Vertex-neighbor-scattering number of graphs

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Abstract

Incorporating the concept of the scattering number and the idea of the vertex-neighbor-connectivity, we introduce a new graph parameter called the vertex-neighbor-scattering number, which measures how easily a graph can be broken into many components with the removal of the neighborhoods of few vertices, and discuss some properties of this parameter. Some tight upper and lower bounds for this parameter are also given.

Keywords: vertex-neighbor-scattering number; vertex-neighbor-connectivity; vulnerability

1 Introduction

The notion of the scattering number of graphs first appeared in literature in a paper of Jung [1]. It turned out that this parameter is convenient in the measure of the connectivity of a graph. In 1978, Gunther and Hartnell [2] introduced, and in 1985-86, Gunther [3, 4] further developed the idea of modeling a spy network by a graph whose vertices represent the stations and whose edges represent lines of communication. If a station is destroyed, the adjacent stations will be betrayed so that the betrayed stations become useless to network as a whole. Therefore, considering the question of not only removing some vertices but also of removing all of their adjacent vertices, Cozzens and Wu introduced the vertex-neighbor-connectivity (VNC) in [5] and the vertex-neighbor-integrity (VNI) in [6] of a graph to measure the vulnerability of a spy network. As shown in section 4, VNC is sensitive to the number of edges present in a graph, and using only VNC or VNI can not distinguish the vulnerability of different networks very well in some situations. Therefore, incorporating the concept of the scattering number and the idea of the vertex-neighbor-connectivity, we introduce a new graph parameter called the vertex-neighbor-scattering number in this paper.

Let $G = (V, E)$ be a graph and u a vertex of G. We call $N(u) = \{v \in V(G) | u \neq v, u$ and v are adjacent$\}$ the *open neighborhood* of u, and $N[u] = N(u) \cup \{u\}$ the *closed neighborhood* of u. A vertex u in G is said to be *subverted* if its closed neighborhood $N[u]$ is deleted from G. A set of vertices $S \subseteq V(G)$ is called a *vertex subversion strategy* of G if each of the vertices in S is subverted from G. By G/S we denote the survival subgraph that remains after each vertex of S is subverted from G. A vertex set S is called a *cut-strategy* of G if the survival subgraph G/S is disconnected, or is a clique, or is empty. The *vertex-neighbor-connectivity* of G, denoted by $K(G)$, is defined to be the minimum size of all cut-strategies of G. A graph G is *m-neighbor-connected* if $K(G) = m$.

Let G be a connected noncomplete graph. The *vertex-neighbor-scattering number* (VNS) of G is defined as

$$VNS(G) = \max_{S \subseteq V(G)} \{\omega(G/S) - |S|\},$$

where the maximum is taken over all S, the cut-strategy of G, and $\omega(G/S)$ is the number of components of G/S. We call $S^* \subseteq V(G)$ a *VNS-set* of G if $VNS(G) = \omega(G/S^*) - |S^*|$. For the complete graph, subverting any one vertex will betray the entire graph, so we define $VNS(K_n) = 1$.

Example 1. Let $K_{n_1,n_2,\ldots,n_p}$ be a complete p-partite graph with a partition $(N_1, N_2, \ldots, N_p)$, where $|N_i| = n_i > 1, i = 1, 2, \ldots, p$. Let v be a vertex in $K_{n_1,n_2,\ldots,n_p}$. If $v \in N_i$, then $K_{n_1,n_2,\ldots,n_p}/\{v\}$ is composed of $n_i - 1$ isolated vertices, $i = 1, 2, \ldots, p$. It is easy to check that a VNS-set has size one. Therefore, $VNS(K_{n_1,n_2,\ldots,n_p}) = \max_i\{n_i - 1\} - 1 = max\{n_1, n_2, \ldots, n_p\} - 2$.

In particular, for a star graph $S_{1,n}$ ($n \geq 2$), we have $VNS(S_{1,n}) = n - 2$.

We use Bondy and Murty [7] for terminology and notation not defined here and consider only finite simple connected graphs. Throughout this paper, $\lceil x \rceil$ denotes the smallest integer greater than or equal to x, and $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.

