On bounds for the norms of circulant matrices with the generalized Fibonacci and Lucas numbers

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Abstract

This paper is an extension of the work [On the norms of circulant matrices with the Fibonacci and Lucas numbers, Appl. Math. and Comp., 160 (2005), 125-132.], in which for some norms of the circulant matrices with classical Fibonacci and Lucas numbers it is obtained the lower and upper bounds. In this new paper, we generalize the results of that work.

Key words: The generalized Fibonacci numbers; the generalized Lucas numbers; the Euclidean norm; the spectral norm.

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1 Introduction

The use of integer sequences, and in particular of Fibonacci sequence and its variations, is rather common in physics [1-11]. The reason may be found in the need, and even in the necessity, to express a physical phenomenon in terms of an effective and comprehensive analytical form for the whole scientific community. Fibonacci numbers also have been studied from different points of view in modern science [1-11, 20-29].

A Fibonacci sequence is a sequence of natural numbers determined by taking each number equal to the sum of the last two. For this reason, this type of sequences are called "secondary Fibonacci sequences", to distinguish them from the ternary Fibonacci sequences, in which each term is a linear combination of the last three.

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Beginning with $F_0 = 0; F_1 = 1$, we have

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots$$

where

$$F_{n+2} = F_{n+1} + F_n. \tag{1}$$

The Fibonacci sequence that obeys condition (1) may be generalized, giving birth to "generalized secondary Fibonacci sequences" (GSFS),

$$a, b, pb + qa, p(pb + qa) + qb, \ldots$$

which satisfy relations of the type

$$G_{n+1} = pG_n + qG_{n-1} \tag{2}$$

with p and q natural numbers [25]. Spinadel [26], Gazale [27], Kappraff [29] and later Stakhov [28], Falcón and Plaza [20, 23] independently introduced the *Generalized Fibonacci and Lucas Numbers of the Order m or simply Fibonacci and Lucas m-numbers.* They are given by the following recurrence relations:

$$F_m(n) = mF_m(n-1) + F_m(n-2); F_m(0) = 0; F_m(1) = 1,$$

$$L_m(n) = mL_m(n-1) + L_m(n-2); L_m(0) = 2; L_m(1) = m,$$

where $m > 0$ is a positive real number and $n = 0, \pm 1, \pm 2, \ldots$.

The generalization given by Falcón and Plaza [20] is defined by the following equation for any given integer number $k \geq 1$

$$F_{k,n+1} = kF_{k,n} + F_{k,n-1} \text{ for } n \geq 1$$

with initial conditions

$$F_{k,0} = 0; \quad F_{k,1} = 1,$$

and called k-Fibonacci numbers. These numbers are partial case of Spinadel's numbers defined in (2) for the case $p = k$ and $q = 1$. Also, k-Fibonacci numbers that generalize, between others, both the classic Fibonacci numbers and the Pell numbers. It is obvious that when $k = 1$, then nth k-Fibonacci number is the nth classic Fibonacci number. In [20], Falcón and Plaza showed the relation between the 4-triangle longest-edge (4TLE) partition and the k-Fibonacci numbers, as another example of the relation between geometry and numbers, and many properties of these numbers are deduced directly from elementary matrix algebra. In [21], many properties of these numbers are deduced and related with the so-called Pascal

2-triangle. In [22], the 3-dimensional k-Fibonacci spirals are studied from a geometric point of view. These curves appear naturally from studying the k-th Fibonacci numbers $\{F_{k,n}\}_{n=0}^{\infty}$ and the related hyperbolic k-Fibonacci functions.

Besides, Fibonacci k-numbers named Fibonacci m-numbers were studied in [28]. The another generalization for Fibonacci numbers is the generalized Fibonacci p-numbers introduced by Stakhov in 1977. Stakhov [30] defined the generalized Fibonacci p-numbers that are given for any integer number $p > 0$ by the following recurrence relation:

$$F_p(n) = F_p(n-1) + F_p(n-p-1), \quad (n > p)$$

with the initial values

$$F_p(0) = 0, F_p(1) = 1, F_p(2) = 1, ..., F_p(p) = 1,$$

where $n = 0, \pm 1, \pm 2, \ldots$. The generalized Fibonacci p-numbers can be represented in the following form:

