THE BINET-LIKE FORMULA OF A FAMILY OF THE CONDITIONAL SEQUENCES BY MATRIX METHODS

MURAT SAHIN

ABSTRACT. Let $a_0, a_1, ...a_{r-1}$ be positive integers and define a conditional sequence $\{q_n\}$, with initial conditions $q_0 = 0$ and $q_1 = 1$, and for all $n \geqslant 2$, $q_n = a_t q_{n-1} + q_{n-2}$ where $n \equiv t \pmod r$. For $r = 2$, the author studied it in [1]. For general $\{q_n\}$, we found a closed form of the generating function for $\{q_n\}$ in terms of the continuant in [2]. In this paper, we give the matrix representation and a Binet-like formula for the conditional sequence $\{q_n\}$ by using the matrix methods.

1. INTRODUCTION

Recently, the authors introduced further generalization of the Fibonacci sequence, namely the generalized Fibonacci sequence, defined by

$$F_0^{(a,b)} = 0,\ F_0^{(a,b)} = 1,\ F_n^{(a,b)} = \begin{cases} aF_{n-1}^{(a,b)} + F_{n-2}^{(a,b)} & , \text{ if } n \text{ is even} \\ bF_{n-1}^{(a,b)} + F_{n-2}^{(a,b)} & , \text{ if } n \text{ is odd} \end{cases}$$

for any two non-zero real numbers a and b (see [1]). They find the generating function and Binet like formula for the generalized Fibonacci sequence. This new generalization produces a distinct sequence for each new choice of a and b. In fact, one can get many famous sequence, such as Fibonacci sequence, Pell numbers, k-Fibonacci numbers,...etc., by altering the values of a and b in the sequence. Also, the following open problem was given in [1].

Remark 1.1 (An Open Problem). *Let $a_0, a_1, ..., a_{r-1}$ be positive integers and define a sequence $\{q_n\}$ as follows. Set*

$$q_0 = 0 \textit{ and } q_1 = 1,$$

and for all $n \geqslant 2$,

$$q_n = a_t q_{n-1} + q_{n-2}$$

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where $n \equiv t \pmod r$. *Note that when* $r = 2$, *we get the sequence* $\left\{F_n^{(a,b)}\right\}$. *The open problem is finding a closed form of the generating function and a Binet-like formula for* $\{q_n\}$.

The author found a closed form of the generating function for $\{q_n\}$ in terms of the continuant in [2]. In this paper, we give the matrix representation of the conditional sequence $\{q_n\}$ and then we solve the second part of this open problem, that is, we give a Binet-like formula for conditional sequence $\{q_n\}$ by using the matrix methods.

2. Binet-like Formula For the Sequence $\{q_n\}$

In algebra, the continuant is a multivariate polynomial representing the determinant of a tridiagonal matrix and having applications in the theory of generalized continued fractions.

Let $\varnothing$ denotes empty set. For positive numbers $a_0, a_1, ..., a_n$, define the continuant $K(a_0, a_1, ...a_n)$ recursively by

$$K(\varnothing) = 1\ ,\ K(a_0) = a_0 \tag{2.1}$$

and for $n \geqslant 2$,

$$K(a_0, a_1, ..., a_n) = a_n K(a_0, a_1, ..., a_{n-1}) + K(a_0, a_1, ..., a_{n-2})\,. \tag{2.2}$$

(See [3] for detailed information).

Let define $K_1 = K(a_0, a_{r-1}, ..., a_2, a_1)$ and $K_2 = K(a_2, a_3, ..., a_{r-1})$ for given positive integer r. In [2], the author linked the sequence $\{q_n\}$ with the continuant as follows.

Theorem 2.1. *If* $n, r \geqslant 2$ *be positive integers then*

$$q_{nr+i} = (K_1 + K_2)\, q_{nr+i-r} + (-1)^{r+1} q_{nr+i-2r}\ ,\ for\ i = 0, 1, ..., r-1.$$

Proof. See [2] for proof. □

Theorem 2.2. *If* $r \geqslant 2$ *be positive integers then* $\{q_n\}$ *satisfies the* $2r$*-order recurrence*

$$q_n = (K_1 + K_2) q_{n-r} + (-1)^{r+1} q_{n-2r}$$

with initial conditions $q_0, q_1, q_2, ..., q_{2r-1}$.

Proof. Taking the values for $n = 2, 3, 4, ...$ in Theorem 2.1, we see that each of the following elements of the sequence $\{q_n\}$

$$q_{2r}, q_{2r+1}, q_{2r+2}, ..., q_{3r}, q_{3r+1}, ..., q_{4r+1}, q_{4r+2}, ...$$

are satisfy the given $2r$ order recurrence

$$q_n = (K_1 + K_2) q_{n-r} + (-1)^{r+1} q_{n-2r}.$$

So, with initial conditions $q_0, q_1, q_2, ..., q_{2r-1}$ we get the desired result. □

By Theorem 2.2, we can see the conditional sequence $\{q_n\}$ as a constant coefficient $2r$ order recurrence for any positive integer r. So, we can use matrix methods to obtain the Binet-like formula for the conditional sequence $\{q_n\}$ by aid of the Theorem 2.2 for given any positive integer r.

