# A Ballot Number Formulary

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## 1 Introduction

In this paper we present a collection of interesting binomial identities and expansions for the ballot numbers $B(n,k)$ with indications of proofs and sources when known. Some of the formulas are old and some apparently new. The Ballot numbers are quite often defined by the formula

$$B(n,k) = \frac{n-k}{n+k}\binom{n+k}{k} = -B(k,n) \tag{1.1}$$

with $0 \le k \le n$, and $n \ge 1$. Note $B(0,0)$ is undefined in (1.1). However, we will adopt, whenever necessary, a customary convention that $B(0,0) = 1$. These numbers have been studied extensively. This is the principal definition which we adopt. For the most part our formulas will be stated using the binomial coefficient notation. Gould [10], [11] has studied identities and expansions involving Rothe polynomials

$$A_k(a,b) = \frac{a}{a+bk}\binom{a+bk}{k} \tag{1.2}$$

so that the Ballot numbers may also be expressed in terms of these as

$$B(n,k) = A_k(n-k,2) = A_k(n,1) - A_n(k,1). \tag{1.3}$$

The numbers are sometimes defined differently by the formula

$$a_{n,k} = \binom{n+k}{k} - \binom{n+k}{k-1} = \frac{n+1-k}{n+1}\binom{n+k}{k}. \tag{1.4}$$

See, e.g. Riordan [20, p.130], so that

$$a_{n-1,k} = B(n,k). \tag{1.5}$$

In this form they arose historically as the solution to the recurrence

$$a_{n,k} = a_{n-1,k} + a_{n,k-1}, a_{n,n} = a_{n-1,n}, \tag{1.6}$$

where $n > k$ and with the boundary condition $a_{1,0} = 1$, and making the convention that $a_{0,0} = 1$. The formula (1.4) as the solution of (1.6) was found by Joseph Bertrand [4] and MacMahon [19].

The notation $a_{n,k}$ is used by Carlitz and Riordan [5]. The $a_{n,k}$ were first called lattice permutations by MacMahon [19, Sect.III, Chap. V]. The term "ballot numbers" arises when we consider an election where two candidates A and B are running and the final vote is $n$ votes for A and $k$ votes for B, and all partial returns correctly predict the winner. The lattice path interpretation is understood in the following way. Let a vote for A be represented by a horizontal line and a vote for B be represented by a vertical line. Then the lattice path that arises starts at $(0, 0)$ and ends at $(n, k)$ with the stipulation that the path never crosses the diagonal connecting these points. Note that when $k = n$, then

$$a_{n,n} = \frac{1}{n+1}\binom{2n}{n} = C(n) \tag{1.7}$$

the so-called Catalan numbers. These enumerate the number of different products of $n + 1$ elements in a given order in nonassociative multiplication. See Gould [13] for a long bibliography and history of these numbers. They were first studied by Catalan, Euler, Segner and Fuss. Kuchinski [18] has studied correspondences between several dozen structures enumerated by the Catalan numbers. Of course it is also well-known that the ordinary generating function for $C(n)$ is

$$\sum_{n=0}^{\infty} C(n)x^n = \frac{1-\sqrt{1-4x}}{2x} \tag{1.8}$$

# 2 List of identities

**Identity 1:** (Ordinary generating function for $B(n, k)$) Let $n \geq 1$, then

$$\sum_{k=0}^{\infty} \frac{n-k}{n+k}\binom{n+k}{k} x^k = \frac{1-2x}{(1-x)^{n+1}}, \tag{2.1}$$

and when $n = 0$,

$$\sum_{k=0}^{\infty} \frac{n-k}{n+k}\binom{n+k}{k} x^k = \frac{1}{1-x}. \tag{2.2}$$

*Proof of Identity 1:* To prove Identity 1, we make use of the negative transformation of the binomial polynomial. For easy reference, we list this transformation below.

$$\binom{-x}{n} = (-1)^n \binom{x+n-1}{n} \tag{2.3}$$

Thus, we have

$$\begin{aligned}
\sum_{k=0}^{\infty} \frac{n-k}{n+k} \binom{n+k}{k} x^k &= \sum_{k=0}^{\infty} \frac{(n-k)(n+k-1)!}{k!n!} x^k \\
&= \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k - \sum_{k=1}^{\infty} \binom{n+k-1}{n} x^k \\
&= \sum_{k=0}^{\infty} (-1)^k \binom{-n}{k} x^k - x^{-n+1} \sum_{j=n}^{\infty} \binom{j}{n} x^j \\
&= (1-x)^{-n} - \frac{x}{(1-x)^{n+1}} = \frac{1-2x}{(1-x)^{n+1}}. \qquad \square
\end{aligned}$$

