A characterization of the planes secant to a non–singular quadric in PG(4,q)

Dedicated to Professor Franco Eugeni on the occasion of his 70th birthday

Stefano Innamorati
Mauro Zannetti
Department of Electrical and Information Engineering
University of L'Aquila
Via G. Gronchi, 18
I–67100 L'Aquila Italy
e-mail: stefano.innamorati@univaq.it
mauro.zannetti@univaq.it

Abstract. In this article, the planes meeting a non-singular quadric of PG(4,q) in a conic are characterized by their intersection properties with points, lines and 3–spaces.

Keywords: Projective space, two character set, non-singular quadric.

Mathematics Subject Classification (2000): 51E20.

1. Introduction and Motivations

Let PG(r,q) denote the projective space of dimension r and order q, where $q=p^h$ is a prime power. Quadrics in PG(4,q) are very interesting objects with many combinatorial properties. One is that planes can only meet a quadric in few ways. So we may consider a family of planes that all meet a non-singular quadric in the same way. This family has remarkable properties. An important question is whether we may use these properties to characterize them. In order to give a better picture of the current interest in this type of characterization, the reader is referred to [1], [2], [3], [4], [5] and [6]. The following result enter into this scheme of things.

Result ([2] D. K. Butler, 2007).- Let K be a set of planes in PG(4,q), such that:

(I) Every points lies on q^4 or q^4+q^2 planes of K;
(II) Every line lies on 0 or q^2 planes of K;

(III) Every 3–space contains at least one plane of K.

Then K is the set of planes meeting a non–singular quadric of PG(4,q) in a conic.

Let K denote a k–set of planes, i.e. a set of k planes of PG(4,q). We call K an *intersection* set if every 3–space, considered as the set of its planes, meets K in at least one plane. We recall that a *star* of planes is the set of all the planes through a same line. Moreover a *hyperstar* of planes is the set of all the planes through a same point. We recall that the *characters* of K, with respect to star (hyperstar) of planes, are the numbers $t_i=t_i(K)$ of stars (hyperstars) meeting K in exactly i planes, $0\le i\le q^2+q+1$ ($0\le i\le q^4+q^3+2q^2+q+1$). Denote by m and n two non-negative integers with $0\le m<n\le q^2+q+1$ ($0\le m<n\le q^4+q^3+2q^2+q+1$). A set K is said to be *of type* (m,n) with respect to star (hyperstar) of planes, if any star (hyperstar) contains either m or n planes of K, and such stars (hyperstars) do exist, see [11] and [12]. A set of type (m,n) is also called *two character set*. If m and n are polynomial in q we call K a *two polynomial character set.*

In this paper we give a characterization of the set of planes of PG(4,q) meeting a non–singular quadric of PG(4,q) in a conic as a polynomial two character set with respect to either stars or hyperstars of planes which intersects every 3–space, considered as the set of its planes. In particular, we prove the following

Theorem.- *In* PG(4,q), *an intersection* (q^6+q^4)*–set of planes having two polynomial character respect to either stars or hyperstars, and exactly* $\dfrac{q^{12}+q^{10}-q^9+q^5}{2}$ *pairs of planes which meet in exactly one point, is the set of planes meeting a non–singular quadric in a conic.*

2. The proof of the Theorem

We first consider the case where our set has two polynomial characters with respect to stars.

Let $m=m(q)$ and $n=n(q)$ denote two non-negative polynomials in q with $0\le m<n\le q^2+q+1$. Suppose that K is a k–set of planes of type (m,n) respect to stars in PG(4,q). By counting in double way the total number of stars, of incident planes–stars pairs (α,S) with $\alpha\in K\cap S$, and triples (α,β,S) with $\alpha,\beta\in K\cap S$, we have what are referred to as the *standard equations* on the integers $t_m=t_m(K)$ and $t_n=t_n(K)$, see [10],

$$(2.1)\quad \begin{cases} t_m + t_n = q^6+q^5+2q^4+2q^3+2q^2+q+1 \\ mt_m + nt_n = k(q^2+q+1) \\ m(m-1)t_m + n(n-1)t_n = k(k-1)-2\tau \end{cases},$$

where τ denotes the number of plane–pairs of K which meet in exactly one point. Thus, a two character set with respect to stars of planes depends by four parameters k, τ, m and n and a complete classification seems to be extremely difficult, see [6], [7], [8], [9], [13] and [14].

