Closed Monophonic and Minimal Closed Monophonic Numbers of a Graph

Ferdinand P. Jamil
Sergio R. Canoy, Jr.
Mathematics Department
MSU-Iligan Institute of Technology
Iligan City, Philippines
Email: *ferdinand.jamil@g.msuiit.edu.ph*
serge_canoy@yahoo.com

Salma N. Marohombsar[1]
Mathematics Department
Mindanao State University
Marawi City, Philippines

Abstract

This paper considered the concepts of monophonic, closed monophonic, and minimal closed monophonic numbers of a connected graph G. It was shown that any positive integers m, n, d and k satisfying the conditions that $4 \leq n \leq m$, $3 \leq d \leq k$, and $k \geq 2m-n+d+1$ are realizable as the monophonic number, closed monophonic number, m-diameter and order, respectively, of a connected graph. Also, any positive integers n, m, d and k with $2 \leq n \leq m$, $d \geq 3$, and $k \geq m+d-1$ are realizable as the closed monophonic number, minimal closed monophonic number, m-diameter and order, respectively, of a connected graph. Further, the closed monophonic number of the composition of connected graphs was also determined.

Mathematics Subject Classification: 05C12

Keywords: monophonic, closed monophonic, minimal closed monophonic, m-diameter

1 Introduction

Let $G = (V(G), E(G)$ be a connected simple graph. The order of G is the cardinality $|V(G)|$ of the vertex set of G. An edge joining the vertices u and v in G is denoted by uv. In this case, u and v are said to be adjacent vertices. For any vertices u and v of G, a u-v geodesic is meant any shortest path in G with endvertices u and v. The symbol $d_G(u, v)$ denotes the length

[1]Research partially funded by the Commission on Higher Education - Faculty Development Program, Philippines

of a u-v geodesic. The set $I_G[u,v]$ is defined as the set of all vertices of G lying on any u-v geodesic. If $S \subseteq V(G)$, then the *geodetic closure* of S is the set $I_G[S] = \cup\{I_G[u,v] : u,v \in S\}$. If $I_G[S] = V(G)$, then S is called a *geodetic set* of G. By a *monophonic path* in G is meant any path P in G such that no two nonconsecutive vertices in P are adjacent. The length of any longest monophonic path is called the *m-diameter* of G, and is denoted by $diam_m(G)$. A *m-diametral path* is any monophonic path with length equal to $diam_m(G)$. For any vertices u and v of G, the *closed monophonic interval* $J_G[u,v]$ is the set of all vertices of G lying in any u-v monophonic path. For any nonempty $S \subset V(G)$, the set $J_G[S] = \cup\{I_G[u,v] : u,v \in S\}$ is the *monophonic closure* of S. If $J_G[S] = V(G)$, then S is called a *monophonic set* of G. Since $I_G[S] \subseteq J_G[S]$, geodetic sets of G are also monophonic sets of G. The minimum cardinality among all monophonic sets of G, denoted by $m(G)$, is called the *monophonic number* of G. This invariant is introduced in [2], and some results concerning this parameter are given in [3] and [5].

2 Closed Monophonic Number of a Graph

Given a connected graph G, a monophonic set S can be obtained in the following way: Choose $v_1 \in V(G)$, and put $S_1 = \{v_1\}$. Then choose $v_2 \in V(G) \setminus S_1$, and put $S_2 = \{v_1, v_2\}$. For $k \geq 3$, choose $v_k \in V(G) \setminus J_G[S_{k-1}]$, and put $S_k = \{v_1, v_2, \ldots, v_k\}$. Since $V(G)$ is a finite set, there is a smallest positive integer k for which $S_k = S$. Any such monophonic set obtained in this way is called a *closed monophonic set* of G. The minimum cardinality among all closed monophonic sets of G is the *closed monophonic number* $m_c(G)$ of G. It is worth mentioning that this new process of selecting vertices and forming new sets actually results with the "monophonic" version of two classes of graphical games called the achievement and avoidance games introduced in [2] (see also [1] and [7]).