**Identity 2:** (Catalan numbers as sums of Ballot numbers)

$$\sum_{k=0}^{n} \frac{n-k}{n+k} \binom{n+k}{k} = C(n) \tag{2.4}$$

*Proof of Identity 2:* The proof of Identity 2 uses the following two binomial coefficient identities. Each of these identities is easily proven by induction on $n$ and can be found in Gould [12] as (1.49) and (1.52) respectively.

$$\sum_{k=0}^{n} \binom{x+k}{k} = \binom{x+n+1}{n} \tag{2.5}$$

$$\sum_{k=j}^{n} \binom{k}{j} = \binom{n+1}{j+1} \tag{2.6}$$

In particular,

$$\begin{aligned}
\sum_{k=0}^{n} \frac{n-k}{n+k} \binom{n+k}{k} &= \sum_{k=0}^{n} \binom{n+k+1}{k} - \sum_{k=1}^{n} \binom{n+k-1}{n} \\
&= \binom{2n}{n} - \sum_{j=n}^{2n-1} \binom{j}{n} \quad \textit{see (2.5) with } x = n-1 \\
&= \binom{2n}{n} - \binom{2n}{n+1} = C(n) \qquad \textit{by (2.6)}. \qquad \square
\end{aligned}$$

**Remark 2.1** *We provide a simple combinatorial explanation that justifies the validity of Identity 2. Each summand in the left hand side of Equation (2.4) counts the number of lattice paths from* $(n-k,0)$ *to* $(n,n)$ *which never cross the diagonal* $y=x$. *By summing over all* $0 \leq k \leq n$, *we count the number of lattice paths from* $(0,0)$ *to* $(n,n)$ *which never cross* $y=x$, *which is* $C(n)$. *Thus, for a fixed* $n$, *Identity 2 provides a partition of* $C(n)$, *which is reminiscent of how, for* $0 \leq k \leq n$, $S(n,k)$, *the Stirling number of the second kind, partitions* $B(n)$, *the* $n^{th}$ *Bell Number.*

**Identity 3:**

$$\sum_{k=0}^{n} \frac{k(n-k)}{n+k} \binom{n+k}{k} = \binom{2n}{n+2} \tag{2.7}$$

*Proof of Identity 3:*

$$\begin{aligned}
\sum_{k=0}^{n} \frac{k(n-k)}{n+k} \binom{n+k}{k} &= \sum_{k=0}^{n} \frac{(n-k)(n+k-1)!}{(k-1)!n!} \\
&= \sum_{k=0}^{n} (n-k) \binom{n+k-1}{n} \\
&= \sum_{j=n}^{2n-1} (2n-j-1) \binom{j}{n} \\
&= 2n \sum_{j=n}^{2n-1} \binom{j}{n} - \sum_{j=n}^{2n-1} (j+1) \binom{j}{n} \\
&= 2n \sum_{j=n}^{2n-1} \binom{j}{n} - (n+1) \sum_{j=n+1}^{2n} \binom{j}{n+1} \\
&= 2n \binom{2n}{n+1} - (n+1) \binom{2n}{n+2} \text{ by (2.6)},
\end{aligned}$$

and, by simple manipulation of the factorials, it is easy to show that

$$2n \binom{2n}{n+1} - (n+1) \binom{2n+1}{n+2} = \binom{2n}{n+2} \qquad \square.$$

**Identity 4:** (in Gould [12])

$$\sum_{k=0}^{n} \frac{n-k}{n+k} \binom{n+k}{k} 2^{n-k} = \binom{2n}{n} \tag{2.8}$$

*Proof of Identity 4:* The proof of Identity 4 uses Pascal's recurrence which, for easy reference, we list below.

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \tag{2.9}$$

It also uses the following binomial coefficient identity which is easily proven by induction on $n$ and can be found in Gould [12] as (1.79).