For $k=(q^6+q^4)$ and $\tau=\dfrac{q^{12}+q^{10}-q^9+q^5}{2}$ the system of equations (2.1) becomes

$$(2.2)\quad \begin{cases} t_m + t_n = q^6+q^5+2q^4+2q^3+2q^2+q+1 \\ mt_m + nt_n = (q^6+q^4)(q^2+q+1) \\ m(m-1)t_m + n(n-1)t_n = q^{10}+q^9+q^8-q^6-q^5-q^4 \end{cases}.$$

From the first two equations of (2.2), we get

$$(2.3)\quad \begin{cases} t_m = \dfrac{n(q^6+q^5+2q^4+2q^3+2q^2+q+1)}{n-m} \\ \quad -\dfrac{(q^8+q^7+2q^6+q^5+q^4)}{n-m} \\ t_n = \dfrac{(q^8+q^7+2q^6+q^5+q^4)}{n-m} \\ \quad -\dfrac{m(q^6+q^5+2q^4+2q^3+2q^2+q+1)}{n-m} \end{cases}.$$

Since $t_n>0$, by the second equation of (2.3) we have that

$$(q^8+q^7+2q^6+q^5+q^4)-m(q^6+q^5+2q^4+2q^3+2q^2+q+1)>0$$

And so,

$$m<q^2-\frac{q^5+q^4+q^3+q^2}{q^6+q^5+2q^4+2q^3+2q^2+q+1}.$$

Therefore, $0\le m\le q^2-1$.

By substituting (2.3) into the third equation of (2.2) we get

$(n+m)(q^8+q^7+2q^6+q^5+q^4)-mn(q^6+q^5+2q^4+2q^3+2q^2+q+1)$
$=q^{10}+q^9+2q^8+q^7+q^6$,

from which we get

$$(2.4)\quad n=\frac{(q^{10}+q^9+2q^8+q^7+q^6)+ \\ -m(q^8+q^7+2q^6+q^5+q^4)}{(q^8+q^7+2q^6+q^5+q^4)+ \\ -m(q^6+q^5+2q^4+2q^3+2q^2+q+1)}.$$

Since $0\le m\le q^2-1$, we can write m as a polynomial function of degree 2, $m=\alpha q^2+\beta q+\gamma$.

By substituting m into (2.4) we get

$$n=\frac{q^4\left[(\alpha-1)q^4+(\alpha+\beta-1)q^3+(\alpha+\beta+\gamma-1)q^2+(\beta+\gamma)q+\gamma\right]}{(\alpha-1)q^6+(\alpha+\beta-1)q^5+(\alpha+\beta+\gamma-1)q^4+(\alpha+\beta+\gamma)q^3+(\alpha+\beta+\gamma)q^2+(\beta+\gamma)q+\gamma}.$$

Putting $N(q)=q^4\left[(\alpha-1)q^4+(\alpha+\beta-1)q^3+(\alpha+\beta+\gamma-1)q^2+(\beta+\gamma)q+\gamma\right]$ and

$D(q)=(\alpha-1)q^6+(\alpha+\beta-1)q^5+(\alpha+\beta+\gamma-1)q^4+(\alpha+\beta+\gamma)q^3$
$+(\alpha+\beta+\gamma)q^2+(\beta+\gamma)q+\gamma$, we have that $n=\frac{N(q)}{D(q)}$.