$$\underbrace{\begin{pmatrix} F_p(n) \\ F_p(n-1) \\ F_p(n-2) \\ \vdots \\ F_p(n-p+1) \\ F_p(n-p) \end{pmatrix}}_{\mathbf{f}_n} = \underbrace{\begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix}}_{Q_p^T} \underbrace{\begin{pmatrix} F_p(n-1) \\ F_p(n-2) \\ F_p(n-3) \\ \vdots \\ F_p(n-p) \\ F_p(n-p-1) \end{pmatrix}}_{\mathbf{f}_{n-1}}$$

where $\mathbf{f}_n$ and $\mathbf{f}_{n-1}$ are $(p+1) \times 1$ vectors and Q_p^T is the transpose of the Fibonacci Q_p matrix:

$$Q_p = \begin{pmatrix} 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Q_p is a $(p+1) \times (p+1)$ matrix which contains a $p \times p$ identity matrix in its upper right corner. Regardless of the value of p, the first column always begins and ends with 1, and has zeros elsewhere. The last row always begins with 1 and has zeros in all other positions [31]. In general the n^{th} power of the Q_p matrix takes the following form [31]:

Theorem 1 *For any integer $p \geq 1$ and $n \in \mathbb{Z}$, the n^{th} power of the Fibonacci Q_p matrix is given by:*

$$Q_p^n = \begin{pmatrix} F_p(n+1) & F_p(n) & \cdots & F_p(n-p+2) & F_p(n-p+1) \\ F_p(n-p+1) & F_p(n-p) & \cdots & F_p(n-2p+2) & F_p(n-2p+1) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ F_p(n-1) & F_p(n-2) & \cdots & F_p(n-p) & F_p(n-p-1) \\ F_p(n) & F_p(n-1) & \cdots & F_p(n-p+1) & F_p(n-p) \end{pmatrix},$$

where $F_p(n)$ is the n^{th} generalized Fibonacci p-number.

The application of the Fibonacci p-numbers in Fibonacci Coding Theory uses the Q_p^n matrix to a great extent.

Remark 2 *Q_p^n is a $(p+1) \times (p+1)$ matrix where the first row is made from a decreasing Fibonacci p-sequence of length $p+1$, starting with the term $F_p(n+1)$:*

$$\ldots, F_p(n+1), F_p(n), \ldots, F_p(n-p+2), F_p(n-p+1), \ldots.$$

The second row consists of the sequence above shifted forwards by p terms. All subsequent rows are found by shifting the sequence in the previous row backwards by a term. In other words, we conclude that the Q_p^n-matrices are circulant matrices.

In 2009, all these results are described in the book *The Mathematics of Harmony: From Euclid to Contemporary Mathematics and Computer Science [32].*

For any given $a_0, a_1, ..., a_{n-1} \in \mathbb{C}$, the circulant matrix $B = (b_{ij})_{n\times n}$ is defined by $b_{ij} = a_{j-i(\text{mod}\, n)}$, i.e.,

$$B = \begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-1} \\ a_{n-1} & a_0 & a_1 & \cdots & a_{n-2} \\ a_{n-2} & a_{n-1} & a_0 & \cdots & a_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1 & a_1 & a_2 & \cdots & a_0 \end{bmatrix}.$$

A circulant matrix is a special kind of Toeplitz matrix where each row vector is rotated one element to the right relative to the preceding row vector. Several problems of physics and electromagnetics involve circulant matrices (CM) and block-circulant matrices (BCM). For example, such matrices appear when the method of moments is used to study the electromagnetic behavior of structures having circular periodicity around an axis (e.g., circular arrays of equally spaced dipoles or cylindrical arrays of equally spaced parallel wires). In such problems, the exploitation of the properties of CM and BCM may lead to a considerable reduction of the computational complexity [29, 30]. CM and BCM have been studied extensively and closed-form expressions for their inversion are well known [31, 32]. In [15], the formulas of inversion have been derived in the CM case by writing a matrix as a combination of permutation matrices, which are expressed in terms of their eigenvalues and eigenvectors. In [14], this procedure has been extended to the BCM case.

In this note, we first present a generalization of Spinadel's numbers defined in (2), which we call the generalized (s,t)-Fibonacci numbers, and then establish bounds for norms of circulant matrices composed of these numbers. Thus, in this note, we give the generalizations of results presented in Solak [24].

