SMALL LOCALLY nK_2 GRAPHS

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ABSTRACT. A locally nK_2 graph G is a graph such that the set of neighbors of any vertex of G induces a subgraph isomorphic to nK_2. We show that a locally nK_2 graph G must have at least $6n - 3$ vertices, and that a locally nK_2 graph with $6n - 3$ vertices exists if and only if $n \in \{1, 2, 3, 5\}$, and in these cases the graph is unique up to isomorphism. The case $n = 5$ is surprisingly connected to a classic theorem of algebraic geometry: The only locally $5K_2$ graph on $6 \times 5 - 3 = 27$ vertices is the incidence graph of the 27 straight lines on any nonsingular complex projective cubic surface.

1. INTRODUCTION

All our graphs are finite, simple and connected. If x is a vertex of the graph G, we denote by $N(x)$ the subgraph of G induced by the neighbors of x in G. A graph G is called *locally homogeneous* if there is a graph H such that $N(x) \cong H$ for all $x \in G$. There is ample literature on locally homogeneous graphs, see for example [1–7], where the problems of realization (given H, find a (finite) graph G which is locally H) and characterization (given H, characterize all locally H graphs) are addressed.

The graph nK_2 is the disjoint union of n copies of the complete graph K_2 on two vertices. For $n = 1$ it is immediate that the only locally nK_2 is K_3. The construction techniques of [3, 6, 7] give rise to an infinite number of locally nK_2 graphs for each $n \geq 2$. We will show in Section 2 that the number of vertices for a locally nK_2 graph is bounded below by $6n-3$, and then our main result:

Theorem. *A locally nK_2 graph G with $6n - 3$ vertices exists if and only if $n \in \{1, 2, 3, 5\}$, and in those cases, G is unique up to isomorphism.*

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We give a simple description of the three graphs, besides K_3, that have this extremal property.

In this paper, a maximal complete subgraph is called a *clique*. We identify each induced subgraph of G with its vertex set. The adjacency relation will be denoted as $\sim$.

2. Proof of the theorem

From now on, assume G to be a locally nK_2 graph and $n \geq 2$. First note that the cliques of G are exactly its triangles, and no edge of G is in two cliques. We will use this property extensively.

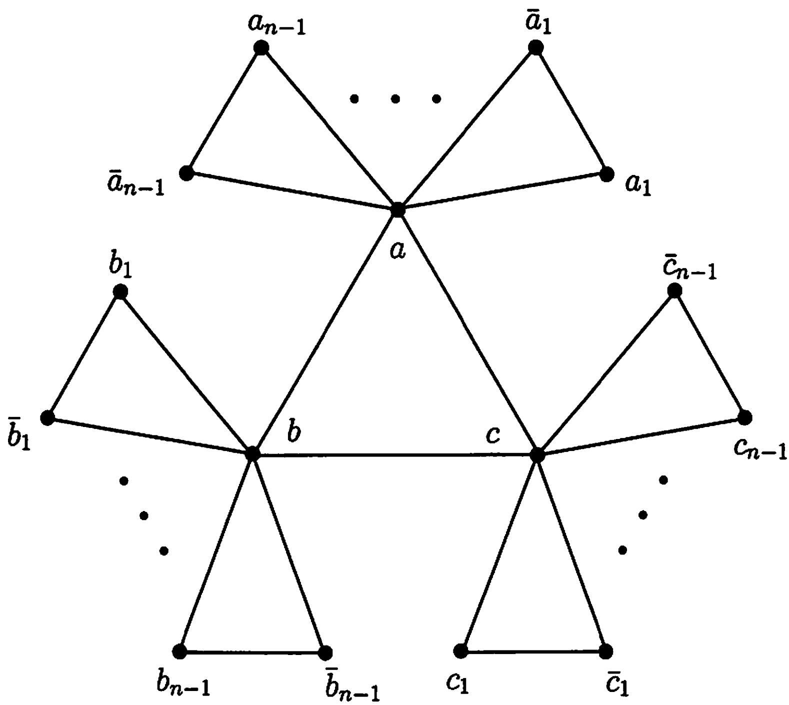

FIGURE 1. A triangle in G and its neighbours.