As n is an integer, then the remainder $R(q)$ in the division of $N(q)$ by $D(q)$ must be zero for any q.

Firstly, suppose $\alpha=1$.

Then $m=q^2+\beta q+\gamma$ and

$$n=\frac{q^4\left[\beta q^3+(\beta+\gamma)q^2+(\beta+\gamma)q+\gamma\right]}{\beta q^5+(\beta+\gamma)q^4+(\beta+\gamma+1)q^3+(\beta+\gamma+1)q^2+(\beta+\gamma)q+\gamma}.$$

Putting $N(q)=q^4\left[\beta q^3+(\beta+\gamma)q^2+(\beta+\gamma)q+\gamma\right]$ and

$D(q)=\beta q^5+(\beta+\gamma)q^4+(\beta+\gamma+1)q^3+(\beta+\gamma+1)q^2+(\beta+\gamma)q+\gamma$, we have that $n=\frac{N(q)}{D(q)}$.

As n is an integer, then the remainder $R(q)$ in the division of $N(q)$ by $D(q)$ must be zero for any q.

If $\beta\neq 0$ the remainder $R(q)$ in the division of $N(q)$ by $D(q)$ is:

$$R(q)=\frac{(\gamma-\beta^2)}{\beta}q^4-\frac{\beta^2+\beta(\gamma-1)-\gamma-1}{\beta}q^3-\frac{\beta(\gamma-1)-\gamma-1}{\beta}q^2+\frac{(\beta+\gamma)}{\beta}q+\frac{\gamma}{\beta}.$$

Since $R(q)$ must be zero for any q, we need that the coefficients of the polynomial $R(q)$ must be zero for any q. We obtain a system of five equations and two variables β and γ which has no solution.

If $\alpha=1$ and $\beta=0$, then $m=q^2+\gamma$ and $n=\frac{\gamma q^4(q^2+q+1)}{\gamma q^4+(\gamma+1)q^3+(\gamma+1)q^2+\gamma q+\gamma}$.

Putting $N(q)=\gamma q^4(q^2+q+1)$ and $D(q)=\gamma q^4+(\gamma+1)q^3+(\gamma+1)q^2+\gamma q+\gamma$ we have that $n=\frac{N(q)}{D(q)}$.

If $\gamma\neq 0$ the remainder $R(q)$ in the division of $N(q)$ by $D(q)$ is:

$$R(q)=-\frac{\gamma^3-\gamma^2+1}{\gamma^2}q^3-\frac{\gamma^3-\gamma^2+\gamma+1}{\gamma^2}q^2+\frac{\gamma-1}{\gamma}q-\frac{1}{\gamma}.$$

Since $R(q)$ must be zero for any q, the coefficients of the polynomial $R(q)$ must be zero for any q.

We obtain a system of four equations in the variable γ which has no solution.
If $\alpha=1$, $\beta=0$ and $\gamma=0$, then $m=q^2$ and $n=0$, a contradiction because $m<n$.
Now assume $\alpha\neq 1$.
Putting $N(q)=q^4\left[(\alpha-1)q^4+(\alpha+\beta-1)q^3+(\alpha+\beta+\gamma-1)q^2+(\beta+\gamma)q+\gamma\right]$ and

$D(q)=(\alpha-1)q^6+(\alpha+\beta-1)q^5+(\alpha+\beta+\gamma-1)q^4+(\alpha+\beta+\gamma)q^3$
$+(\alpha+\beta+\gamma)q^2+(\beta+\gamma)q+\gamma$, we have that $n=\dfrac{N(q)}{D(q)}$.