2 The vertex-neighbor-scattering number of some specific graphs

Theorem 1. *Let P_n be a path of order n (≥ 3). Then*

$$VNS(P_n) = \begin{cases} 0, & \text{if } n = 3, 4; \\ 1, & \text{if } n \geq 5. \end{cases}$$

Proof. The cases $n = 3$ and 4 are trivial, so we assume $n \geq 5$.

It is clear that for any $S \subseteq V(P_n)$, we have $\omega(P_n/S) \leq |S| + 1$. Thus,

$$VNS(P_n) = \max_{S \subseteq V(P_n)} \{\omega(P_n/S) - |S|\} \leq |S| + 1 - |S| = 1.$$

On the other hand, there exists a vertex v in P_n such that $\omega(P_n/\{v\}) = 2$, so we have

$$VNS(P_n) = \max_{S \subseteq V(P_n)} \{\omega(P_n/S) - |S|\} \geq \omega(P_n/\{v\}) - |\{v\}| = 2 - 1 = 1.$$

Therefore, $VNS(P_n) = 1$. □

A *wheel*, denoted by $W_{1,n}$, is a graph obtained from a cycle C_n ($n \geq 4$) by adding a new vertex and edges joining it to all vertices of the cycle; the new vertex is called the *center* of the wheel.

Theorem 2. *Let $W_{1,n}$ be a wheel, where $n \geq 4$. Then*

$$VNS(W_{1,n}) = \begin{cases} -1, & \text{if } n = 6, 7; \\ 0, & \text{if } n = 4, 5 \text{ or } n \geq 8. \end{cases}$$

Proof. The cases $n = 4, 5, 6$ and 7 are trivial, so we assume $n \geq 8$.

If a non-center vertex is subverted from $W_{1,n}$, then the survival subgraph is a path of order $n - 3$ (≥ 5). If the center is subverted from $W_{1,n}$, then the survival subgraph is empty. By the definition of VNS and Theorem 1, we have $VNS(W_{1,n}) = \max\{VNS(P_{n-3}) - 1, 0 - 1\} = 0$. □

Theorem 3. *Let C_n be a cycle of order n (≥ 4). Then*

$$VNS(C_n) = \begin{cases} -1, & \text{if } n = 6, 7; \\ 0, & \text{if } n = 4, 5 \text{ or } n \geq 8. \end{cases}$$

Proof. The cases $n = 4, 5, 6$ and 7 are trivial, so we assume $n \geq 8$.

Note that for any vertex $v \in V(C_n)$, we have $|N[v]| = 3$, and $C_n/\{v\}$ is a path of order $n-3$ (≥ 5). Then, for any $S \subseteq V(C_n)$, we have $\omega(C_n/S) \leq |S|$ and

$$VNS(C_n) = \max_{S \subseteq V(C_n)} \{\omega(C_n/S) - |S|\} \leq |S| - |S| = 0.$$

On the other hand, choose two vertices u and v in C_n such that $d_{C_n}(u, v) \geq 4$. Then $\omega(C_n/\{u, v\}) = 2$ and $\omega(C_n/\{u, v\}) - |\{u, v\}| = 0$. Therefore, $VNS(C_n) = 0$. □

A *comet* $C_{t,r}$ is a graph obtained by identifying one end of a path P_t ($t \geq 2$) with the center of a star $S_{1,r}$ ($r \geq 2$). The center of $S_{1,r}$ is called the *center* of $C_{t,r}$.

Theorem 4. *Let $C_{t,r}$ be a comet, both t and r are at least 2. Then*

$$VNS(C_{t,r}) = \begin{cases} r-1, & \text{if } t = 2,3; \\ r, & \text{if } t \geq 4. \end{cases}$$

Proof. Suppose $V(P_t) = \{v_1, v_2, \ldots, v_t\}$, and v_1 is the center of $C_{t,r}$. Clearly, $C_{t,r}$ is 1-neighbor-connected. In other words, if S is a VNS-set of $C_{t,r}$, then $|S| \geq 1$. If $t = 2,3$, then $C_{t,r}/\{v_2\}$ is a graph composed of r isolated vertices. If $t \geq 4$, then $C_{t,r}/\{v_2\}$ is a graph composed of a path of order $t-3$ and r isolated vertices. So we have

$$VNS(C_{t,r}) \geq \begin{cases} r-1, & \text{if } t = 2,3; \\ r, & \text{if } t \geq 4. \end{cases}$$