Let $T = \{a, b, c\}$ be a triangle of G (see Figure 1). If there is some $x \in N(a) \cap N(b)$ with $x \neq c$, vertex b would have a path on three vertices in its neighbourhood. Hence $N(a) \cap N(b) = \{c\}$ and since our situation is symmetrical on a, b and c, it follows that $|G| \geq |N(a) \cup N(b) \cup N(c)| = 6n-3$. Hence:

Lemma 1. [9] *A locally nK_2 graph G has at least $6n-3$ vertices.* □

Assume from now on that G has $6n-3$ vertices. Fix a triangle $T = \{a, b, c\}$ of G. By the previous argument, $V(G) = N(a) \cup N(b) \cup N(c)$. Label the vertices of G as in Figure 1, that is, set $N(a) = \{b, c\} \cup \{a_1, a_2, \ldots, a_{n-1}\} \cup \{\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_{n-1}\}$ with $a_i \sim \bar{a}_i$; and also label the neighbors of b and c in a similar way. We will use letters x, y, z to refer to (generic) elements in $\{a, b, c\}$. Likewise, (say) x_i and $\bar{x}_i$ are generic elements in $V(G) - T$, but we must always have that $\{x, x_i, \bar{x}_i\}$ is a triangle. Unless otherwise stated we assume that no two of x, y and z are equal.

Since $N(x) \cong nK_2$, it follows that the $2n-2$ vertices of $N(x_i)$ besides x and $\bar{x}_i$ are exactly half of the $4n-4$ vertices in $G - N(x) - \{x\} = \{y_1, \bar{y}_1, \ldots, y_{n-1}, \bar{y}_{n-1}\} \cup \{z_1, \bar{z}_1, \ldots, z_{n-1}, \bar{z}_{n-1}\}$. But we can not have both $x_i \sim y_j$ and $x_i \sim \bar{y}_j$, for otherwise, the edge $y_j\bar{y}_j$ would be in two triangles. Likewise, we could not have both $x_i \sim y_j$ and $\bar{x}_i \sim y_j$. Hence, it follows that for all j, the vertex x_i is adjacent to exactly one of y_j, $\bar{y}_j$ and exactly one of z_j, $\bar{z}_j$ and that $\bar{x}_i$ is adjacent to exactly the other two. So we just proved that:

Lemma 2. *The edges connecting $\{x_i, \bar{x}_i\}$ to $\{y_j, \bar{y}_j\}$ form a perfect matching.* □

Therefore, we know that the connections from $\{x_i, \bar{x}_i\}$ to $\{y_j, \bar{y}_j\}$ can only be either *straight* ($x_i \sim y_j$ and $\bar{x}_i \sim \bar{y}_j$) or *twisted* ($x_i \sim \bar{y}_j$ and $\bar{x}_i \sim y_j$). This allows us to use an auxiliary representation for those connections of G: Set $A_i = \{a_i, \bar{a}_i\}$, $B_i = \{b_i, \bar{b}_i\}$ and $C_i = \{c_i, \bar{c}_i\}$. Let H_n be the complete tripartite graph on $3n-3$ vertices with $n-1$ vertices in each part. We assume without loss that the vertices of H_n are precisely the aforementioned A_i, B_i and C_i for $i \in I := \{1, \ldots, n-1\}$. Moreover we set the parts of H_n to be precisely $\{A_i\}_{i \in I}$, $\{B_i\}_{i \in I}$ and $\{C_i\}_{i \in I}$. Now, whenever the connections from $X_i := \{x_i, \bar{x}_i\}$ to $Y_j := \{y_i, \bar{y}_i\}$ are straight in G, we paint the edge X_iY_j blue in H_n, otherwise (when the connections are twisted) we paint it red. It should be clear that the possible adjacencies of G (compatible with Lemma 2) are in bijection with the edge-colorings of H_n.