As n is an integer, then the remainder $R(q)$ in the division of $N(q)$ by $D(q)$ must be zero for any q.
The remainder $R(q)$ in the division of $N(q)$ by $D(q)$ is $R(q)=-\alpha q^5-(\alpha+\beta)q^4-(\beta+\gamma)q^3-\gamma q^2$.
Since $R(q)$ must be zero for any q, the coefficients of the polynomial $R(q)$ must be zero for any q.
We obtain a system of four linear equations in three variables α, β and γ which has one solution $(\alpha,\beta,\gamma)=(0,0,0)$.
Therefore, $m=0$ and $n=q^2$. Thus K is a (q^6+q^4)–set of type $(0,q^2)$ with respect to stars of planes in PG(4,q).
We now consider the case where our set has two polynomial characters with respect to hyperstars.
Let $a=a(q)$ and $b=b(q)$ denote two non-negative polynomials in q with $0\leq a<b\leq q^4+q^3+2q^2+q+1$. Suppose that K is a k–set of type (a,b) with respect to hypestars in PG(4,q). By counting in double way the total number of hyperstars and of incident planes–hyperstars pairs (α,h) with $\alpha\in K\cap h$, we have what are referred to as the *standard equations* on the integers $t_a=t_a(K)$ and $t_b=t_b(K)$, see [10],

$$(2.5)\qquad \begin{cases} t_a+t_b=q^4+q^3+q^2+q+1 \\ at_a+bt_b=k(q^2+q+1) \end{cases}.$$

In order to make the proof clearer, we call *a–point* (resp. *b–point*) the centre of a hyperstar which intersects K in a (resp. b) planes. Since K is a set of type $(0, q^2)$ with respect to stars of planes, we call *0–line* (resp. *q^2–line*) the centre of a star which intersects K in 0 (resp. q^2) planes. Consider a point P. We show that the number $x=x(P)$ of q^2–lines through P does not depend from P, but only from the parameters a and b. Indeed, by counting in double way the number of pairs (r,α) where $P\in r\subset\alpha$, r is a q^2–line and α is a plane belonging to K, we get
$q^2x=(q+1)a$ if P is an a–point,
$q^2x=(q+1)b$ if P is a b–point.
Therefore, the number of q^2–lines through one a–point (resp. b–point) is $\dfrac{(q+1)}{q^2}a$ (resp. $\dfrac{(q+1)}{q^2}b$).

In order to have a third equation on the integers $t_a=t_a(K)$ and $t_b=t_b(K)$, independent from the two of (2.5), we count the number τ of plane–pairs of K which meet in exactly one point. We get

(2.6) $$\left[\binom{a}{2}-\frac{q+1}{q^2}a\binom{q^2}{2}\right]t_a+\left[\binom{b}{2}-\frac{q+1}{q^2}b\binom{q^2}{2}\right]t_b=\tau,$$ substituting $\tau=\dfrac{q^{12}+q^{10}-q^9+q^5}{2}$ and simplifying yields:

(2.7) $a^2t_a+b^2t_b=q^{12}+q^{11}+3q^{10}+q^9+2q^8$.

For $k=(q^6+q^4)$ the system of equations (2.5) becomes

(2.8) $$\begin{cases} t_a+t_b=q^4+q^3+q^2+q+1 \\ at_a+bt_b=(q^6+q^4)(q^2+q+1) \end{cases}.$$

From the two equations of (2.8), we get

(2.9) $$\begin{cases} t_a=\dfrac{b(q^4+q^3+q^2+q+1)-(q^8+q^7+2q^6+q^5+q^4)}{b-a} \\ t_b=\dfrac{q^8+q^7+2q^6+q^5+q^4-a(q^4+q^3+q^2+q+1)}{b-a} \end{cases}.$$

Since $t_b>0$, by the second equation of (2.9) we have that

$$(q^8+q^7+2q^6+q^5+q^4)-a(q^4+q^3+q^2+q+1)>0$$

and so,

$$a<q^4+q^2-q+\frac{q}{q^4+q^3+q^2+q+1}.$$

Therefore, $0\le a\le q^4+q^2-q$.

