple property involving powers of a given positive integer. Curiously, we did not succeed as yet in finding a proof of it in the literature. This is the reason why we present a proof. Let us denote by $\mathbf{N}$ the set of non-negative integers.

Proposition 2.1. Let n be a positive integer. Among all the sums of the form $\sum_i a_i 2^i$ yielding n and such that $a_i \in \mathbf{N}$ for all i, the sum corresponding to the binary representation of n minimizes the quantity $\sum_i a_i$, and it is the only one to have this property.

Proof. We denote by $w(n)$ the number of 1s in the binary representation of n (that is, the minimum sum in the claim). In the proof we use induction on n. If $n = 1$ the claim holds. Let us assume that n is odd and that $\sum_{0\leq i\leq I} b_i 2^i = n$ for some coefficients b_i which minimize the sum of all coefficients over all such decompositions. Then b_0 (which is certainly odd, thus positive) is equal to 1, for otherwise we would obtain a smaller sum of coefficients yielding n after decreasing b_0 by 2 and increasing b_1 by 1. We can therefore write $\frac{n-1}{2}$ as $\sum_{1\leq i\leq I} b_i 2^{i-1}$. By the induction hypothesis, the b_i's sum up to a number larger than $w\left(\frac{n-1}{2}\right)$ unless they are the coefficients of the binary representation of $\frac{n-1}{2}$. Due to the minimizing property of the b_i's, related to n, the former possibility can be excluded, so the conclusion is reached. If n is even we reason in a similar way, by considering the smallest i such that $b_i \neq 0$. □

What we are mainly interested to is the following consequence.

Corollary 2.2. Let I be a positive integer. If

$$\sum_{1\leq i\leq I} 2^i + \sum_{1\leq i\leq I} \left(\frac{1}{2}\right)^i = \sum_{1\leq j\leq J} a_j 2^j + \sum_{1\leq k\leq K} a'_k \left(\frac{1}{2}\right)^k$$

for some positive integers J, K and with all a_j, a'_k in $\mathbf{N}$, then $\sum_j a_j + \sum_k a'_k \neq 2I$ unless the above equality is trivial.

Proof. After multiplying each side of the equality by $2^{\max(I,K)}$, on the left side we obtain the binary representation of a positive integer, with as many 1s as $2I$; the sum on the right side now has no fractions, and Proposition 2.1 is applicable. □

Now we are ready for the main definition. Notice that the number 2 (which will play a major role in the whole paper) could be replaced by other integers, thus leading to a different definition of real-gracefulness.

Definition 2.3. Let us consider a simple graph $G = (V, E)$ with no loops, and denote $|E|$ by e. An injective map $\gamma : V \longrightarrow [0, e]$ is a *real-graceful labelling* if

$$\sum_{\{u,v\}\in E} 2^{\gamma(u)-\gamma(v)} + 2^{\gamma(v)-\gamma(u)} = 2^{e+1} - 2^{-e} - 1 .$$

As mentioned in the Introduction, the above definition can be considered a generalisation of the graceful labelling by virtue of the following result.

Theorem 2.4. If γ is a real-graceful labelling whose labels are all integers, then γ is a graceful labelling.

Proof. If the absolute values of the differences $\gamma(u) - \gamma(v)$ cover the set $\{1, 2, \dots, e\}$ when $\{u, v\}$ varies in E – that is, if γ is a classical graceful labelling – then the sum in the above definition is equal to

$$2^1 + 2^2 + \cdots + 2^e + 2^{-1} + 2^{-2} + \cdots + 2^{-e} =$$

$$= \left(\frac{2^{e+1} - 1}{2 - 1} - 2^0\right) + \left(\frac{(2^{-1})^{e+1} - 1}{2^{-1} - 1} - (2^{-1})^0\right)$$

which, after a brief calculation, yields the number on the right side of the equality. Due to Corollary 2.2, such number can be obtained only through a labelling that generates the differences 1,2,...,*e*.

□

3 Real-gracefulness of cycles and complete graphs

Let us analyse the cycle C_{4t+2} for some fixed integer $t \geq 1$. We are in the presence of an Eulerian graph whose number of edges does not allow for a graceful labelling. Let us then look for a real-graceful labelling of it. After writing its vertices (circularly ordered by adjacency) as $v_1, \dots, v_{4t+2}$, we label v_{2i+1} by $4t + 2 - i$ $(0 \leq i \leq 2t)$, then v_{2i+2} by i if $0 \leq i \leq t$ and by $i + 1$ if $t + 1 \leq i \leq 2t - 1$. We leave the label of v_{4t+2} as an unknown, x. At this stage we have obtained $4t$ differences, namely all the integers from 2 to $4t + 2$ except $2t$. In order to obtain a real-graceful labelling of C_{4t+2}, we should find a real number $x \in [0, 4t + 2]$ different from any label and such that

$$2^{x-(4t+2)} + 2^{(4t+2)-x} + 2^{x-(2t+2)} + 2^{(2t+2)-x} = 2^1 + 2^{-1} + 2^{2t} - 2^{-2t} .$$