$$\sum_{k=0}^{n} \binom{n+k}{k} 2^{-k} = 2^{n} \tag{2.10}$$

In particular

$$\sum_{k=0}^{n} \frac{n-k}{n+k} \binom{n+k}{k} 2^{n-k} =$$

$$\sum_{k=0}^{n} \binom{n+k-1}{k} 2^{n-k} - \sum_{k=0}^{n} \binom{n+k-1}{k-1} 2^{n-k}$$

$$= \sum_{k=0}^{n} \binom{n+k}{k} 2^{n-k} - 2\sum_{k=0}^{n} \binom{n+k-1}{k-1} 2^{n-k} \quad \textit{by (2.9)}$$

$$= 2^{n}(2^{n}) - 2\sum_{k=0}^{n} \binom{n+k-1}{k-1} 2^{n-k} \quad \textit{by (2.10)}$$

$$= 2^{2n} - 2\sum_{j=0}^{n} \binom{n+j}{j} 2^{n-j-1} + \binom{2n}{n}$$

$$= 2^{2n} - 2(2^{n-1})2^{n} + \binom{2n}{n} = \binom{2n}{n} \quad \textit{by (2.10).} \quad \square$$

**Identity 5:**

$$\sum_{k=0}^{n} \frac{n-k}{n+k} \binom{n+k}{k} (n+1-k) 2^{n-k} = 4^{n} \tag{2.11}$$

*Proof of Identity 5:*

$$\sum_{k=0}^{n} \frac{n-k}{n+k} \binom{n+k}{k} (n+1-k)2^{n-k}$$

$$= (n+1) \sum_{k=0}^{n} \frac{n-k}{n+k} \binom{n+k}{k} 2^{n-k} - \sum_{k=0}^{n} \frac{(n+k-1)!}{(n-1)!(k-1)!} 2^{n-k}$$

$$+ \sum_{k=0}^{n} \frac{k(n+k-1)!}{(k-1)!n!} 2^{n-k}$$

$$= (n+1) \binom{2n}{n} - \sum_{k=0}^{n} \frac{(n+k-1)!}{(n-1)!(k-1)!} 2^{n-k}$$

$$+ \sum_{k=0}^{n} \frac{k(n+k-1)!}{(k-1)!n!} 2^{n-k} \text{ by } (2.8)$$

$$= (n+1) \binom{2n}{n} + (-n+1) \sum_{k=0}^{n} \binom{n+k-1}{k-1} 2^{n-k}$$

$$+ (n+1) \sum_{k=0}^{n} \binom{n+k-1}{k-2} 2^{n-k}$$

$$= (n+1) \left[ \binom{2n}{n} + \sum_{k=0}^{n} \left[ \binom{n+k-1}{k-1} + \binom{n+k-1}{k-2} \right] 2^{n-k} \right]$$

$$- 2n \sum_{k=0}^{n} \binom{n+k-1}{k-1} 2^{n-k}$$

$$= (n+1) \binom{2n}{n} + (n+1) \sum_{k=0}^{n} \binom{n+k}{k-1} 2^{n-k}$$

$$- 2n \sum_{k=0}^{n} \binom{n+k-1}{k-1} 2^{n-k} \text{ by } (2.9)$$

$$= (n+1) \binom{2n}{n} + (n+1) \sum_{j=0}^{n-1} \binom{n+j+1}{j} 2^{n-j-1}$$

$$- 2n \sum_{j=0}^{n} \binom{n+j}{j} 2^{n-j-1} + n \binom{2n}{n}$$

$$= (n+1) \binom{2n}{n} + (n+1) \sum_{j=0}^{n-1} \binom{n+j+1}{j} 2^{n-j-1}$$

$$- 4^n n + n \binom{2n}{n} \text{ by } (2.10)$$

Now let $m = n + 1$ in the remaining sum. Then, the previous line becomes

$$(2n+1)\binom{2n}{n} - 4^n n + m\sum_{j=0}^{m}\binom{m+j}{j}2^{m-j-2}$$

$$-\frac{m}{4}\binom{2m}{m} - \frac{m}{2}\binom{2m-1}{m-1}$$

$$= (2n+1)\binom{2n}{n} - 4^n n - \frac{m}{4}\binom{2m}{m}$$

$$-\frac{m}{2}\binom{2m-1}{m-1} + m4^{m-2} \text{ by } (2.10)$$

$$= 4^n + (2n+1)\binom{2n}{n} - \frac{n+1}{4}\binom{2n+2}{n+1} - \frac{n+1}{2}\binom{2n+1}{n} = 4^n.$$

The last equality comes by writing the binomial coefficients as factorials and simplifying. □

**Identity 6:** (The $m^{th}$ difference operator applied to Ballot numbers)

$$\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}\frac{n-k}{n+k}\binom{n+k}{k} = \frac{n-2m}{n}\binom{n}{m} \tag{2.12}$$

**Remark 2.2** *If $m = n$, Identity 6 simply becomes*

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\frac{n-k}{n+k}\binom{n+k}{k} = -1, \tag{2.13}$$

