# STAIRCASE VISIBILITY AND KRASNOSEL'SKII-TYPE RESULTS FOR PLANAR COMPACT SETS

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ABSTRACT. Some Krasnosel'skii-type results previously established for a simply connected orthogonal polygon may be extended to a nonempty compact planar set $S$ having connected complement. In particular, if every two points of $S$ are visible via staircase paths from a common point of $S$, then $S$ is starshaped via staircase paths. For $n$ fixed, $n \geq 1$, if every two points of $S$ are visible via staircase $n$-paths from a common point of $S$, then $S$ is starshaped via staircase $(n+1)$-paths. In each case, the associated staircase kernel is orthogonally convex.

## 1. INTRODUCTION.

We begin with some definitions from [2]-[6]. Let $\lambda$ be a simple polygonal path in the plane whose edges $[v_{i-1}, v_i], 1 \leq i \leq n$, are parallel to the coordinate axes. Path $\lambda$ is a *staircase path* if and only if the associated vectors alternate in direction. That is, for an appropriate labeling, for $i$ odd the vectors $\overrightarrow{v_{i-1}v_i}$ have the same horizontal direction, and for $i$ even the vectors $\overrightarrow{v_{i-1}v_i}$ have the same vertical direction. Edge $[v_{i-1}, v_i]$ will be called north, south, east, or west according to the direction of vector $\overrightarrow{v_{i-1}v_i}$. Similarly, we use the terms north, south, east, west, northeast, northwest, southeast, southwest to describe the relative position of points. For $n \geq 1$, if the staircase path $\lambda$ is a union of at most $n$ edges, then $\lambda$ is called a *staircase n-path.* If the staircase path $\lambda$ is a union of exactly $n$ edges, then $n$ is the *length* of $\lambda$.

Let $S \subseteq \mathbb{R}^2$. For points $x$ and $y$ in set $S$, we say *$x$ sees $y$* ($x$ is *visible* from $y$) via staircase $n$-paths if and only if there is a staircase $n$-path in $S$ which contains both $x$ and $y$. Set $S$ is called *staircase n-convex* (orthogonally $n$-convex) provided for every $x, y$ in $S$, $x$ sees $y$ via staircase $n$-paths. Similarly, set $S$ is *starshaped via staircase n-paths* if and only if, for some point $p$ in $S, p$ sees each point of $S$ via staircase $n$-paths, and the set of all such points $p$ is

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the *staircase n-kernel* of $S$, denoted $Ker_n S$. Of course, parallel definitions hold for staircase paths.

Many results in convexity that involve the usual notion of visibility via straight line segments have analogues that employ the idea of visibility via staircase paths. A natural choice for a set in which to use staircase visibility is the orthogonal polygon, a connected union of finitely many planar boxes whose edges are parallel to the coordinate axes. (See [2]-[6].) However, recent work by Magazanik and Perles ([11], [12]) extends the use of staircase visibility from an orthogonal polygon to an arbitrary nonempty compact planar set. In this paper, we use a similar approach to extend to a more general setting some previously established Krasnosel'skii-type staircase results.

Throughout the paper, $cl\, S$ and $bdry\, S$ will denote the closure and boundary, respectively, for set $S$. If $\lambda$ is a simple path containing points $x$ and $y$, $\lambda(x,y)$ will represent the subpath of $\lambda$ from $x$ to $y$. We call $\lambda(x,y)$ an *$x-y$ path.* If $\lambda(x,y)$ is a staircase path, we call it an *$x-y$ staircase.* When $x$ and $y$ are distinct, $L(x,y)$ will represent their corresponding line. Readers may refer to Valentine [14], to Lay [10], to Danzer, Grünbaum, Klee [7], and to Eckhoff [8] for discussions on Helly and Krasnosel'skii-type theorems, visibility via straight line segments, and starshaped sets. Readers may refer to Nadler [13] for information on the Hausdorff metric.

## 2. The results

The first theorem is an analogue of an orthogonal polygon result in [4, Corollary 1].

**Theorem 1.** Let $S$ be a nonempty compact set in the plane, with $\mathbb{R}^2 \setminus S$ connected. If every two points of $S$ are visible via staircase paths from a common point of $S$, then $S$ is starshaped via staircase paths. Moreover, the associated staircase kernel of $S$ will be orthogonally convex. The number two is best possible.

*Proof.* The proof relies on the following preliminary result.

Proposition 1. For all $x$ in the set $S$ above, the associated visibility set $V_x = \{y : x \text{ sees } y \text{ via staircase paths}\}$ is closed.