2 Backgrounds for matrix norms

There are many norms that could be defined for vectors. One type of norm is called an l_p norm, often denoted as $\|.\|_p$. For $p \geq 1$, it is defined as

$$\|x\|_p = \left(\sum_i |x_i|^p\right)^{1/p}.$$

This is also sometimes called the *Minkowski norm* and also the *Hölder norm.*

It is well known that some matrix norms are defined in terms of vector norms. For clarity, we will denote a vector norm as $\|.\|_v$ and a matrix norm as $\|.\|_M$. (This notation is meant to be generic; that is, $\|.\|_v$ represents any vector norm.) The matrix norm $\|.\|_M$ induced by $\|.\|_v$ is defined for the $n \times m$ matrix A and $m \times 1$ vector x by

$$\|A\|_M = \max_{x \neq 0} \frac{\|Ax\|_v}{\|x\|_v}. \tag{3}$$

Also, it can be said that an induced norm is indeed a matrix norm. For any vector norm and its induced matrix norm, we see from equation (3) that

$$\|Ax\| \leq \|A\| \, \|x\|$$

because $\|x\| \geq 0$. The matrix norms that correspond to the l_p vector norms are defined for the $n \times m$ matrix A as

$$\|A\|_p = \max_{\|x\|_p = 1} \|Ax\|_p.$$

It is clear that the l_p matrix norms satisfy the consistency property, because they are induced norms. The l_1 and l_∞ norms have interesting simplifications of equation (3):

$$\|A\|_1 = \max_j \sum_i |a_{ij}| \,, \tag{4}$$

so the l_1 is also called the *column-sum norm*; and

$$\|A\|_\infty = \max_i \sum_j |a_{ij}| \,, \tag{5}$$

so the l_∞ is also called the *row-sum norm.* The Frobenius and the spectral norms are defined as

$$\|A\|_F = \sqrt{\sum_{i,j} |a_{ij}|^2}, \tag{6}$$

and

$$\|A\|_2 = \sqrt{\max_{1 \le i \le n} \lambda_i \left(A^* A\right)},$$

respectively. Here A^* is conjugate transpose of matrix A. It is easy to see that this measure has the consistency property, as a norm must. The Frobenius norm is sometimes called the Euclidean matrix norm and denoted by $\|.\|_E$, although the l_2 matrix norm is more directly based on the Euclidean vector norm, as we mentioned above.

If $\|.\|_a$ and $\|.\|_b$ are matrix norms, then there are positive numbers r and s such that, for any matrix A,

$$r \|A\|_b \le \|A\|_a \le s \|A\|_b \,, \tag{7}$$

(for more details see [16]). If A is an $n \times m$ real matrix we have some specific instances of (7):

$$\|A\|_\infty \le \sqrt{m} \, \|A\|_F \,, \tag{8}$$

$$\|A\|_F \le \sqrt{\min(n, m)} \, \|A\|_2 \,, \tag{9}$$

$$\|A\|_2 \le \sqrt{m} \, \|A\|_1 \,, \tag{10}$$

$$\|A\|_1 \le \sqrt{n} \, \|A\|_2 \,, \tag{11}$$

$$\|A\|_2 \le \|A\|_F \,, \tag{12}$$

$$\|A\|_F \le \sqrt{n} \, \|A\|_\infty \,. \tag{13}$$

Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be $m \times n$ matrices. The Hadamard product of A and B is defined by $A \circ B = [a_{ij} b_{ij}]_{m \times n}$ (for more details see [17]). Therefore if $\|.\|$ is any norm on $m \times n$ matrices, then [18]

$$\|A \circ B\| \leq \|A\| \, \|B\|, \tag{14}$$

and if $A = B \circ C$, then [19]

$$\|A\|_2 \leq r_1(B) \, c_1(C), \tag{15}$$

where $r_1(B) = \max_i \sqrt{\sum_j |b_{ij}|^2}$ and $c_1(C) = \max_j \sqrt{\sum_i |c_{ij}|^2}$.

3 Results and Proofs

Now we introduce a generalization of the Fibonacci k-numbers. It should be noted that the recurrence formula of these numbers depends on four real parameters.

Definition 3 *For any nonzero real numbers s, t, the nth (s,t)-Fibonacci $\{F_{s,t,n}\}_{n \in \mathbb{N}}$ sequence is defined recurrently by*

$$F_{s,t,n+1} = sF_{s,t,n} + tF_{s,t,n-1} \quad \text{for } n \geq 1, \tag{16}$$

with

$$F_{s,t,0} = a, \; F_{s,t,1} = b,$$

where a and b are any real numbers.