If $|S| = 1$ and $S \neq \{v_2\}$, then $C_{t,r}/S$ is connected, or $\omega(C_{t,r}/S) \leq r$, i.e. $\omega(C_{t,r}/S) - |S| \leq r-1$. If $|S| > 1$, then it is easy to check that $\omega(C_{t,r}/S) - |S| < r-1$ (if $t = 2,3$) and $\omega(C_{t,r}/S) - |S| \leq r$ (if $t \geq 4$). Therefore,

$$VNS(C_{t,r}) = \begin{cases} r-1, & \text{if } t = 2,3; \\ r, & \text{if } t \geq 4. \end{cases}$$

□

Let G_1 and G_2 be two graphs. The *Cartesian product* of G_1 and G_2, denoted by $G_1 \times G_2$, is a simple graph, where $V(G_1 \times G_2) = V(G_1) \times V(G_2)$, and for $u_1, v_1 \in V(G_1)$, $u_2, v_2 \in V(G_2)$, $((u_1, u_2), (v_1, v_2)) \in E(G_1 \times G_2)$ if and only if $u_1 = v_1$ and $(u_2, v_2) \in V(G_2)$ or $u_2 = v_2$ and $(u_1, v_1) \in V(G_1)$.

Lemma 1. ([8]) *Let K_m and K_n be two complete graphs. Then $K_m \times K_n$ is r-neighbor-connected, where $r = min\{m-1, n-1\}$.*

Theorem 5. *Let K_m and K_n be two complete graphs. Then $VNS(K_m \times K_n) = 2 - min\{m, n\}$.*

Proof. Since $K_m \times K_n$ is $r = min\{m-1, n-1\}$-neighbor-connected, we have $VNS(K_m \times K_n) \leq 1 - r = 2 - min\{m, n\}$.

Denote $V(K_m) = \{u_1, u_2, \ldots, u_m\}$ and $V(K_n) = \{v_1, v_2, \ldots, v_n\}$. Assume $S \subset V(K_m \times K_n)$, $S = \{(u_{i_1}, v_{j_1}), (u_{i_2}, v_{j_2}), \ldots, (u_{i_s}, v_{j_s})\}$, and the number of different subscripts of u and v is p, q, respectively, where $1 \leq p, q \leq s$. From the structure of $K_m \times K_n$, it is not difficult to see that $K_m \times K_n/S = K_{m-p} \times K_{n-q}$. Let $M = \{(u_{i_1}, v_{i_1}), (u_{i_2}, v_{i_2}), \ldots, (u_{i_r}, v_{i_r})\}$, where $r = min\{m-1, n-1\}$. It is easy to check that M is a smallest cut-strategy of $K_m \times K_n$ and $\omega(K_m \times K_n/M) = 1$. Therefore, $\omega(K_m \times K_n/M) - |M| = 1 - r = 2 - min\{m, n\}$. □

3 Bounds for the vertex-neighbor-scattering number

In this section we give several lower and upper bounds for the vertex-neighbor-scattering number.

Theorem 6. *Let G be a connected noncomplete graph. Then $VNS(G) \geq -K(G)$.*

Proof. Assume X is a smallest cut-strategy of G. Then $|X| = K(G)$ and $\omega(G/X) \geq 0$. Therefore,

$$\begin{aligned} VNS(G) &= \max_{S \subseteq V(G)} \{\omega(G/S) - |S|\} \\ &\geq \omega(G/X) - |X| \\ &= -K(G). \end{aligned}$$

□

Theorem 7. *Let G be a connected noncomplete graph. Then $VNS(G) \geq -\sigma(G)$, where $\sigma(G)$ is the vertex dominating number of G.*

Proof. Let X be a smallest vertex dominating set of G. Then $|X| = \sigma(G)$ and $G/S = \phi$. So we have

$$\begin{aligned} VNS(G) &= \max_{S \subseteq V(G)} \{\omega(G/S) - |S|\} \\ &\geq \omega(G/X) - |X| \\ &= -\sigma(G) \end{aligned}$$

□

Remark 1. The lower bounds of VNS in Theorem 6 and Theorem 7 are tight. This can be shown by $W_{1,6}$ or $W_{1,7}$.