*which is the $n^{th}$ difference operator applied to the Ballot numbers.*

*Proof of Identity 6:* To prove Identity 6, we use the following two well known facts about binomial polynomials. The first is the negative transformation given by (2.3) and the second is the Vandermonde Convolution, both of which are found in [12]. The Vandermonde Convolution is listed below.

$$\sum_{k=0}^{n}\binom{x}{k}\binom{y}{n-k} = \binom{x+y}{n} \tag{2.14}$$

In particular,

$$\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}\frac{n-k}{n+k}\binom{n+k}{k}$$
$$=\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}\binom{n+k-1}{k}$$
$$-\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}\binom{n+k-1}{k-1}$$
$$=(-1)^m\sum_{k=0}^{m}\binom{m}{m-k}\binom{-n}{k}$$
$$-\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}\binom{n+k-1}{k-1} \quad \textit{see } (2.3)$$
$$=(-1)^m\binom{m-n}{m}+(-1)^m\sum_{k=0}^{m}\binom{m}{k}\binom{-n-1}{k-1} \quad \textit{by } (2.14),\ (2.3)$$
$$=(-1)^m\binom{m-n}{m}+(-1)^m\sum_{k=0}^{m}\binom{m}{m-k}\binom{-n}{k}$$
$$+(-1)^{m+1}\sum_{k=0}^{m}\binom{m}{m-k}\binom{-n-1}{k} \quad \textit{by } (2.9)$$
$$=2(-1)^m\binom{m-n}{m}+(-1)^{m+1}\binom{m-n+1}{m} \qquad \textit{by } (2.14)$$
$$=2\binom{n-1}{m}-\binom{n}{m}=\frac{n-2m}{n}\binom{n}{m} \quad \textit{by } (2.3). \qquad \square$$

**Identity 7:** (Generalized Larcombe Ballot number identity due to Gould [14])

$$\sum_{j=0}^{m}(-1)^j\binom{x}{j}\binom{x+n-1+k-j}{k-j}\frac{x+j}{x}=\frac{n-k}{n+k}\binom{n+k}{k} \qquad (2.15)$$

**Identity 8:** (Inverse generalized Larcombe Ballot number identity due to Gould [14])

$$\sum_{k=0}^{m}(-1)^{n-k}\binom{x+n}{m-k}\frac{n-k}{n+k}\binom{n+k}{k}=(-1)^n\frac{x+m}{x}\binom{x}{m} \qquad (2.16)$$

**Remark 2.3** *The proofs of Identities 7 and 8 can be found, in complete detail, in Gould [14].*

**Identity 9:** (Chebyshev Inversion Formula) Given any two sequences $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$, note that

$$a_n = \sum_{k=0}^{\lceil \frac{n}{2} \rceil} \binom{n-k}{k} b_{n-k} \tag{2.17}$$

if and only if

$$b_n = \sum_{k=0}^{n} (-1)^k \frac{n-k}{n+k} \binom{n+k}{k} a_{n-k}. \tag{2.18}$$

**Remark 2.4** *This inverse pair and its proof is stated in Riordan [20, Pp.61-63].*

**Identity 10:** (A Fibonacci number relation)

$$\sum_{k=0}^{n} (-1)^k \frac{n-k}{n+k} \binom{n+k}{k} F_{n+1-k} = 1 \tag{2.19}$$

with

$$F_{n+1} = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} \tag{2.20}$$

where $F_n$ is the $n$-th Fibonacci number.

*Proof of Identity 10:* Use the Chebyshev Inversion Formula with $a_n = F_{n+1}$ and $b_n = 1$.

**Identity 11:** (Binomial identity (3.54) in Gould [12])

$$\sum_{k=0}^{n} (-1)^k \frac{n-k}{n+k} \binom{n+k}{k} \binom{x}{n-k} \frac{1}{2^{n+k}} = \binom{x/2}{n} \tag{2.21}$$

*Proof of Identity 11:* Let $D \equiv \frac{d}{dx}$. We look at the operator $\left(\frac{1}{x}D\right)^n$ acting on a differentiable function $f(x)$. In particular, the first few iterations of this operator give us

$$\left(\frac{1}{x}D\right) f(x) = \frac{f'(x)}{x}$$

$$\left(\frac{1}{x}D\right)^2 f(x) = \frac{f''(x)}{x^2} - \frac{f'(x)}{x^3}$$

$$\left(\frac{1}{x}D\right)^3 f(x) = \frac{f^{(3)}(x)}{x^3} - \frac{3f''(x)}{x^4} + \frac{3f'(x)}{x^5}$$

$$\left(\frac{1}{x}D\right)^4 f(x) = \frac{f^{(4)}(x)}{x^4} - \frac{6f^{(3)}(x)}{x^5} + \frac{15f''(x)}{x^6} - \frac{15f'(x)}{x^7}.$$