Define $2r \times 2r$ matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & \dots & 0 & 1 \\ (-1)^{r+1} & 0 & 0 & \dots & (K_1+K_2) & \dots & 0 & 0 \end{bmatrix}$$

where $(K_1 + K_2)$ is the entry in $(2r)\,th$ row and $(r+1)\,th$ column. In fact, A is the companion matrix for the polynomial

$$p(x) = x^{2r} - (K_1 + K_2)\,x^r - (-1)^{r+1}.$$

$p(x)$ is both characteristic and minimal polynomial for A. By using an inductive argument, we can give the matrix representation of the conditional sequence $\{q_n\}$ as

$$A^n \begin{bmatrix} q_0 \\ q_1 \\ \vdots \\ q_{2r-1} \end{bmatrix} = \begin{bmatrix} q_n \\ q_{n+1} \\ \vdots \\ q_{n+2r-1} \end{bmatrix}. \tag{2.3}$$

This matrix representation is important since it may be used to derive many interesting properties of the conditional sequence $\{q_n\}$.

Theorem 2.3. *The polynomial $p(x)$ has no multiple root.*

Proof. A polynomial has a multiple root if and only if its discriminant zero. The discriminant of the polynomial $p(x)$ is

$$D(p) = (-1)^{\frac{r(r-1)}{2}} R(p, p'),$$

where $R(p,p')$ is the resultant of the polynomial p and the derivative of p. The resultant can be given by the determinant of the Sylvester matrix of the polynomials p and the derivative of it. By calculating this determinant we can find the discriminant as follows

$$D(p) = r^{2r}(-1)^{r+1}\left((K_1+K_2)^2 + (-1)^r 4\right)^r.$$

So, the discriminant is zero if and only if

$$\begin{aligned} (K_1+K_2)^2 &= -4 \quad , \quad \text{if } r \text{ is even} \\ (K_1+K_2)^2 &= 4 \quad , \quad \text{if } r \text{ is odd} \end{aligned}$$

for positive integer $r \geqslant 2$. But, this is impossible because of the definition of the continuant. Therefore, the polynomial $p(x)$ has no multiple root. □

Let $\lambda_1, \lambda_2, ..., \lambda_{2r}$ be eigenvalues of the matrix A. These eigenvalues are all distinct by Theorem 2.3, so A can be diagonalized by using the Vandermode matrix

$$V(\lambda_1, \lambda_2, ..., \lambda_{2r}) = \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_{2r-1} & \lambda_{2r} \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_{2r-1}^2 & \lambda_{2r}^2 \\ \vdots & \vdots & & \vdots & \vdots \\ \lambda_1^{2r-1} & \lambda_2^{2r-1} & \cdots & \lambda_{2r-1}^{2r-1} & \lambda_{2r}^{2r-1} \end{bmatrix}.$$

So, we can give the following theorem by using the matrix methods in [4].

Theorem 2.4. *The Binet-like formula for the conditional sequence* $\{q_n\}$ *is*

$$q_n = \sum_{i=1}^{2r} \frac{\lambda_i^n}{p'(\lambda_i)}.$$

Proof. By (2.3), we can write a formula for q_n as follow

$$q_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix} A^n \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ \vdots \\ q_{2r-1} \end{bmatrix}. \tag{2.4}$$

By using the Vandermode matrix V, we can diagonalize A as

$$A = VDV^{-1}$$

where D is a diagonal matrix. We substitute for A in (2.4) , we can get

$$q_n = \sum_{i=1}^{2r} \lambda_i^n y_i \tag{2.5}$$

where y_i's satisfy

$$V \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{2r} \end{bmatrix} = \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ \vdots \\ q_{2r-1} \end{bmatrix}.$$

We can determine y_i's by using Cramer's rule. Then substituting y_i's in (2.5) and simplifying the obtained expression, we can get the desired result

$$q_n = \sum_{i=1}^{2r} \frac{\lambda_i^n}{p'(\lambda_i)}.$$

□

Example 2.1. *Let's find the Binet-like formula for the conditional sequence $\{q_n\}$ for given integer $r = 3$. Since $r = 3$, we have*

$$\begin{aligned} K_1 &= K(a_0, a_2, a_1) \\ &= a_1 K(a_0, a_2) + K(a_0) \\ &= a_1(a_0 + a_2 + 1) + a_0 \\ &= a_0 a_1 a_2 + a_0 + a_1 \end{aligned}$$

and

$$\begin{aligned} K_2 &= K(a_2) \\ &= a_2 \end{aligned}$$

by the definition of the continuant. So,

$$K_1 + K_2 = a_0 a_1 a_2 + a_0 + a_1 + a_2.$$

By Theorem 2.2 , the conditional sequence $\{q_n\}$ satisfy 6 order recurrence

$$q_n = (a_0 a_1 a_2 + a_0 + a_1 + a_2) q_{n-3} + q_{n-6}$$

with initial conditions q_0, q_1, q_2, q_3, q_4 and q_5. So,

$$p(x) = x^6 - (a_0 a_1 a_2 + a_0 + a_1 + a_2) x^3 - 1.$$

Let λ_1, λ_2, λ_3, λ_4, λ_5 and λ_6 are roots of the polynomial $p(x)$. These roots are distinct by Theorem 2.3. So, the Binet-like formula for the conditional sequence $\{q_n\}$ for given integer $r = 3$ is

$$q_n = \sum_{i=1}^{6} \frac{\lambda_i^n}{p'(\lambda_i)}$$

by Theorem 2.4.

For given any r, we can similarly construct the Binet-like formula the conditional sequence $\{q_n\}$.

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ANKARA UNIVERSITY, FACULTY OF SCIENCE, DEPARTMENT OF MATHEMATICS, TANDOGAN TR-06100, ANKARA, TURKEY.

E-mail address: muratsahin1907@gmail.com