The following table summarizes special cases of $F_{s,t,n}$:

(a,b)	(s,t)	$F_{s,t,n}$
$(0,1)$	$(s,1)$	the Fibonacci k-numbers [20]
$(0,1)$	$(1,1)$	the classical Fibonacci numbers
$(0,1)$	$(2,1)$	the classical Pell numbers
$(0,1)$	$(1,2)$	the classical Jacobsthal numbers
$(0,1)$	$(3,-2)$	the classical Mersenne numbers
$(2,1)$	$(1,1)$	the classical Lucas numbers
$(2,2)$	$(2,1)$	the classical Pell-Lucas numbers
$(2,1)$	$(1,2)$	the classical Jacobsthal-Lucas numbers
$(2,3)$	$(3,-2)$	the classical Fermat numbers

Lemma 4 *For $s+t-1\neq 0$,*

$$\sum_{j=0}^{n} F_{s,t,j} = \frac{1}{s+t-1}\left(tF_{s,t,n} + F_{s,t,n+1} + sa - a - b\right). \tag{17}$$

Proof. Let $W = F_{s,t,0} + F_{s,t,1} + F_{s,t,2} + F_{s,t,3} + \ldots + F_{s,t,n}$. Then,

$$\begin{aligned} sW &= sF_{s,t,0} + sF_{s,t,1} + sF_{s,t,2} + sF_{s,t,3} + \ldots + sF_{s,t,n-1} + sF_{s,t,n} \\ +tW &= \qquad + tF_{s,t,0} + tF_{s,t,1} + tF_{s,t,2} + \ldots + tF_{s,t,n-1} + tF_{s,t,n} \end{aligned}$$

yields (by using the generalized (s,t)-Fibonacci recursion on the above columns)

$$\begin{aligned} sW + tW &= sF_{s,t,0} + (F_{s,t,2} + F_{s,t,3} + \ldots + F_{s,t,n}) + F_{s,t,n+1} + tF_{s,t,n} \\ &= sF_{s,t,0} + (W - F_{s,t,0} - F_{s,t,1}) + tF_{s,t,n} + F_{s,t,n+1} \\ &= W + tF_{s,t,n} + F_{s,t,n+1} + sa - a - b \text{ as } F_{s,t,0} = a \text{ and } F_{s,t,1} = b. \end{aligned}$$

From where,

$$W = \frac{1}{s+t-1}\left(tF_{s,t,n} + F_{s,t,n+1} + sa - a - b\right).$$

Thus the result is obtained. ■

Binet's formulas are well known in the Fibonacci numbers theory [[1], [8]]. Binet's formula allows us to express the nth generalized (s,t)-Fibonacci number as functions of the roots α and β of the characteristic equation $x^2 = sx + t$ associated with recurrence relation (16).

Proposition 5 (Binet's formula) *If α and β are distinct, the nth generalized (s,t)-Fibonacci number is given by*

$$F_{s,t,n} = \frac{b-\beta a}{\alpha-\beta}\alpha^n - \frac{b-\alpha a}{\alpha-\beta}\beta^n. \tag{18}$$

Proof. We now are assuming that α and β are distinct and $|\alpha| > |\beta|$. Thus, α^n and β^n are independent solutions to the difference equation (16). And since we already know that the null space is two dimensional, that makes $\{\alpha^n, \beta^n\}$ a basis. In this case, $F_{s,t,n}$ is characterized as the set of linear combinations of these two geometric progressions. Thus, we can express $F_{s,t,n}$ in the form

$$F_{s,t,n} = c_1\alpha^n + c_2\beta^n,$$

where the constants c_1 and c_2 are determined by the initial conditions

$$\begin{aligned} a &= c_1 + c_2 \\ b &= c_1\alpha + c_2\beta. \end{aligned}$$

Since we are assuming that α and β are distinct, so this system has the solution

$$\begin{aligned} c_1 &= \frac{b-\beta a}{\alpha-\beta}, \\ c_2 &= -\frac{b-\alpha a}{\alpha-\beta}, \end{aligned}$$

which completes the proof. ■

If $a=0$, $b=1$ and $t=1$, for the Fibonacci k-sequence, we have: $\alpha = \frac{k+\sqrt{k^2+4}}{2}$ and $\beta = \frac{k-\sqrt{k^2+4}}{2}$. Therefore the general term of the Fibonacci k-sequence is given by

$$F_{k,n} = \frac{\left(\frac{k+\sqrt{k^2+4}}{2}\right)^n - \left(\frac{k-\sqrt{k^2+4}}{2}\right)^n}{\sqrt{k^2+4}} \text{ [20]}. \tag{19}$$