Inspection of these four equations leads to the formation of the following sum, which is easily proven by the induction technique of [22].

$$\left(\frac{1}{x}D\right)^n f(x) = \sum_{k=0}^{n} a_{n,k} x^{-2n+k} D^k f(x), \tag{2.22}$$

where $a_{0,0} = 1$, $a_{n,0} = 0$ for $n \geq 1$, $a_{n,k} = 0$ for $k > n$, and

$$a_{n+1,k+1} = (-2n + k + 1)a_{n,k+1} + a_{n,k}. \tag{2.23}$$

Using (2.23), we are able to form a table of values for the $a_{n,k}$. Inspection of these values leads to an alternative formula for $a_{n,k}$, namely,

$$a_{n,k} = \frac{(-1)^{n-k}(2n-k-1)!}{(k-1)!(n-k)!2^{n-k}} = \frac{(-1)^{n-k}kn!}{(2n-k)k!2^{n-k}} \binom{2n-k}{n}. \tag{2.24}$$

Note that Equation (2.24) can be proven by induction on Recurrence (2.23). Now, let $f(x) = x^p$. Through repeated iterations, we can easily show that

$$\left(\frac{1}{x}D\right)^n x^p = x^{p-2n} \prod_{k=0}^{n-1}(p-2k) = x^{p-2n}(-2)^n n! \binom{\frac{-p}{2}+n-1}{n}. \tag{2.25}$$

But from (2.22), we have

$$\left(\frac{1}{x}D\right)^n x^p = \sum_{k=0}^{n} a_{n,k} x^{-2n+k} D^k x^p = x^{p-2n} \sum_{k=0}^{n} a_{n,k} \frac{p!}{(p-k)!}. \tag{2.26}$$

Thus, comparing (2.25) with (2.26), we have

$$\sum_{k=0}^{n} a_{n,k} k! \binom{p}{k} = (-2)^n n! \binom{\frac{-p}{2}+n-1}{n}. \tag{2.27}$$

which is a polynomial identity of degree $n$ in $p$, and hence the identity is true for all real or complex values of $p$. Now we let $x = p$ and put (2.24) into the left sum of (2.27). Thus, we obtain

$$\sum_{k=0}^{n}(-1)^k \binom{x}{k} \binom{2n-k}{n} 2^k \frac{k}{2n-k} = 2^{2n} \binom{\frac{-x}{2}+n-1}{n}. \tag{2.28}$$

If we let $k \to n-k$ and use (2.3) on the right hand term of Equation (2.28), we obtain Equation (2.21). □

**Definition:** Ballot number polynomial studied by Aval, Bergeron and Bergeron [2]

$$F_n(t) = \sum_{k=0}^{n} \frac{n-k}{n+k} \binom{n+k}{k} t^k \tag{2.29}$$

They arrive at $F_n(t)$ as the Hilbert series of the space of super-covariant polynomials.

**Identity 12:** (Generating function for the Ballot number polynomial (2.29))

$$\sum_{n=1}^{\infty} F_n(t)x^n = \frac{1-2x-\sqrt{1-4tx}}{2(t+x-1)} \tag{2.30}$$

*Proof of Identity 12:*

$$\sum_{n=0}^{\infty} x^n \sum_{k=0}^{n} B(n,k)t^k = \sum_{n=1}^{\infty} x^n \sum_{k=1}^{n} B(n,k)t^k + \sum_{n=1}^{\infty} x^n B(n,0) + B(0,0)$$

$$= \sum_{k=1}^{\infty} t^k \sum_{n=k}^{\infty} B(n,k)x^n + \frac{x}{1-x} + B(0,0)$$

$$= \sum_{k=1}^{\infty} t^k \sum_{n=0}^{\infty} B(n+k,k)x^{n+k} + \frac{x}{1-x} + B(0,0)$$

$$= \sum_{k=1}^{\infty} (xt)^k \sum_{n=1}^{\infty} \frac{n}{n+2k} \binom{n+2k}{k} x^n + \frac{x}{1-x} + B(0,0)$$

$$= \sum_{n=1}^{\infty} x^n \sum_{k=1}^{\infty} \frac{n}{n+2k} \binom{n+2k}{k} (tx)^k + \frac{x}{1-x} + B(0,0)$$

$$= \sum_{n=1}^{\infty} x^n \sum_{k=0}^{\infty} \frac{n}{n+2k} \binom{n+2k}{k} (tx)^k + B(0,0)$$

$$= \sum_{n=1}^{\infty} x^n \sum_{k=0}^{\infty} A_k(n,2)(tx)^k + B(0,0).$$

In [11], Gould showed that

$$\sum_{k=0}^{\infty} A_k(a,b)z^k = u^a, \qquad where\ z = \frac{u-1}{u^b}. \tag{2.31}$$