Definition 1. Let n be an integer at least 2. We construct a graph of order n, denoted by $\tilde{G}_n$, as follows:
(1) If $\lfloor\sqrt{n}\rfloor^2 \leq n < \lfloor\sqrt{n}\rfloor\lceil\sqrt{n}\rceil$, let $n = n_1 + n_2 + \cdots + n_{\lfloor\sqrt{n}\rfloor}$ such that $|n_i - n_j| \leq 1$ for $i \neq j$. Replace the vertices of the complete graph $K_{\lfloor\sqrt{n}\rfloor}$ by cliques $K_{n_1}, K_{n_2}, \ldots, K_{n_{\lfloor\sqrt{n}\rfloor}}$, respectively, then add edges such that:
(a) K_{n_i} and $K_{n_j} (i \neq j)$ are joined by an edge exactly.
(b) Each vertex in K_{n_i} is incident to, at most, one edge not entirely contained in K_{n_i}, $i = 1, 2, \ldots, \lfloor\sqrt{n}\rfloor$.
(2) If $\lfloor\sqrt{n}\rfloor\lceil\sqrt{n}\rceil \leq n < \lceil\sqrt{n}\rceil^2$, let $n = n_1 + n_2 + \cdots + n_{\lceil\sqrt{n}\rceil}$ such that $|n_i - n_j| \leq 1$ for $i \neq j$. Replace the vertices of the complete graph $K_{\lceil\sqrt{n}\rceil}$ by cliques $K_{n_1}, K_{n_2}, \ldots, K_{n_{\lceil\sqrt{n}\rceil}}$, respectively, then add edges as in (1).

For example, when $n = 18$, we decompose $18 = 4+4+5+5$. The graph $\tilde{G}_{18}$ is shown as the following figure.

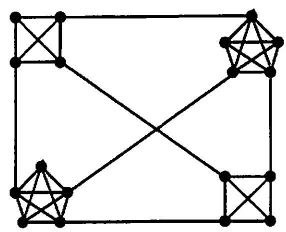

Figure 1: The graph $\tilde{G}_{18}$

Theorem 8. $VNS(\tilde{G}_n) = \begin{cases} 2 - \lfloor\sqrt{n}\rfloor, & \textit{if } \lfloor\sqrt{n}\rfloor^2 \le n < \lfloor\sqrt{n}\rfloor\lceil\sqrt{n}\rceil; \\ 2 - \lceil\sqrt{n}\rceil, & \textit{if } \lfloor\sqrt{n}\rfloor\lceil\sqrt{n}\rceil \le n < \lceil\sqrt{n}\rceil^2. \end{cases}$

Proof. Observe that deleting any $\lfloor\sqrt{n}\rfloor - 2$ (or $\lceil\sqrt{n}\rceil - 2$) neighborhoods in $\tilde{G}_n$ is equivalent to deleting the $\lfloor\sqrt{n}\rfloor - 2$ (or $\lceil\sqrt{n}\rceil - 2$) corresponding vertices in $K_{\lfloor\sqrt{n}\rfloor}$ (or $K_{\lceil\sqrt{n}\rceil}$). Since $K_{\lfloor\sqrt{n}\rfloor}$ is $(\lfloor\sqrt{n}\rfloor - 1)$-connected and $K_{\lceil\sqrt{n}\rceil}$ is $(\lceil\sqrt{n}\rceil - 1)$-connected, we have

$$K(\tilde{G}_n) = \begin{cases} \lfloor\sqrt{n}\rfloor - 1, & \text{if } \lfloor\sqrt{n}\rfloor^2 \le n < \lfloor\sqrt{n}\rfloor\lceil\sqrt{n}\rceil; \\ \lceil\sqrt{n}\rceil - 1, & \text{if } \lfloor\sqrt{n}\rfloor\lceil\sqrt{n}\rceil \le n < \lceil\sqrt{n}\rceil^2. \end{cases}$$

Since $\tilde{G}_n$ contains $\lfloor\sqrt{n}\rfloor$ (or $\lceil\sqrt{n}\rceil$) cliques, it is not difficult to know that, for any cut strategy S of $\tilde{G}_n$, $|S| \ge K(\tilde{G}_n)$ and $\omega(\tilde{G}_n/S) \le 1$. Then $VNS(\tilde{G}_n) \le 1 - K(\tilde{G}_n)$.