For, the two remaining differences must contribute to the expected quantity, $2^{4t+3} - 2^{-(4t+2)} - 1$, together with all the already existing differences that yield the other powers of 2. Let us denote 2^x and 2^{2t} by X and T respectively. After some routine calculations we have the following equation:

$$X^2(1 + T) - 4X\left(T^3 + \frac{5}{2}T^2 + T\right) + 16T^3(T + 1) = 0 .$$

Its solutions are $\frac{T}{T+1}\left(2T^2 + 5T + 2 \pm \sqrt{4T^4 + 4T^3 + T^2 + 4T + 4}\right)$. For example if $t = 3$ we obtain, approximately, $x_1 = 14.01$ and $x_2 = 7.99$. The first value is slightly too large because we are considering the cycle C_{14}. The real-graceful labelling given by the second value is, in circular order, $(14, 0, 13, 1, 12, 2, 11, 3, 10, 5, 9, 6, 8, x_2)$. More generally, the admissible solution is $O(4T)$ and does never reach $4T$ (the number in parentheses is smaller than $2T^2+5T+2-(2T^2+T)$, and $\frac{T}{T+1}(4T+2) < 4T$). As to the other solution, it is $O(4T^2)$ and larger than this limit. The corresponding labels tend to $2t+2$ and $4t+2$ respectively. The labellings we obtain for any t have two numbers that almost coincide: as t increases, x tends to the label adjacent to x itself.

We can reason in a similar way when dealing with any cycle of the form C_{4t+1}. Notice that if we use a different labelling and leave again an unknown x, we might find no solution. Let us indeed go back to C_{4t+2} and consider the ingenuous labelling that does not avoid $t+1$ in the middle. In this case the requirement is

$$2^{x-(4t+2)} + 2^{(4t+2)-x} + 2^{x-(2t+2)} + 2^{(2t+2)-x} = 2^1 + 2^{-1} + 2^2 + 2^{-2} ,$$

because we have generated all the differences larger than 2. The corresponding equation is $X^2(1+T) - 27XT^2 + 16T^4 + 16T^3$, whose discriminant is negative for all the admissible values of T. This experiment might indicate that real-graceful labellings require a certain care, in the same spirit as for classical graceful labellings. We do not possess a universal (and too mechanical) remedy.

Now we are going to analyse the behaviour of some complete graphs, the first being K_5 – the smallest non-graceful. Let us label four of its vertices by 10, 0, 1, 8. The remaining label, whatever it be, cannot generate the differences from 3 to 6, so we write down the equation for a real-graceful labelling:

$$2^{x-10} + 2^{10-x} + 2^{x-8} + 2^{8-x} + 2^{x-1} + 2^{1-x} + 2^x + 2^{-x} = \sum_{3\le i\le 6} (2^i + 2^{-i}) .$$

After replacing, as usual, 2^x with X, we obtain: $1541X^2 - 123120X + 1313792 = 0$. The logarithms to the base 2 of its solutions are approximately equal to 3.66 and 6.07. Both numbers are suitable for achieving a real-graceful labelling of K_5.

In the case of K_6 we can start with 15, 0, 1, 13, 4 (notice that we are following the same recursion as above, trying to cover all the largest differences and leaving as few empty spaces as possible). Let us denote these labels by $p_1, ..., p_5$. In the present case we still have to obtain the integers in $J = \{5, 6, 7, 8, 10\}$. The corresponding equation is

$$X^2 \sum_{1\le i\le 5} 2^{15-p_i} - 2^{15-10} X\Big(2^{10}\sum_{j\in J} 2^j + \sum_{j\in J} 2^{10-j}\Big) + \sum_{1\le i\le 5} 2^{15+p_i} = 0 .$$

Here 10 has a special role because it is the largest label to obtain (the equation related to K_5 is, clearly, of the same form, as long as we take

$J = \{3, 4, 5, 6\}$ and use the old four labels). Both the solutions, appr. 5.81 and 10.87, are acceptable. Similarly, we obtain two suitable solutions also for K_7 endowed with the labels 21, 0, 1, 19, 4, 14; they are appr. 5.19 and 15.50.

4 Some final remarks

When the order of the complete graph grows it is necessary to increase the number of unknowns for the labels (see the related problem of Golomb rulers, in [4]). In the next future it might be interesting to devise an algorithm for real-graceful labellings of complete graphs that require the smallest number of labels in $\mathbf{R} \setminus \mathbf{N}$. From this it might follow a certain knowledge of the asymptotical behaviour of such labellings. Another question is about the too large coefficients in the above equations for complete graphs. It would be nice to find some shortcut that relieves from long calculations. Finally, we are confident that the present generalisation can be, sooner or later, connected to real-life problems similar to those which accompanied the introduction of graceful labellings, many years ago.

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