Thus, by (2.31),

$$\sum_{k=0}^{\infty} A_k(n,2)(tx)^k = u^n, \qquad tx = \frac{u-1}{u^2}.$$

This implies

$$u = \frac{1-\sqrt{1-4tx}}{2tx}.$$

Hence,

$$\sum_{n=1}^{\infty} x^n \sum_{k=0}^{\infty} A_k(n,2)(tx)^k + B(0,0) = B(0,0) - 1 + \sum_{n=0}^{\infty} \left(\frac{1-\sqrt{1-4tx}}{2t}\right)^n$$
$$= B(0,0) - 1 + \frac{2t-1-\sqrt{1-4tx}}{2(x+t-1)}.$$

Therefore,

$$\sum_{n=1}^{\infty} x^n F_n(t) = \sum_{n=0}^{\infty} x^n \sum_{k=0}^{n} B(n,k)t^k - B(0,0)$$
$$= -1 + \frac{2t-1-\sqrt{1-4tx}}{2(x+t-1)} = \frac{1-2x-\sqrt{1-4tx}}{2(t+x-1)}. \qquad \square$$

**Identity 13:** (A double series generating function of Stanley and Gessel [8])

$$\sum_{k=0,n=0}^{\infty} \frac{|n-k|}{n+k} \binom{n+k}{k} x^n y^k = F_0(y) + F_0(x) - 1 + \frac{\sqrt{1-4xy}}{1-x-y} \quad (2.32)$$

*Proof of Identity 13:*

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{|n-k|}{n+k} \binom{n+k}{k} x^n y^k$$
$$= \sum_{n=0}^{\infty} x^n \sum_{k=0}^{n} \frac{n-k}{n+k} \binom{n+k}{k} y^k + \sum_{n=0}^{\infty}\sum_{k=n}^{\infty} \frac{k-n}{n+k} \binom{n+k}{n} x^n y^k$$
$$= \sum_{n=0}^{\infty} x^n F_n(y) + \sum_{k=0}^{\infty} y^k \sum_{n=0}^{k} \frac{k-n}{n+k} \binom{n+k}{n} x^n$$
$$= F_0(y) + \sum_{n=1}^{\infty} x^n F_n(y) + F_0(x) + \sum_{k=1}^{\infty} y^k F_k(x)$$
$$= F_0(y) + \frac{1-2x-\sqrt{1-4xy}}{2(x+y-1)} + F_0(x) + \frac{1-2y-\sqrt{1-4xy}}{2(x+y-1)} \qquad \text{by } (2.30)$$
$$= F_0(y) + F_0(x) + \frac{1-x-y-\sqrt{1-4xy}}{(x+y-1)}$$
$$= F_0(y) + F_0(x) - 1 + \frac{\sqrt{1-4xy}}{1-x-y}. \ \square$$

**Identity 14:** (Another double series generating function)

$$\sum_{k=0,n=0}^{\infty} \frac{n-k}{n+k} \binom{n+k}{k} x^n y^k = \frac{1-2x}{1-x-y} \tag{2.33}$$

*Proof of Identity 14:*

$$\begin{aligned}
&\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{n-k}{n+k} \binom{n+k}{k} x^k y^n \\
&= B(0,0) - \sum_{k=0}^{\infty} x^k + \sum_{n=1}^{\infty}\sum_{k=0}^{\infty} \frac{n-k}{n+k} \binom{n+k}{k} x^k y^n \\
&= B(0,0) + \frac{-1}{1-x} + \sum_{n=1}^{\infty} \frac{1-2x}{(1-x)^{n+1}} y^n, \ by\ (2.1) \\
&= B(0,0) + \frac{-1}{1-x} + \frac{1-2x}{1-x}\frac{y}{1-x-y} \\
&= B(0,0) + \frac{y-x}{1-y-x}.
\end{aligned}$$