As immediate consequence of Binet's formula given in Eq. (18), for the generalized (s,t)-Fibonacci sequence, a new idendity which is a generalization of Catalan's identity, is derived below:

Lemma 6 *If α and β are distinct, for the nth generalized (s,t)-Fibonacci number*

$$F_{s,t,n-r}F_{s,t,n+r} - F_{s,t,n}^2 = (-t)^{n+1-r}\, xyF_{k,r}^2, \tag{20}$$

where $x = b-\beta a$ and $y = b-\alpha a$.

Proof. Let $x = b-\beta a$ and $y = b-\alpha a$. By using Eq. (18) in the left hand side (LHS) of Eq. (20), and taking into account that $\alpha\beta = -t$ it is obtained

$$\begin{aligned}
\text{(LHS)} &= \frac{x\alpha^{n-r}-y\beta^{n-r}}{\alpha-\beta}\frac{x\alpha^{n+r}-y\beta^{n+r}}{\alpha-\beta} - \left(\frac{x\alpha^n-y\beta^n}{\alpha-\beta}\right)^2 \\
&= \frac{x^2\alpha^{2n} - xy\alpha^{n-r}\beta^{n+r} - xy\alpha^{n+r}\beta^{n-r}}{(\alpha-\beta)^2} + \\
&\quad + \frac{y^2\beta^{2n} - x^2\alpha^{2n} + 2xy\alpha^n\beta^n - y^2\beta^{2n}}{(\alpha-\beta)^2} \\
&= \frac{1}{(\alpha-\beta)^2}\left[-xy(\alpha\beta)^n\left(\frac{\beta}{\alpha}\right)^r - xy(\alpha\beta)^n\left(\frac{\alpha}{\beta}\right)^r + 2xy(\alpha\beta)^n\right] \\
&= \frac{xy(\alpha\beta)^{n+1}}{(\alpha-\beta)^2}\left[\frac{\alpha^{2r}+\beta^{2r}}{(\alpha\beta)^r} - 2\right] \\
&= (-t)^{n+1-r}\, xyF_{k,r}^2 \text{ by Eq. (19)}.
\end{aligned}$$

Thus, Eq. (20) is proven. ■

Note that for $r = 1$, Eq. (20) gives a generalization of Cassini's identity [12] for the generalized (s,t)-Fibonacci sequence

$$F_{s,t,n-1}F_{s,t,n+1} - F_{s,t,n}^2 = (-t)^n xy. \tag{21}$$

Lemma 7 *For all $n \geq 1$,*

$$(-1)^n - 2\sum_{i=1}^{n}(-1)^i = 1. \tag{22}$$

Proof. Using mathematical induction method, it can be easily proven this Lemma. ■

Lemma 8 *Let $(s+t-1)(s-t+1) \neq 0$. If α and β are distinct, then*

$$\begin{aligned}\sum_{j=0}^{n} F_{s,t,j}^2 &= \frac{1}{(s+t-1)(s-t+1)}\left[F_{s,t,n}^2 - t^2F_{s,t,n-1}^2 + \right. \\ &\quad \left. +t^2(b-a)^2 - a^2 + t^{j+1}xy\left(1+(-1)^{n+1}\right)\right],\end{aligned} \tag{23}$$

where $x = b - \beta a$ and $y = b - \alpha a$.

Proof. Let $T = \sum_{j=0}^{n} F_{s,t,j}^2$. Then, we have that

$$\begin{aligned}T &= \sum_{j=0}^{n} F_{s,t,j}^2 \\ &= \sum_{j=0}^{n}\left(\frac{F_{s,t,j+1} - tF_{s,t,j-1}}{s}\right)^2 \quad \text{by (16)} \\ &= \frac{1}{s^2}\left[\sum_{j=0}^{n} F_{s,t,j+1}^2 + t^2\sum_{j=0}^{n} F_{s,t,j-1}^2 - 2t\sum_{j=0}^{n} F_{s,t,j+1}F_{s,t,j-1}\right] \\ &= \frac{1}{s^2}\left[T - F_{s,t,0}^2 + F_{s,t,n+1}^2 + t^2F_{s,t,-1}^2 + \right. \\ &\quad \left. +t^2T - t^2F_{s,t,n}^2 - 2t\sum_{j=0}^{n}\left((-t)^j xy + F_{s,t,j}^2\right)\right] \quad \text{by (21)} \\ &= \frac{1}{s^2}\left[T\left(t^2 - 2t + 1\right) - a^2 + F_{s,t,n+1}^2 + \right. \\ &\quad \left. +t^2(b-a)^2 - t^2F_{s,t,n}^2 - 2t^{j+1}xy\sum_{j=0}^{n}(-1)^j\right],\end{aligned}$$