*Proof.* Assume that $\{y_n\}$ is a sequence in $V_x$ and converging to $y$, to show that $y \in V_x$. Observe that if $x$ and $y$ lie on a horizontal or vertical line, then the associated sequence of $x - y_n$ staircase paths in $S$ will converge (in the Hausdorff metric) to $[x,y]$, and $[x,y] \subseteq S$, finishing the proof. Thus we will assume that $[x,y]$ is not horizontal, not vertical.

Clearly $y \in S$, so by hypothesis $x$ and $y$ are visible via staircase paths from a common point of $S$. For convenience, assume that this point is the origin $\theta$, and let $\lambda_x(x,\theta), \lambda_y(y,\theta)$ denote associated staircase paths in $S$ from $x$ to $\theta$, $y$ to $\theta$, respectively.

We consider cases according to possible locations for $x$ and $y$.

Case 1. Assume that points $x$ and $y$ are in the same quadrant, say quadrant 1.

Case 1a. Assume that $x$ is strictly northeast of $y$. Then the $x-\theta$ staircase $\lambda_x(x,\theta)$ either contains a point north of $y$ or contains a point east of $y$. Without loss of generality, assume that the former situation occurs, and let $z$ be such a point.

We assert that $[z,y] \subseteq S$. Suppose on the contrary that $(z,y)\backslash S \neq \phi$, to obtain a contradiction. (See Figure 1.) Choose a point $p$ in $\mathbb{R}^2\backslash S$ and strictly northeast of $x$. Since $S$ is compact, we may select $p$ so that the horizontal line $H$ at $p$ lies in $\mathbb{R}^2\backslash S$. Since $\mathbb{R}^2\backslash S$ is open and connected, it is polygonally connected, so there is a simple polygonal path $\mu = \mu(a,p)$ in $\mathbb{R}^2\backslash S$ from some $a$ in $(z,y)\backslash S$ to $p$. Certainly there is a last point of $\mu$ on $[z,y]$, so we may assume that $a$ is the only point of $\mu \cap [z,y]$. Letting $T$ denote the closed, simply connected region bounded by $[z,y] \cup \lambda_y(y,\theta) \cup \lambda_x(x,\theta)$, since $\mu$ is disjoint from $\lambda_y(y,\theta) \cup \lambda_x(x,\theta)$, certainly $\mu \cap T = \{a\}$.

Let $L = L(z,y)$. Then $x$ lies strictly east of $L$, say in the open halfplane $L_1$ determined by $L$. Also, by comments above, the first segment of $\mu$ lies in $L_1 \cup \{a\}$. Similarly, point $x$ lies strictly south of $H$, say in the open halfplane $H_1$ determined by $H$. Since $H \subseteq \mathbb{R}^2\backslash S$, we may assume that $p$ is the only point of $\mu \cap H$. Hence $\mu\backslash\{p\} \subseteq H_1$ as well.

Moreover, point $x$ lies in a component $C$ of $H_1 \cap L_1\backslash\mu$. Clearly *bdry* $C$ lies in $H \cup L \cup \mu$ (and possibly just in $L \cup \mu$). Of course, $y \notin C$. In fact, since the first segment of $\mu$ lies in $L_1 \cup \{a\}, \mu \cap [z,y] = \{a\}$, and $\mu \cap \lambda_x(x,z) = \phi$, by examining possible intersections of $\mu$ with $L$, it is easy to show that $[y,a) \cap cl\, C = \phi$, also. We will use this to obtain a contradiction.

Since $\{y_n\}$ converges to $y$, for $n$ sufficiently large, $y_n \in \mathbb{R}^2\backslash cl\, C$. For convenience, assume that this is true for all $n$. Similarly, by passing to a subsequence if necessary, we may assume that either $\{y_n\} \subseteq L_1, \{y_n\} \subseteq L$, or $\{y_n\} \subseteq L_2$ (where $L_2$ is the opposite open halfplane determined by $L$). We consider each possibility.

If $\{y_n\} \subseteq L_1$, choose any $n$ and any $y_n - x$ staircase path $\delta_n$ in $S$. Then $\delta_n \subseteq L_1 \cap H_1$. Since $\delta_n$ passes from $y_n \in \mathbb{R}^2\backslash cl\, C$ to $x \in C$, $\delta_n$ contains a boundary point of $C$ in $L_1 \cap H_1$, hence a point of $\mu$. However, $\delta_n \subseteq S$ while $\mu \subseteq \mathbb{R}^2\backslash S$, so $\delta_n \cap \mu = \phi$. We have a contradiction, and this situation cannot occur.