By substituting (2.9) into the equation (2.7) we get

$(b+a)(q^8+q^7+2q^6+q^5+q^4)-ab(q^4+q^3+q^2+q+1)=q^{12}+q^{11}+3q^{10}+q^9+2q^8$,

from which we get

(2.10) $$b=\frac{q^{12}+q^{11}+3q^{10}+q^9+2q^8-a(q^8+q^7+2q^6+q^5+q^4)}{q^8+q^7+2q^6+q^5+q^4-a(q^4+q^3+q^2+q+1)}.$$

Since $0\le a\le q^4+q^2-q$, we can write a as a polynomial function of degree 4, $a=\alpha q^4+\beta q^3+\gamma q^2+\delta q+\varepsilon$.

By substituting a into (2.10) we get

$$b=\left[\frac{(\alpha-1)q^8+(\alpha+\beta-1)q^7+(2\alpha+\beta+\gamma-3)q^6+(\alpha+2\beta+\gamma+\delta-1)q^5+(\alpha+\beta+2\gamma+\delta+\varepsilon-2)q^4+(\beta+\gamma+2\delta+\varepsilon)q^3+(\gamma+\delta+2\varepsilon)q^2+(\delta+\varepsilon)q+\varepsilon}{(\alpha-1)q^8+(\alpha+\beta-1)q^7+(\alpha+\beta+\gamma-2)q^6+(\alpha+\beta+\gamma+\delta-1)q^5+(\alpha+\beta+\gamma+\delta+\varepsilon-1)q^4+(\beta+\gamma+\delta+\varepsilon)q^3+(\gamma+\delta+\varepsilon)q^2+(\delta+\varepsilon)q+\varepsilon}\right]q^4.$$

Putting

$$N(q)=\left[(\alpha-1)q^8+(\alpha+\beta-1)q^7+(2\alpha+\beta+\gamma-3)q^6\right.$$
$$+(\alpha+2\beta+\gamma+\delta-1)q^5+(\alpha+\beta+2\gamma+\delta+\varepsilon-2)q^4$$
$$\left.+(\beta+\gamma+2\delta+\varepsilon)q^3+(\gamma+\delta+2\varepsilon)q^2+(\delta+\varepsilon)q+\varepsilon\right]q^4$$

and

$$D(q)=(\alpha-1)q^8+(\alpha+\beta-1)q^7+(\alpha+\beta+\gamma-2)q^6$$
$$+(\alpha+\beta+\gamma+\delta-1)q^5+(\alpha+\beta+\gamma+\delta+\varepsilon-1)q^4$$
$$+(\beta+\gamma+\delta+\varepsilon)q^3+(\gamma+\delta+\varepsilon)q^2+(\delta+\varepsilon)q+\varepsilon.$$

We have that $b=\dfrac{N(q)}{D(q)}$.

As b is an integer, then the remainder $R(q)$ in the division of $N(q)$ by $D(q)$ must be zero for any q.

Firstly, suppose $\alpha\neq 1$, then the remainder $R(q)$ in the division of $N(q)$ by $D(q)$ is:
$R(q)=-q^7+(\alpha-1)q^5+\beta q^4+\gamma q^3+\delta q^2+\varepsilon q$, which is not zero.

Therefore $\alpha=1$, then $a=q^4+\beta q^3+\gamma q^2+\delta q+\varepsilon$ and

$$b=\frac{q^4(q^2+1)\left[\beta q^5+(\beta+\gamma-1)q^4+(\beta+\gamma+\delta)q^3+(\gamma+\delta+\varepsilon)q^2+(\delta+\varepsilon)q+\varepsilon\right]}{\beta q^7+(\beta+\gamma-1)q^6+(\beta+\gamma+\delta)q^5+(\beta+\gamma+\delta+\varepsilon)q^3(q+1)+(\gamma+\delta+\varepsilon)q^2+(\delta+\varepsilon)q+\varepsilon}.$$

Putting

$$N(q)=q^4(q^2+1)\left[\beta q^5+(\beta+\gamma-1)q^4+(\beta+\gamma+\delta)q^3+(\gamma+\delta+\varepsilon)q^2+(\delta+\varepsilon)q+\varepsilon\right.$$

and

$$D(q)=\beta q^7+(\beta+\gamma-1)q^6+(\beta+\gamma+\delta)q^5+(\beta+\gamma+\delta+\varepsilon)q^3(q+1)$$
$$+(\gamma+\delta+\varepsilon)q^2+(\delta+\varepsilon)q+\varepsilon,$$

we have that $b=\dfrac{N(q)}{D(q)}$.