Now if $B(0,0) = 0$, the previous line gives the double variable generating function provided by Gessel [9, P.185]. On the other hand, if $B(0,0) = 1$, the previous line becomes

$$1 + \frac{y-x}{1-y-x} = \frac{1-2x}{1-y-x}. \qquad \square$$

**Identity 15:** (Convolution of Ballot numbers) For $n \geq 1$,

$$\sum_{k=0}^{n} B(a,k)B(b,n-k) = B(a+b+1,n) - 2B(a+b+1,n-1) \tag{2.34}$$

*Proof of Identity 15:* Let $a$ and $b$ be fixed integers greater than or equal to 1. Using Identity 1, we can form the following ordinary generating functions $F(x,a)$ and $F(x,b)$ where

$$F(x,a) = \sum_{n=0}^{\infty} B(a,n)x^n = \frac{1-2x}{(1-x)^{a+1}} \tag{2.35}$$

$$F(x,b) = \sum_{n=0}^{\infty} B(b,n)x^n = \frac{1-2x}{(1-x)^{b+1}}. \tag{2.36}$$

Thus, the Cauchy Convolution implies that

$$F(x,a)F(x,b) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} B(a,k)B(b,n-k)x^n. \tag{2.37}$$

However, we also note that

$$F(x,a)F(x,b) = \frac{(1-2x)^2}{(1-x)^{a+b+2}} = (1-2x)\sum_{n=0}^{\infty} B(a+b+1,n)x^n, \; by \; (2.1) \tag{2.38}$$

$$= \sum_{n=0}^{\infty} B(a+b+1,n)x^n - 2\sum_{n=0}^{\infty} B(a+b+1,n-1)x^n. \tag{2.39}$$

Comparing the coefficients of (2.38) with (2.39) proves the identity. □

**Identity 16:** (Another Ballot number convolution) Let $r \geq 1$.

$$\sum_{k=0}^{r} B(x+k,k)B(y+r-k,r-k) = B(x+y+r,r). \tag{2.40}$$

*Proof of Identity 16:* In [11], Gould showed that

$$\sum_{k=0}^{r} A_k(x,z)A_{r-k}(y,z) = A_r(x+y,z), \tag{2.41}$$

for all complex values of $x, y$, and $z$. If we let $z = 2$, Equation (2.41) becomes

$$\sum_{k=0}^{r} A_k(x,2)A_{r-k}(y,2) = A_r(x+y,2). \tag{2.42}$$

Recall from Equation (1.3) that $B(n,k) = A_k(n-k,2)$. Thus, it is easy to show that $A_k(x,2) = B(x+k,k)$, $A_{r-k}(y,2) = B(y+r-k,r-k)$, and $A_r(x+y,2) = B(x+y+r,r)$. Substituting these three idendities into (2.42) gives us the desired result. □

# 3 The Ballot Number Matrix and Its Inverse

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 2 | 2 | 0 | 0 | 0 | 0 | 0 |
| 1 | 3 | 5 | 5 | 0 | 0 | 0 | 0 |
| 1 | 4 | 9 | 14 | 14 | 0 | 0 | 0 |
| 1 | 5 | 14 | 28 | 42 | 42 | 0 | 0 |
| 1 | 6 | 20 | 48 | 90 | 132 | 132 | 0 |
| 1 | 7 | 27 | 75 | 165 | 297 | 429 | 429 |

Table 1: Values of $B(n,k)$. Rows correspond to $n = 1, 2, 3, \ldots$ and columns to $k = 0, 1, \ldots, n$.

We use Table 1 to define, for a given sequence $\{f(n)\}_{n=0}^{\infty}$, a new sequence $\{g(n)\}_{n=0}^{\infty}$, where