On the other hand, we can find a cut strategy S in $\tilde{G}_n$ such that $|S| = K(\tilde{G}_n)$ and $\omega(\tilde{G}_n/S) = 1$. Therefore,

$$VNS(\tilde{G}_n) = 1 - K(\tilde{G}_n) = \begin{cases} 2 - \lfloor\sqrt{n}\rfloor, & \text{if } \lfloor\sqrt{n}\rfloor^2 \le n < \lfloor\sqrt{n}\rfloor\lceil\sqrt{n}\rceil; \\ 2 - \lceil\sqrt{n}\rceil, & \text{if } \lfloor\sqrt{n}\rfloor\lceil\sqrt{n}\rceil \le n < \lceil\sqrt{n}\rceil^2. \end{cases}$$

The proof is complete. □

Gunther et al. [3, 8, 9] described a number of special families of k-neighbor-connected graphs. All of these are k-regular and k-neighbor-connected. In particular, Gunther [3] found a classification of all minimal k-regular, k-neighbor-connected graphs which contain k-cliques. These turn out to have order $k(k+1)$. If $n = k(k+1)$, then $\tilde{G}_n$ is a k-regular, k-neighbor-connected graph which contain $k+1$ k-cliques.

Gunther [3] proved that $K(G) \leq \delta(G)$ for any graph G, where $\delta(G)$ is the minimum vertex degree of G. Then for any graph G of order $k(k+1)$, we have $K(G) \leq k$. Otherwise, if $K(G) > k$, then $\delta(G) > k$ and $\sigma(G) < \frac{k(k+1)}{\delta(G)}$. But $\sigma(G) \geq K(G) > k$, a contradiction. Therefore, we have $VNS(G) \geq 1 - K(G) = 1 - k$ for any noncomplete graph G of order $k(k+1)$.

We conjecture this conclusion holds for any noncomplete graph G of order n, i.e., $VNS(G) \geq VNS(\tilde{G}_n) = 1 - VNS(\tilde{G}_n)$.

Conjecture. Let G be a noncomplete graph of order n. Then

$$VNS(G) \geq \begin{cases} 2 - \lfloor\sqrt{n}\rfloor, & \text{if } \lfloor\sqrt{n}\rfloor^2 \leq n < \lfloor\sqrt{n}\rfloor\lceil\sqrt{n}\rceil; \\ 2 - \lceil\sqrt{n}\rceil, & \text{if } \lfloor\sqrt{n}\rfloor\lceil\sqrt{n}\rceil \leq n < \lceil\sqrt{n}\rceil^2. \end{cases}$$

Remark 3. If the conjecture is true, then the lower bound is tight. This can be shown by $\tilde{G}_n$.

The following Theorem gives an upper bound for the vertex-neighbor-scattering number.

Theorem 9. *Let G be a connected noncomplete graph of order n. Then $VNS(G) \leq n - K(G)(\delta(G)+2)$, where $\delta(G)$ is the minimum vertex degree of G.*

Proof. For any VNS-set S of G, we have $|S| \geq K(G)$. Since G is connected, we know that for any $v \in V(G)$, $|N[v]| \geq \delta(G) + 1$. So G/S contains at most $n - K(G)(\delta(G) + 1)$ vertices, i.e., $\omega(G/S) \leq n - K(G)(\delta(G) + 1)$. Therefore,

$$\begin{aligned} VNS(G) &= \max_{S \subseteq V(G)} \{\omega(G/S) - |S|\} \\ &\leq n - K(G)(\delta(G)+1) - K(G) \\ &= n - K(G)(\delta(G)+2). \end{aligned}$$

□

Remark 4. The upper bound in Theorem 9 is tight. This can be achieved by the star graphs.

4 The vulnerability and the vertex-neighbor-scattering number of graphs

In this section, the notation vulnerability of a graph is considered under the neighbor sense, that is when deleting a vertex from a graph, all its adjacent vertices are deleted at same time.