and hence, by (22), it follows that

$$T = \frac{1}{s^2}\left[T\left(t^2-1\right)^2 - a^2 + F_{s,t,n+1}^2 + t^2(b-a)^2 - \right. \\ \left. -t^2F_{s,t,n}^2 + t^{j+1}xy\left(1+(-1)^{n+1}\right)\right]. \tag{24}$$

Therefore, from Eq. (24) we obtain

$$T(s+t-1)(s-t+1) = F_{s,t,n+1}^2 - t^2F_{s,t,n}^2 + t^2(b-a)^2 - \\ -a^2 + t^{j+1}xy\left(1+(-1)^{n+1}\right),$$

which completes the proof. ■

Theorem 9 *Let $s+t-1 \neq 0$ and let the $n\times n$ matrix A be as $A=[a_{ij}]$ such that $a_{ij} = F_{s,t,\mathrm{mod}(j-i,n)}$. Then,*

$$\|A\|_1 = \|A\|_\infty = \frac{n}{s+t-1}\left(tF_{s,t,n-1} + F_{s,t,n} + sa - a - b\right),$$

$$\frac{n^{1/2}}{s+t-1}\left(tF_{s,t,n-1} + F_{s,t,n} + sa - a - b\right) \leq \|A\|_2,$$

$$\|A\|_2 \leq \frac{n^{3/2}}{s+t-1}\left(tF_{s,t,n-1} + F_{s,t,n} + sa - a - b\right),$$

$$\frac{n^{1/2}}{s+t-1}\left(tF_{s,t,n-1} + F_{s,t,n} + sa - a - b\right) \leq \|A\|_F,$$

$$\|A\|_F \leq \frac{n^{3/2}}{s+t-1}\left(tF_{s,t,n-1} + F_{s,t,n} + sa - a - b\right).$$

Proof. The matrix A is of the form

$$A = \begin{bmatrix} F_{s,t,0} & F_{s,t,1} & \cdots & F_{s,t,n-1} \\ F_{s,t,n-1} & F_{s,t,0} & \cdots & F_{s,t,n-2} \\ \vdots & \vdots & \vdots & \vdots \\ F_{s,t,1} & F_{s,t,2} & \cdots & F_{s,t,0} \end{bmatrix}. \tag{25}$$

Since $s+t-1\neq 0$, from Eq. (4), (5) and (17) we have

$$\|A\|_1 = \|A\|_\infty = n\sum_{j=0}^{n-1} F_{s,t,j} \\ = \frac{n}{s+t-1}\left(tF_{s,t,n-1} + F_{s,t,n} + sa - a - b\right). \tag{26}$$

From (10), (11) and (26) we get

$$\frac{n^{1/2}}{s+t-1}\left(tF_{s,t,n-1}+F_{s,t,n}+sa-a-b\right)\leq\|A\|_2\,,$$

$$\|A\|_2\leq\frac{n^{3/2}}{s+t-1}\left(tF_{s,t,n-1}+F_{s,t,n}+sa-a-b\right).$$

From (8), (13) and (26) we obtain

$$\frac{n^{1/2}}{s+t-1}\left(tF_{s,t,n-1}+F_{s,t,n}+sa-a-b\right)\leq\|A\|_F\,,$$

$$\|A\|_F\leq\frac{n^{3/2}}{s+t-1}\left(tF_{s,t,n-1}+F_{s,t,n}+sa-a-b\right)$$

■

Corollary 10 *Let* $s+t-1\neq 0$ *and let the* $n\times n$ *matrix* A *be as* $A=[a_{ij}]$ *such that* $a_{ij}=F_{s,t,\text{mod}(i+j,n)}$. *Then,*

$$\|A\|_1=\|A\|_\infty=\frac{n}{s+t-1}\left(tF_{s,t,n}+F_{s,t,n+1}+sa-a-b\right),$$