$$g(n) = \sum_{k=0}^{n} B(n,k) f(k). \tag{3.1}$$

**Remark 3.1** *In (3.1), we assume* $B(0,0) = 1$.

By inverting Equation (3.1), we have

$$f(n) = \sum_{k=0}^{n} A(n,k) g(k), \tag{3.2}$$

where the $A(n,k)$ is the inverse of the Ballot number $B(n,k)$.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| -1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\frac{1}{2}$ | -1 | $\frac{1}{2}$ | 0 | 0 | 0 | 0 | 0 |
| $\frac{-1}{10}$ | $\frac{2}{5}$ | $\frac{-1}{2}$ | $\frac{1}{5}$ | 0 | 0 | 0 | 0 |
| $\frac{-1}{140}$ | $\frac{-3}{70}$ | $\frac{5}{28}$ | $\frac{-1}{5}$ | $\frac{1}{14}$ | 0 | 0 | 0 |
| $\frac{1}{420}$ | $\frac{-1}{105}$ | $\frac{-1}{84}$ | $\frac{1}{15}$ | $\frac{-1}{14}$ | $\frac{1}{42}$ | 0 | 0 |
| $\frac{3}{3080}$ | $\frac{-1}{1540}$ | $\frac{-1}{264}$ | $\frac{-1}{330}$ | $\frac{1}{44}$ | $\frac{-1}{42}$ | $\frac{1}{132}$ | 0 |
| $\frac{1}{8008}$ | $\frac{5}{12012}$ | $\frac{-17}{24024}$ | $\frac{-1}{858}$ | $\frac{-3}{4004}$ | $\frac{2}{273}$ | $\frac{-1}{132}$ | $\frac{1}{429}$ |

Table 2: Values of $A(n,k)$. Rows correspond to $n = 1, 2, 3, \ldots$ and columns to $k = 0, 1, \ldots, n$.

If we let $g(n) \equiv 1$, then Equation (3.1) implies that $f(0) = 1$ and $f(n) = 0$, whenever $n \geq 1$. Hence, Equation (3.2) becomes

$$0 = \sum_{k=0}^{n} A(n,k), \qquad whenever \qquad n \geq 2. \tag{3.3}$$

In other words, from the second row onward, the row sum of the entries in Table 2 is zero.

**Remark 3.2** *In Equation (2.3), we have (3.1) for $f(n) \equiv 1$. Thus, Equation (3.2) implies*

$$\sum_{k=0}^{n} A(n,k)C(k) = 1. \tag{3.4}$$

**Remark 3.3** *In Equation (2.6), we have $f(n) = n$. Thus, its inverse has the form*

$$\sum_{k=0}^{n} A(n,k) \binom{2k}{k+2} = n. \tag{3.5}$$

**Remark 3.4** *To find the inverse of Equation (2.7), we note that an equivalent form of (2.7) is*

$$\sum_{k=0}^{n} B(n,k)2^{-k} = 2^{-n} \binom{2n}{n}.$$

*Thus, the inverse becomes*

$$\sum_{k=0}^{n} A(n,k) \binom{2n}{n} 2^{-k} = 2^{-n}, \qquad i.e. \tag{3.6}$$

$$\sum_{k=0}^{n} A(n,k) \binom{2n}{n} 2^{n-k} = 1. \tag{3.7}$$

# 4 Variations on Ballot Numbers

The most common variation on a ballot number is the rotated form given (1.4). Table 3 give the values of these rotated ballot numbers.

| | | | | | |
|---|---|---|---|---|---|
| 1 | | | | | |
| 1 | 1 | | | | |
| 1 | 2 | | | | |
| 1 | 3 | 2 | | | |
| 1 | 4 | 5 | | | |
| 1 | 5 | 9 | 5 | | |
| 1 | 6 | 14 | 14 | | |
| 1 | 7 | 20 | 28 | 14 | |
| 1 | 8 | 27 | 48 | 42 | |
| 1 | 9 | 35 | 75 | 90 | 42 |
| 1 | 10 | 44 | 110 | 165 | 132 |

Table 3: Values of $a_{n-k+1,k} = \binom{n}{k} - \binom{n}{k-1}$. In this table, rows correspond to $n = 1, 2, 3, \ldots$ and columns to $k = 0, 1, \ldots, \lfloor \frac{n}{2} \rfloor$.

**Remark 4.1** *Note that diagonals in Table 1 correspond to rows in Table 3 and conversely.*

Various authors, e.g.Carlitz [6],[7] have studied the Ballot numbers arranged as in Table 3. Ordman [21] posed the identity

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \left[ \binom{n}{k} - \binom{n}{k-1} \right]^2 = C(n). \tag{4.1}$$

The proof follows easily from identity (3.76) in Gould [11].
This sum of squares should be compared with Identity 2 where the sum of the Ballot numbers gives the Catalan numbers in a different manner.
Another way to express (4.1) is the identity

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} B^2(n-k,k) = C(n). \tag{4.2}$$

Another variation involving Ballot numbers is the following alternating sum. The numbers generated by the alternating sum

$$\sum_{k=0}^{n}(-1)^k B(n,k), \tag{4.3}$$

viz. 1, 1, 0, 1, -2, 6, 18, 57, -186, 622, -2120, . . . are essentially, except for signs, tabulated as Seq. No. A000957 in Neil Sloane's Online Encyclopedia of Integer Sequences, and are called Fine Numbers (after Nathan J. Fine). Certainly one avenue for future research involves exploring the algebraic and combinatorial structure of these Fine Numbers.

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