The definition of the vertex-neighbor-scattering number shows that the parameter considers not only the amount of work done to damage the network but also how badly the network is damaged. Graphs with large VNS are more vulnerable. For example, when $n \geq 8$, $VNS(P_n) = 1$ and $VNS(C_n) = 0$. In fact, P_n is more vulnerable than C_n.

It is true that the vulnerability of a graph may be determined by the vertex-neighbor-connectivity. For a graph, the smaller its vertex-neighbor-connectivity is, the more vulnerable it is and vice-versa. But the property of being k-neighbor-connected is sensitive to the number of edges present in the graph. The complete graph K_n is highly connected, but is not even 1-neighbor-connected. It seems that if there are too many edges present in the graph, the neighbor-connectivity actually decreases. In other words, the graph is easier to attack. Unfortunately, it is not at all clear what constitutes too many edges. As pointed out in [8], the complicated relationship that holds between vertex-neighbor-connectivity and the number of edges present is manifested by the sequence of diagrams in the Figure 2. The first diagram shows C_{12}, which is 2-neighbor-connected. In the second diagram, we add an edge to C_{12}, and its vertex-neighbor-connectivity is now 1. If, however, we add the third edge, then we again re-established 2-neighbor-connectivity in the resulting graph.

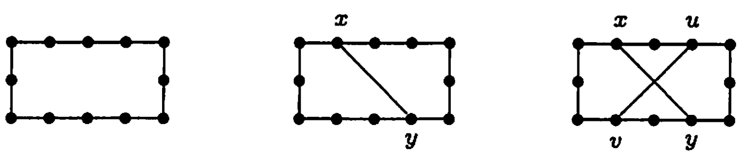

Figure 2: The VNC is sensitive to the number of edges present in a graph

One finds that the VNS value of a graph is closely related to its VNC. Generally, the smaller the VNC is, the larger the VNS is and vice-versa. But graphs with the same VNC may have different VNS. The following examples show that the VNS is independent to the VNC and the former is better than the latter in measuring the vulnerability of graphs in some situations.

Example 2. When $n \geq 8$, $K(K_3 \times K_4) = K(C_n) = 2$, but $VNS(K_3 \times K_4) = -1$, $VNS(C_n) = 0$.

Example 3. When $n \geq 5$ and $t \geq 4$, $K(C_{t,r}) = K(P_n) = 1$, but $VNS(C_{t,r}) = r$, $VNS(P_n) = 1$.

The vertex-neighbor-integrity, introduced by Cozzens and Wu [6], is defined as $VNI(G) = \min\{|X| + \tau(G/X) : X \subseteq V(G)\}$, where $\tau(G/X)$ stands for the maximum order of the components of G/X. The following examples show that the VNS is better than the VNI in measuring the vulnerability of graphs in some situations.

Example 4. $VNS(C_{4,6}) = 6$, $VNS(K_{5,5}) = 3$, but $VNI(C_{4,6}) = VNI(K_{5,5}) = 2$.

Example 5. $VNS(P_7) = 1$, $VNS(C_9) = 0$, $VNS(K_3 \times K_3) = -1$, but $VNI(P_7) = VNI(C_9) = VNI(K_3 \times K_3) = 3$.

As one can see, when $n \geq 4$, $VNS(W_{1,n}) = VNS(C_n)$. But $K(W_{1,n}) = 1, K(C_n) = 2$, and $VNI(W_{1,n}) = 1$, $VNI(C_n) = \lceil 2\sqrt{n} \rceil - 3$. This example means that VNS has its defects too. So using the above three parameters rather than only one or two is more desirable in measuring the vulnerability of a graph. An example is given in the following table. Among them, $\tilde{G}_{12}$ is the most stable and $S_{1,11}$ is the most vulnerable. As for C_{12} and $W_{1,11}$, we can say the former is more stable than the latter.

Table 1: The VNS, VNI and VNC of 7 graphs of order 12

	$\tilde{G}_{12}$	$K_3 \times K_4$	C_{12}	$W_{1,11}$	P_{12}	$C_{6,6}$	$S_{1,11}$
VNS	-2	-1	0	0	1	6	9
VNI	3	4	4	1	4	3	1
VNC	3	2	2	2	1	1	1

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