$$\frac{n^{1/2}}{s+t-1}\left(tF_{s,t,n}+F_{s,t,n+1}+sa-a-b\right)\leq\|A\|_2\,,$$

$$\|A\|_2\leq\frac{n^{3/2}}{s+t-1}\left(tF_{s,t,n}+F_{s,t,n+1}+sa-a-b\right),$$

$$\frac{n^{1/2}}{s+t-1}\left(tF_{s,t,n}+F_{s,t,n+1}+sa-a-b\right)\leq\|A\|_F\,,$$

$$\|A\|_F\leq\frac{n^{3/2}}{s+t-1}\left(tF_{s,t,n}+F_{s,t,n+1}+sa-a-b\right).$$

Theorem 11 *Let* $(s+t-1)(s-t+1)\neq 0$, α *and* β *be distinct and the matrix* A *be matrix given in* (25). *Then,*

$$\|A\|_F=\sqrt{\begin{array}{c}\frac{n}{(s+t-1)(s-t+1)}\left[F_{s,t,n}^2-t^2F_{s,t,n-1}^2+t^2(b-a)^2-\right.\\ \left.-a^2+t^{j+1}xy\left(1+(-1)^{n+1}\right)\right]\end{array}}\,,$$

$$\sqrt{\frac{1}{(s+t-1)(s-t+1)}\left[F_{s,t,n}^2 - t^2 F_{s,t,n-1}^2 + t^2(b-a)^2 - -a^2 + t^{j+1}xy\left(1+(-1)^{n+1}\right)\right]} \leq \|A\|_2,$$

$$\|A\|_2 \leq \frac{1}{(s+t-1)(s-t+1)}\left[F_{s,t,n}^2 - t^2 F_{s,t,n-1}^2 + t^2(b-a)^2 - -a^2 + t^{j+1}xy\left(1+(-1)^{n+1}\right)\right],$$

$$\|A\|_\infty \leq n\sqrt{\frac{1}{(s+t-1)(s-t+1)}\left[F_{s,t,n}^2 - t^2 F_{s,t,n-1}^2 + t^2(b-a)^2 - -a^2 + t^{j+1}xy\left(1+(-1)^{n+1}\right)\right]},$$

$$\sqrt{\frac{1}{(s+t-1)(s-t+1)}\left[F_{s,t,n}^2 - t^2 F_{s,t,n-1}^2 + t^2(b-a)^2 - -a^2 + t^{j+1}xy\left(1+(-1)^{n+1}\right)\right]} \leq \|A\|_\infty.$$

Proof. By the definitions of Frobenius norm and the matrix A, we have that

$$\|A\|_F^2 = n\sum_{j=0}^{n-1} F_{s,t,j}^2.$$

From Eq. (23), we obtain

$$\|A\|_F = \sqrt{\frac{n}{(s+t-1)(s-t+1)}\left[F_{s,t,n}^2 - t^2 F_{s,t,n-1}^2 + t^2(b-a)^2 - -a^2 + t^{j+1}xy\left(1+(-1)^{n+1}\right)\right]}, \quad (27)$$

where $x = b - \beta a$ and $y = b - \alpha a$. Since $\frac{1}{\sqrt{n}}\|A\|_F \leq \|A\|_2$, from (27) one has

$$\sqrt{\frac{1}{(s+t-1)(s-t+1)}\left[F_{s,t,n}^2 - t^2 F_{s,t,n-1}^2 + t^2(b-a)^2 - -a^2 + t^{j+1}xy\left(1+(-1)^{n+1}\right)\right]} \leq \|A\|_2.$$

From (8), (13) and (27) we get

$$\|A\|_\infty \leq n\sqrt{\frac{1}{(s+t-1)(s-t+1)}\left[F_{s,t,n}^2 - t^2 F_{s,t,n-1}^2 + t^2(b-a)^2 - -a^2 + t^{j+1}xy\left(1+(-1)^{n+1}\right)\right]},$$

and

$$\sqrt{\frac{1}{(s+t-1)(s-t+1)}\left[F_{s,t,n}^2 - t^2 F_{s,t,n-1}^2 + t^2 (b-a)^2 - -a^2 + t^{j+1} xy \left(1 + (-1)^{n+1}\right)\right]} \leq \|A\|_\infty ,$$

respectively. On the other hand, let matrices B and C be as

$$B = (B_{ij}) = \begin{cases} F_{s,t,\text{mod}(j-i,n)}, & i \geq j \\ 1, & i < j \end{cases},$$

and

$$C = (C_{ij}) = \begin{cases} F_{s,t,\text{mod}(j-i,n)}, & i < j \\ 1, & i \geq j \end{cases}.$$