Now, for any triangle $A_iB_jC_k$ of H_n, we observe that $A_i \cup B_j \cup C_k$ induces a disjoint union of two triangles in G if an only if $A_iB_jC_k$ has an even number of red edges. In such case, we say that $A_iB_jC_k$ is a *good* triangle.

We say that an edge-coloring of H_n with colors blue and red is *valid* if every edge of H_n is contained in exactly one good triangle. Since every edge of G is contained in exactly one triangle, we have:

Lemma 3. *Any locally nK_2 graph G with $6n-3$ vertices determines a valid edge-coloring of H_n and conversely, a valid edge-coloring leads to a locally nK_2 graph.* □

However, different colorings can lead to isomorphic locally nK_2 graphs, and in this case, we say that the colorings are *equivalent.* For example, interchanging the names of the two vertices of a pair $X_i = \{x_i, \bar{x}_i\}$ in G would mean that all edges of H_n incident to vertex X_i would switch colors. In this case, we say that we applied *a twist* to the vertex X_i. We can also reorder some vertices of one of the parts of H_n and obtain an equivalent coloring. Twists and reorderings (of the kind described) will be the only two operations that we will use to reduce any edge-coloring of H_n to one, specific, canonical one.

Lemma 4. *For each $n \geq 2$ any two valid edge-colorings of H_n are equivalent. In particular, for each $n \geq 2$ there is, up to isomorphism, at most one locally nK_2 graph on $6n-3$ vertices.*

Proof: Start with a valid coloring of H_n.

Step 1. The edges A_1B_i for $i = 1, \ldots, n-1$ can be assumed to be all blue, by applying a twist to some of the B_i if necessary. Similarly, all edges of the form A_1C_i for $i = 1, \ldots, n-1$ and A_iB_1 for $i = 2, \ldots, n-1$ can be assumed to be blue.

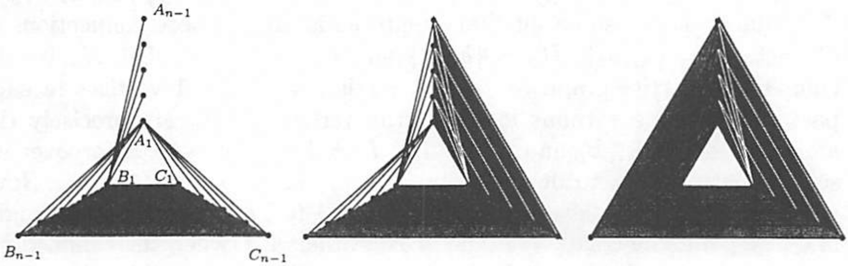

FIGURE 2. Edge colorings of the tripartite graph H_n. Black lines and dark-gray regions are blue. White lines and light-gray regions are red.

Step 2. Since each edge of the form A_1B_i for $i = 1, \ldots, n-1$ is in exactly one good triangle, for each vertex in $\{B_1, \ldots, B_{n-1}\}$ there must be exactly a blue edge joining it to a vertex in $\{C_1, \ldots, C_{n-1}\}$. Moreover, all such blue edges form a perfect matching, since otherwise there would be an edge in two good triangles. By reordering $\{C_1, \ldots, C_{n-1}\}$ if necessary, we can assume that the edges B_iC_i for $i = 1, \ldots, n-1$ are blue, and so all edges B_iC_j with $i \neq j$ are red, see Figure 2 (left).

Note that this settles the case $n = 2$, so we can assume $n \geq 3$ in what follows.

Step 3. The edges C_1A_i for $i = 2, \ldots n-1$ are now forced to be red, for otherwise the edge B_1C_1 would be in two good triangles.

Step 4. Since each edge of the form A_iB_1 for $i = 2, \ldots, n-1$ is in exactly one good triangle, for each vertex in $\{C_2, \ldots, C_{n-1}\}$ there must be exactly a red edge joining it to a vertex in $\{A_2, \ldots, A_{n-1}\}$, and in fact, these red edges form a perfect matching between these two sets (if, for example, the edges C_2A_2 and C_3A_2 were red, the edge A_2B_1 would be in two good triangles), so by reordering $\{A_2, \ldots, A_{n-1}\}$ we can assume without loss that the edges C_iA_i for $i = 2, \ldots, n-1$ are red, and so all edges C_iA_j with $2 \leq i, j \leq n-1$, $i \neq j$ are blue. See Figure 2 (middle).