If $\{y_n\} \subseteq L$, choose $n$ sufficiently large that $[y_n, y] \subseteq \mathbb{R}^2 \backslash cl\, C$. Again let $\delta_n$ denote any $y_n - x$ staircase path in $S$. Observe that $\delta_n \cap L$ either is a degenerate segment containing only $y_n$ or is a nondegenerate north segment with endpoint $y_n$. Either way, $\delta_n \cap L$ is strictly south of $a$ and lies in $[y_n, y] \cup [y, a) \subseteq \mathbb{R}^2 \setminus cl\, C$. Remaining points of $\delta_n$ lie in $L_1 \cap H_1$. Again $\delta_n$ passes from $\mathbb{R}^2 \setminus cl\, C$ to $C$ at a boundary point of $C$ in $L_1 \cap H_1$. That is, $\delta_n$ meets $\mu$. However, this is impossible, so the second situation cannot occur either.

Finally, if $\{y_n\} \subseteq L_2$, for each $n$ choose an $x - y_n$ staircase path $\delta_n$ (ordered from $x$ to $y_n$). Certainly $\delta_n \cap L \neq \phi$ for each $n$, so we may let $w_n$ denote the first point of $\delta_n$ on $L$. By the argument above, $\{w_n\}$ cannot converge to $y$, so we may assume that $\{w_n\}$ is bounded away from $y$ and lies on one of the rays emanating from $y$ in $L$. Since $\{y_n\}$ converges to $y$, it is clear that $\{w_n\}$ cannot be south of $y$ on $L$, so $\{w_n\}$ must be north of $y$. If any $w_n$ were on $(a, y] \subseteq \mathbb{R}^2 \setminus cl\, C$, again a previous argument would yield a contradiction. Therefore, $\{w_n\}$ must be north of $a$ on $L$. However, then a subsequence of $\{\delta_n(w_n, y_n)\}$ will converge (in the Hausdorff metric) to a segment containing $[a, y]$, forcing $a$ to lie in $S$. Again we have a contradiction, and the third situation cannot occur.

Since none of these three situations can occur, our original supposition must be false. That is, $[z, y] \subseteq S$, establishing the assertion. We conclude that $S$ contains the $x - y$ staircase $\lambda_x(x, z) \cup [z, y]$, and $y \in V_x$, finishing the argument for Case 1a.

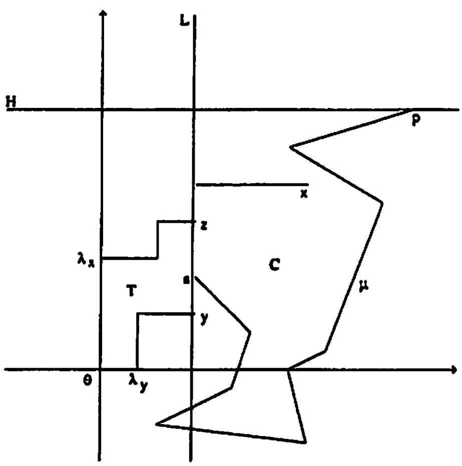


Figure 1

Case 1b. Assume that $y$ is strictly northeast of $x$. With the roles of $x$ and $y$ reversed, repeat the argument in Case 1a to obtain point $z$, point $a \in [z, x] \setminus S$ line $L = L(x, z)$, path $\mu(a, p)$, and set $C$, with $y \in C$ and

$[x, a) \subseteq \mathbb{R}^2 \setminus cl\, C$. For $n$ sufficiently large, $y_n \epsilon C$ and $y_n$ is strictly northwest of $x$. For such an $n$ and for any $x - y_n$ staircase path $\delta_n$ in $S$, an argument like the one in Case 1a (the second situation) shows that $\delta_n \cap L \subseteq \mathbb{R}^2 \setminus cl\, C$ and $\delta_n \cap \mu \neq \phi$, impossible. Thus $[z, x] \subseteq S$, producing the $y - x$ staircase $\lambda_y(y, z) \cup [z, x] \subseteq S$. This completes Case 1b.

Case 1c. Assume that one of the points $x, y$ is strictly southeast of the other. Without loss of generality, assume that $x$ is strictly southeast of $y$. Select point $z$ strictly south of $y$ and strictly west of $x$. We will show that the staircase 2-path $[y, z] \cup [z, x]$ lies in $S$.