As b is an integer, then the remainder $R(q)$ in the division of $N(q)$ by $D(q)$ must be zero for any q.

Suppose $\beta\neq 0$, the remainder $R(q)$ in the division of $N(q)$ by $D(q)$ is:

$$R(q)=\frac{\beta+\gamma-1}{\beta}q^6+\frac{\beta+\gamma+\delta}{\beta}q^5+\frac{\beta^2+\beta+\gamma+\delta+\varepsilon}{\beta}q^4$$
$$+\frac{\beta(\gamma+1)+\gamma+\delta+\varepsilon}{\beta}q^3+\frac{\beta\delta+\gamma+\delta+\varepsilon}{\beta}q^2+\frac{\beta\varepsilon+\delta+\varepsilon}{\beta}q+\frac{\varepsilon}{\beta}.$$

Since $R(q)$ must be zero for any q, we need that the coefficients of the polynomial $R(q)$ must be zero for any q.

Thus we obtain a system of seven equations in four variables β, γ, δ and ε which has no solution.

Therefore $\alpha=1$ and $\beta=0$, then $a=q^4+\gamma q^2+\delta q+\varepsilon$ and

$$b=\frac{q^4(q^2+1)[(\gamma-1)q^4+(\gamma+\delta)q^3+(\gamma+\delta+\varepsilon)q^2+(\delta+\varepsilon)q+\varepsilon]}{(\gamma-1)q^6+(\gamma+\delta)q^5+(\gamma+\delta+\varepsilon)q^3(q+1)+(\gamma+\delta+\varepsilon)q^2+(\delta+\varepsilon)q+\varepsilon}.$$

Putting $N(q)=q^4(q^2+1)[(\gamma-1)q^4+(\gamma+\delta)q^3+(\gamma+\delta+\varepsilon)q^2+(\delta+\varepsilon)q+\varepsilon]$
and
$D(q)=(\gamma-1)q^6+(\gamma+\delta)q^5+(\gamma+\delta+\varepsilon)q^3(q+1)+(\gamma+\delta+\varepsilon)q^2+(\delta+\varepsilon)q+\varepsilon$,
we have that $b=\frac{N(q)}{D(q)}$.

As b is an integer, then the remainder $R(q)$ in the division of $N(q)$ by $D(q)$ must be zero for any q.
Suppose $\gamma\neq 1$, the remainder $R(q)$ in the division of $N(q)$ by $D(q)$ is:

$$R(q)=\frac{\gamma(\varepsilon-\delta-1)-\delta^2-\delta-\varepsilon}{(\gamma-1)^2}q^5-\frac{(\gamma+\delta+\varepsilon)(\delta+1)}{(\gamma-1)^2}q^4$$
$$+\frac{\gamma(\gamma-1)^2-\gamma(\delta+1)-\delta^2-\delta(\varepsilon+1)-\varepsilon}{(\gamma-1)^2}q^3$$
$$-\frac{(\gamma-1)^2(1-\delta)+(\delta+2)\gamma+\delta^2+\delta(\varepsilon+1)+\varepsilon-1}{(\gamma-1)^2}q^2$$
$$+\frac{\gamma^2\varepsilon-\gamma(\delta+2\varepsilon)-\delta(\delta+\varepsilon)}{(\gamma-1)^2}q-\frac{\varepsilon(\gamma+\delta)}{(\gamma-1)^2}.$$

Since $R(q)$ must be zero for any q, we need that the coefficients of the polynomial $R(q)$ must be zero for any q. Thus we obtain a system of six equations in three variables γ, δ and ε which has one solution $(\gamma,\delta,\varepsilon)=(0,0,0)$, for which we get $a=q^4$ and $b=q^4+q^2$.