Step 5. If $n = 3$, then the only edge we still have to consider is A_2B_2, and this has to be blue for C_1A_2 to be in a good triangle. So assume $n \geq 4$. An edge A_iB_j for $2 \leq i, j \leq n-1$, $i \neq j$ has to be red, since if it were blue it would be in the good triangles $A_iB_jC_1$ and $A_iB_jC_j$. This forces, for each $i = 2, \ldots, n-1$, that the edge A_iB_i is blue, since otherwise the edge C_1A_i is in no good triangle. See Figure 2 (right).

Hence, any valid edge-coloring of H_n is equivalent to the specific coloring obtained. □

However, we claim that if $n = 4$ or $n \geq 6$, the coloring just described is not valid. Consider the edge A_3B_2, which is painted red. If $n = 4$, it is not contained in any good triangle (the triangles which contain such edge have the other two edges both red or both blue). If $n \geq 6$, it is contained in at least two good triangles ($A_3B_2C_4$ and $A_3B_2C_5$).

It only remains to be shown that in the remaining cases $n = 2, 3, 5$, the described coloring is valid, thus producing the required locally nK_2 graphs. Of course, this can be done by carefully checking the coloring of the associated graph H_n, but it is also enough to show for each n *any* such locally nK_2 graph G_n on $6n-3$ vertices. This latter approach happens to be more direct: Using Nešetril's notation for products [8], we obtain

$G_2 := K_3 \Box K_3 \cong K_3 \times K_3$. Also, $G_3 := \overline{L(K_6)}$ is the the complement of the line graph of K_6, also known as the Kneser graph $KG_{6,2}$, its vertices are $\{ A \subseteq \{1,2,3,4,5,6\} \mid |A| = 2 \}$ and AA' is an edge whenever $A \cap A' = \varnothing$. The remaining graph G_5 can be described combinatorially by its triangles:

$$\begin{array}{llllll}
\{1,2,25\}, & \{1,9,17\}, & \{1,11,19\}, & \{1,13,21\}, & \{1,15,23\}, & \{2,10,18\}, \\
\{2,12,20\}, & \{2,14,22\}, & \{2,16,24\}, & \{3,4,25\}, & \{3,9,20\}, & \{3,11,18\}, \\
\{3,14,23\}, & \{3,16,21\}, & \{4,10,19\}, & \{4,12,17\}, & \{4,13,24\}, & \{4,15,22\}, \\
\{5,6,25\}, & \{5,9,22\}, & \{5,12,23\}, & \{5,13,18\}, & \{5,16,19\}, & \{6,10,21\}, \\
\{6,11,24\}, & \{6,14,17\}, & \{6,15,20\}, & \{7,8,25\}, & \{7,9,24\}, & \{7,12,21\}, \\
\{7,14,19\}, & \{7,15,18\}, & \{8,10,23\}, & \{8,11,22\}, & \{8,13,20\}, & \{8,16,17\}, \\
\{9,10,26\}, & \{11,12,26\}, & \{13,14,26\}, & \{15,16,26\}, & \{17,18,27\}, & \{19,20,27\}, \\
\{21,22,27\}, & \{23,24,27\}, & \{25,26,27\}. & & &
\end{array}$$

It can also be described as the complement of the Schläfli graph. But, perhaps the easiest way to verify the existence of such a graph is just to recall the classical 27 straight line theorem from algebraic geometry, which says (see for instance Proposition 7.3 in [10]) that any nonsingular cubic surface $S \subset \mathbb{C}P^3$ contains exactly 27 lines and that any of them intersects exactly 10 of the other lines, which in turn intersect each other in pairs. Hence our G_5 is simply the incidence graph of these straight lines.

This completes the proof of the main theorem.

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