To show that $[y, z] \subseteq S$, suppose on the contrary that $(y, z]\backslash S \neq \phi$ to obtain a contradiction. (See Figure 2.) As in Case 1a, select point $p$ in $\mathbb{R}^2 \backslash S$ so that $p$ is strictly northeast of both $x$ and $y$ and so that the associated horizontal line $H$ lies in $\mathbb{R}^2\backslash S$. Let $\mu(a, p)$ denote a simple polygonal path in $\mathbb{R}^2\backslash S$ from $a \epsilon (y, z]\backslash S$ to $p$, with $\mu \cap [y, z] = \{a\}$ and $\mu \cap H = \{p\}$. Label open halfplanes determined by $H$ and $L = L(y, z)$ so that $x \epsilon H_1 \cap L_1$. For $C$ the component of $H_1 \cap L_1 \backslash \mu$ which contains $x, [y, a) \cap clC = \phi$. We may assume that $\{y_n\} \cap clC = \phi$ as well. Considering possible locations for $\{y_n\}$, arguments like those in Case 1a (situations 1, 2, 3) show that our original supposition is false and $[y, z] \subseteq S$.

A simplified version of the same argument (much like the argument in Case 1b) shows that $[z, x] \subseteq S$. We conclude that $[y, z] \cup [z, x] \subseteq S$, finishing Case 1c and completing Case 1.

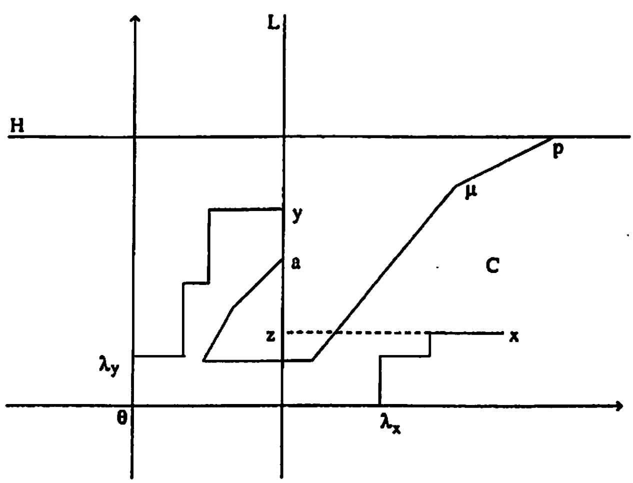


Figure 2

Case 2. Assume that points $x$ and $y$ are not in the same quadrant. If $x$ and $y$ are in opposing quadrants, then $\lambda_x(x, \theta) \cup \lambda_y(\theta, y)$ will be an $x - y$ staircase path in $S$, finishing the argument. Hence we may assume that $x$ and $y$ are in consecutive quadrants, say in quadrants 4 and 1, respectively.

In case $x$ is strictly southeast of $y$, then $\lambda_x(x, \theta)$ contains a point $z$ strictly south of $y$, and an argument like the one in Case1a shows that $[z, y] \subseteq S$. Hence $S$ contains the $x - y$ staircase $\lambda_x(x, z) \cup [z, y]$.

If $x$ is strictly southwest of $y$, then $\lambda_y(y, \theta)$ contains a point $z$ strictly north of $x$. An argument like the one in Case 1b shows that $[z, x] \subseteq S$, and $S$ contains the $y - x$ staircase $\lambda_y(y, z) \cup [z, x]$.

This finishes Case 2 and completes the proof of Proposition 1.

Now we are ready to prove Theorem 1. Using Proposition 1, the $V_x$ sets are closed. hence compact. By arguments like those in [3, Theorem 1], each set $V_x$ is simply connected and every two of the $V_x$ sets have a path connected intersection. Further, using an argument like the one in [4, Theorem 1], every three of the $V_x$ sets meet. Hence we may use a version of Molnár's theorem by Karimov, Repovš, and Željko (see [9, Theorem 1.2] and concluding remarks of this paper) to conclude that every finite subfamily of $\{V_x : x \text{ in } S\}$ has a nonempty intersection. Since the $V_x$ sets are compact, this implies that $\cap\{V_x : x \text{ in } S\} \neq \phi$. For $z$ in this intersection, $z$ sees every point of $S$ via staircase paths. We conclude that $S$ is starshaped via staircase paths, establishing the main result in the theorem.