Now suppose $\gamma=1$. If $\alpha=1$, $\beta=0$ and $\gamma=1$, then $a=q^4+q^2+\delta q+\varepsilon$ and

$$b=\frac{q^4(q^2+1)[(\delta+1)q^3+(\delta+\varepsilon+1)q^2+(\delta+\varepsilon)q+\varepsilon]}{(\delta+1)q^5+(\delta+\varepsilon+1)q^3(q+1)+(\delta+\varepsilon+1)q^2+(\delta+\varepsilon)q+\varepsilon}.$$

Putting $N(q)=q^4(q^2+1)[(\delta+1)q^3+(\delta+\varepsilon+1)q^2+(\delta+\varepsilon)q+\varepsilon]$ and
$D(q)=(\delta+1)q^5+(\delta+\varepsilon+1)q^3(q+1)+(\delta+\varepsilon+1)q^2+(\delta+\varepsilon)q+\varepsilon$, we have that $b=\frac{N(q)}{D(q)}$.

As b is an integer then the remainder $R(q)$ in the division of $N(q)$ by $D(q)$ must be zero for any q.
Suppose $\delta\neq-1$, the remainder $R(q)$ in the division of $N(q)$ by $D(q)$ is:

$$R(q)=\frac{(\delta+\varepsilon+1)\varepsilon^2}{(\delta+1)^3}q^4+\frac{\delta^3+2\delta^2+\delta(\varepsilon^2+1)+\varepsilon^2(\varepsilon+1)}{(\delta+1)^3}q^3$$
$$+\frac{\delta^4+2\delta^3+\delta^2+\delta\varepsilon(\varepsilon+1)+\varepsilon(\varepsilon^2+\varepsilon+1)}{\beta}q^2$$
$$+\frac{\varepsilon\left[\delta^3+3\delta^2+\delta(\varepsilon+2)+\varepsilon^2\right]}{(\delta+1)^3}q+\frac{\varepsilon^2(\delta+\varepsilon+1)}{(\delta+1)^3}.$$

Since $R(q)$ must be zero for any q, we need that the coefficients of the polynomial $R(q)$ must be zero for any q.

Thus we obtain a system of five equations in two variables δ and ε, which has the unique solution $(\delta,\varepsilon)=(0,0)$ for which we get $a=q^4+q^2$ and $b=q^4$, a contradiction because $a<b$.

Therefore $\delta=-1$. If $\alpha=1, \beta=0, \gamma=1$ and $\delta=-1$, then $a=q^4+q^2-q+\varepsilon$ and

$$b=\frac{q^4(q^2+1)\left[\varepsilon q^2+(\varepsilon-1)q+\varepsilon\right]}{\varepsilon q^3(q+1)+\varepsilon q^2+(\varepsilon-1)q+\varepsilon}.$$

Putting $N(q)=q^4(q^2+1)\left[\varepsilon q^2+(\varepsilon-1)q+\varepsilon\right]$ and

$D(q)=\varepsilon q^3(q+1)+\varepsilon q^2+(\varepsilon-1)q+\varepsilon$, we have that $b=\dfrac{N(q)}{D(q)}$.

As b is an integer then the remainder $R(q)$ in the division of $N(q)$ by $D(q)$ must be zero for any q.

Suppose $\varepsilon\neq 0$, the remainder $R(q)$ in the division of $N(q)$ by $D(q)$ is

$R(q)=\dfrac{(\varepsilon+2)}{\varepsilon}q^3+\dfrac{(1-2\varepsilon)}{\varepsilon}q^2+\dfrac{(\varepsilon^2+\varepsilon-1)}{\varepsilon}q+\dfrac{1}{\varepsilon}$, which is not zero.