It is seen easily that $A = B \circ C$. Therefore,

$$\begin{aligned} r_1(B) &= \max_i \sqrt{\sum_j |b_{ij}|^2} \\ &= \sum_{j=0}^{n-1} F_{s,t,j}^2 \\ &= \sqrt{\frac{1}{(s+t-1)(s-t+1)}\left[F_{s,t,n}^2 - t^2 F_{s,t,n-1}^2 + t^2 (b-a)^2 - -a^2 + t^{j+1} xy \left(1 + (-1)^{n+1}\right)\right]} \end{aligned} \tag{28}$$

and

$$\begin{aligned} c_1(C) &= \max_j \sqrt{\sum_i |c_{ij}|^2} \\ &= \sum_{j=0}^{n-1} F_{s,t,j}^2 \\ &= \sqrt{\frac{1}{(s+t-1)(s-t+1)}\left[F_{s,t,n}^2 - t^2 F_{s,t,n-1}^2 + t^2 (b-a)^2 - -a^2 + t^{j+1} xy \left(1 + (-1)^{n+1}\right)\right]} \end{aligned} \tag{29}$$

Thus, from (15), (28) and (29) we get

$$\|A\|_2 \leq \frac{1}{(s+t-1)(s-t+1)} \left[F_{s,t,n}^2 - t^2 F_{s,t,n-1}^2 + t^2 (b-a)^2 - -a^2 + t^{j+1} xy \left(1 + (-1)^{n+1}\right)\right],$$

which completes the proof. ■

Corollary 12 *Let $(s+t-1)(s-t+1) \neq 0$, α and β be distinct, and the $n \times n$ matrix A be as $A = [a_{ij}]$ such that $a_{ij} = F_{s,t,\text{mod}(i+j,n)}$. Then,*

$$\|A\|_F = \sqrt{\begin{array}{c} \frac{n}{(s+t-1)(s-t+1)} \left[F_{s,t,n}^2 - t^2 F_{s,t,n-1}^2 + t^2 (b-a)^2 - \right. \\ \left. -a^2 + t^{j+1} xy \left(1 + (-1)^{n+1}\right)\right] \end{array}},$$

$$\sqrt{\begin{array}{c} \frac{1}{(s+t-1)(s-t+1)} \left[F_{s,t,n}^2 - t^2 F_{s,t,n-1}^2 + t^2 (b-a)^2 - \right. \\ \left. -a^2 + t^{j+1} xy \left(1 + (-1)^{n+1}\right)\right] \end{array}} \leq \|A\|_2,$$

$$\begin{aligned} \|A\|_2 \leq \; & \frac{1}{(s+t-1)(s-t+1)} \left[F_{s,t,n}^2 - t^2 F_{s,t,n-1}^2 + t^2 (b-a)^2 - \right. \\ & \left. -a^2 + t^{j+1} xy \left(1 + (-1)^{n+1}\right)\right], \end{aligned}$$

$$\|A\|_\infty \leq n \sqrt{\begin{array}{c} \frac{1}{(s+t-1)(s-t+1)} \left[F_{s,t,n}^2 - t^2 F_{s,t,n-1}^2 + t^2 (b-a)^2 - \right. \\ \left. -a^2 + t^{j+1} xy \left(1 + (-1)^{n+1}\right)\right] \end{array}}$$

and

$$\sqrt{\begin{array}{c} \frac{1}{(s+t-1)(s-t+1)} \left[F_{s,t,n}^2 - t^2 F_{s,t,n-1}^2 + t^2 (b-a)^2 - \right. \\ \left. -a^2 + t^{j+1} xy \left(1 + (-1)^{n+1}\right)\right] \end{array}} \leq \|A\|_\infty.$$

4 Conclusion

In this study, we have defined the generalized (s,t)-Fibonacci sequence, and then presented Binet's formulas for this sequence. Using Binet's formula, for these numbers Catalan and Cassini's identities have been given. Computing the sum of the first terms of the generalized (s,t)-Fibonacci sequence, we have established bounds for some norms of circulant matrices composed of these sequences.

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