Moreover, it is easy to show that the staircase kernel of $S$, $Ker\, S$, is orthogonally convex. Let $x, y$ belong to $Ker\, S$ and let $\lambda$ be any $x - y$ staircase in $S$, to show that $\lambda \subseteq Ker\, S$. For $z$ in $S, x$ and $y$ see $z$ via staircase paths $\delta_x(x, z)$ and $\delta_y(y, z)$, respectively. By [6, Lemma 2], the simply connected region $T$ bounded by $\lambda \cup \delta_x \cup \delta_y$ is orthogonally convex, and from our hypothesis, $T \subseteq S$. Hence $z$ sees every point of $\lambda$ via staircase paths in $S$, and $\lambda \subseteq V_z$. Since this is true for every $z$ in $S, \lambda \subseteq Ker\, S$, and $Ker\, S$ is orthogonally convex.

Finally, it is clear that the number two in the theorem is best possible, and the proof is complete.

An example in [2, Example 2] illustrates that Theorem 1 fails without the requirement that $\mathbb{R}^2 \backslash S$ be connected. Further, [5, Example 2] shows that no Krasnosel'skii number exists without this condition.

The following easy example shows that a compact connected set $S$ with connected complement need not have closed visibility sets $V_x$.

Example 1. For $n \geq 1$, let $\lambda_n$ denote the east-north staircase path with vertices $\left(\frac{k}{k+1}, \frac{k}{k+1}\right)$ and $\left(\frac{k+1}{k+2}, \frac{k}{k+1}\right), 0 \leq k \leq n$, and let $S = cl(\cup\{\lambda_n : n \geq 1\})$. (See Figure 3.) Letting $\theta$ represent the origin, the associated staircase visibility set $V_\theta$ contains every staircase path $\lambda_n$ yet fails to contain the limit point (1,1). Hence $V_\theta$ is not closed

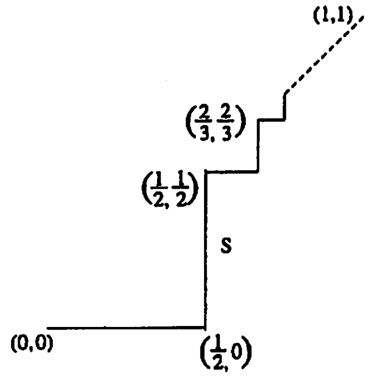


Figure 3

Of course, even when set $S$ satisfies the full hypothesis of Theorem 1, there need not be a ceiling on the lengths of the associated staircase paths. That is, $S$ need not be starshaped via staircase $n$-paths for any $n$. (See [11, Remark 2] for an example.) However, if we restrict the lengths of the staircase paths, we obtain the following analogue of [2, Theorem 1].

**Theorem 2.** Let $S$ be a nonempty compact set in the plane, with $\mathbb{R}^2 \backslash S$ connected. Let $n$ be fixed, $n \geq 1$. If every two points of $S$ are visible via staircase $n$-paths from a common point of $S$, then $S$ is starshaped via staircase $(n+1)$-paths. Moreover, $Ker_{n+1}\, S$ is staircase $(n+1)$-convex. The number two is best possible and the number $n+1$ is best possible for $n \geq 2$.

*Proof.* As in [2, Theorem 1], for each point $x$ in $S$, we define the associated visibility set $W_x = \{y : x \text{ sees } y \text{ via staircase } (n+1) - \text{paths in } S\}$. A standard convergence argument shows that $W_x$ is closed. The arguments in [2, Theorems 1 and 2] finish the proof.

## 3. Concluding remarks.

A very interesting recent paper by Karimov, Repovš, and Željko [9] has revealed that results by the writer in [1], as well as the statement of the classical Molnár theorem in [7], need some additional hypotheses.

Arguments in [1] require that the simply connected sets in question have connected complements. Molnár's theorem, used in [1], requires a stronger hypothesis as well, as in the following version from [9]: Let $\mathcal{F}$ be a finite family of simply connected compact or open sets in the plane. If every two members of $\mathcal{F}$ have a path-connected intersection and every three members of $\mathcal{F}$ have a nonempty intersection, then $\cap\{F : F \text{ in } \mathcal{F}\} \neq \phi$. An example in [9] shows that the word *path* cannot be omitted. (Of course, when members of $\mathcal{F}$ are compact, we may delete the word *finite*.)

These comments suggest the following question: In the version of Molnár's theorem above, if we require members of $\mathcal{F}$ to have connected complements, may we delete the word *path*?

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