If $\alpha=1$, $\beta=0$, $\gamma=1$, $\delta=-1$ and $\varepsilon=0$, then $a=q^4+q^2-q$ and $b=q^6+q^4$, a contradiction because $b\leq q^4+q^3+2q^2+q+1$.

Therefore, $a=q^4$ and $b=q^4+q^2$ and K is a (q^6+q^4)–set of type (q^4,q^4+q^2) with respect to hyperstars of planes in PG(4,q).

Thus K is an intersection (q^6+q^4)–set of planes of type $(0,q^2)$ with respect to stars and of type (q^4,q^4+q^2) with respect to hyperstars in PG(4,q). Then, by the Result, K is the set of planes meeting a non–singular quadric of PG(4,q) in a conic and the Theorem is completely proved.

3. Conclusion

In this paper the planes meeting a non-singular quadric of PG(4,q) in a conic are characterized by incidence properties with respect to points, lines and 3-spaces. The arguments leading to these results are combinatorial arguments based largely on the integrality of the parameters at stake. By requiring the existence of an appropriate set of planes enjoying the same properties for all q, sporadic cases are not considered.

References

[1] S. G. Barwick, D. K. Butler, A Characterisation of the Lines External to an Oval Cone in PG(3,q), q even, *J. Geom.* **93** (2009) 21–27.

[2] D. K. Butler, A characterisation of the planes meeting a non-singular quadric of PG(4,q) in a conic, http://www.maths.adelaide.edu.au/david.butler/sctplanesP4E.pdf (2007), submitted.

[3] D. K. Butler, S. G. Barwick, A Characterisation of the Lines External to a Quadric Cone in PG(3, q), q odd, *Innovations in Incidence Geometry,* **8** (2009) 39-48.

[4] R. Di Gennaro, N. Durante and D. Olanda, A characterization of the family of lines external to a hyperbolic quadric of PG(3, q), *J. Geom.* **80** (2004) 65–74.

[5] R. Di Gennaro, N. Durante, D. Olanda, A characterization of the family of external lines to a quadratic cone of *PG*(3, q), q odd, *J. Geom.* **96** (2009), 63–70.

[6] N. Durante, D. Olanda, A characterization of the family of secant or external lines of an ovoid of PG(3,q), *Bull. Belg. Math. Soc. Simon Stevin* **12** (2005), 1–4.

[7] D. G. Glynn, On the characterization of certain sets of points in finite projective geometry of dimension three, *Bull. London Math. Soc.* **15** (1983), 31–34.

[8] J. W. P. Hirschfeld, *Finite projective spaces of three dimensions,* Clarendon Press Oxford 1985.

[9] J. W. P. Hirschfeld, J. A. Thas, Sets of type (1,n,q+1) in PG(d,q), *Proc. London Math. Soc.,* **41** (1980), 254–278.

[10] G. Tallini, The Geometry on Grassmann manifolds representing subspaces in a Galois space, *Annals of Discrete Mathematics,* **14** (1982), 9–38.

[11] G. Tallini, On line k–sets of type (0,n) with respect to pencil of lines in PG(d,q), *Annals of Discrete Mathematics,* **14** (1982), 283–292.

[12] M. Tallini Scafati, The k–sets of PG(r,q) from the character point of view, *Finite Geometries (Winnipeg, Man., 1984), Lecture Notes in Pure and Appl. Math.* **103**, Dekker, New York (1985), 321–326.

[13] J.A.Thas, A combinatorial problem, *Geom. Dedicata* 1 (1973), 236–240.

[14] A. Venezia, On a characterization of the set of lines which either belong to or are tangent to a non-singular quadric in PG(3,q), q odd, *Rend. Sem. Mat. Brescia,* 7 (1984), 617–623.