Let G be a connected graph. Choose $u, v \in V(G)$ such that the $diam_m(G)$ is the length of some u-v monophonic path. Construct a closed monophonic set S with $v_1 = u$ and $v_2 = v$. Then

$$|V(G)| \geq (diam_m(G) + 1) + (m_c(G) - 2) = m_c(G) + diam_m(G) - 1,$$

so that

$$m_c(G) \leq |V(G)| - diam_m(G) + 1.$$

If, in particular, G is obtained by joining n distinct edges $[v_d, u_i]$, $i = 1, 2, \ldots, n$, to the path $P = [v_1, v_2, \ldots, v_d]$ involving $d = diam_m(G)$ vertices, then $m_c(G) = n+1$, and is determined by the closed monophonic set $\{v_1, u_1,$

$\ldots, u_n\}$ of G. With this graph G, $|V(G)| = d+n = m_c(G)+d-1$, showing that the above inequality is sharp.

For a connected graph G, $m_c(G) = |V(G)|$ if and only if G is complete. For $k = 2, 3$, $m(G) = k$ if and only if $m_c(G) = k$. It will be shown that the same is also true for $k = |V(G)| - 1$, but not necessarily true for any other k.

A *neighborhood* of a vertex v in a connected graph G is the set $N(v)$ of all vertices of G adjacent to v. A vertex v is an *extreme* vertex if for every pair $u, w \in N(v)$, u and w are adjacent. The symbol $Ext(G)$ denotes the set of all extreme vertices of G. It is known that $Ext(G) \subseteq S$ for every monophonic set S of G. Consequently, we have

Lemma 2.1 *For any connected graph G, $Ext(G) \subseteq S$ for every closed monophonic set S of G.*

Theorem 2.2 *Let G be a connected graph of order n and with $diam_m(G) = d$. Then $m_c(G) = n-d+1$ if and only if G has a u-v m-diametral path P, $u, v \in V(G)$, such that $\{u, v\} \cup (V(G) \setminus V(P)) \subseteq Ext(G)$.*

Proof: Let $u, v \in V(G)$ and P be a m-diametral u-v path in G with $u, v, x \in Ext(G)$ for all $x \in V(G) \setminus V(P)$. By Lemma 2.1, $\{u, v, x : x \in V(G) \setminus V(P)\} \subseteq S$ for all closed monophonic sets S of G. Thus $m_c(G) \geq n - d + 1$. Therefore, $m_c(G) = n - d + 1$.

Conversely, suppose that $m_c(G) = n - d + 1$, and let $u, v \in V(G)$ such that there is a m-diametral path P in G connecting u and v. Then $J_G[u, v] = V(P)$. For suppose that there exists $x \in J_G[u, v] \setminus V(P)$. Construct a closed monophonic set $S = \{v_1, v_2, \ldots, v_k\}$ with $v_1 = u$ and $v_2 = v$. Since $x \in J_G[u, v]$, $k < n - d + 1$, a contradiction. Moreover, similar arguments show that $J_G[\{u, v, x\}] = V(P) \cup \{x\}$ for all $x \in V(G) \setminus V(P)$.

We first show that $d_G(x, V(P)) = min\{d_G(x, w) : w \in V(P)\} = 1$ for all $x \in V(G) \setminus V(P)$. Suppose that, in the contrary, there exist $x \in V(G) \setminus V(P)$ and $w \in V(P)$ such that $d_G(x, w) = d_G(x, V(P)) = 2$. Let $[x, z, w]$ be a x-w geodesic in G. Construct a closed monophonic set $S_k = \{v_1, v_2, \ldots, v_k\}$ of G with $v_1 = u$, $v_2 = v$ and $v_3 = x$. Then $z \in J_G[S_3]$, yielding $|J_G[S_3]| \geq d+3$. This implies that $k < n - d + 1$, a contradiction.

Now let $P = [u = u_1, u_2, \ldots, u_d, u_{d+1} = v]$.

Claim 1: $u, v \in Ext(G)$. If $|N(u)| = 1$, then $[u, u_2]$ is a pendant in G, and so $u \in Ext(G)$. Suppose that $|N(u)| \geq 2$, and let $x \in N(u) \setminus V(P)$. Since u is an endvertex of a m-diametral path and $J_G[\{u, v, x\}] = V(P) \cup \{x\}$, x is adjacent to v_j for some $j = 2, 3, \ldots, d + 1$. If $j \neq 2$, then $x \in J_G[u, v]$, a contradiction. Thus, $d_G(x, u_2) = 1$. Let x, y be distinct vertices in $N(u) \setminus V(P)$. Then $d_G(x, u_2) = 1 = d_G(y, u_2)$. If $d_G(x, y) = 2$, then both $[x, u_2, u_3, \ldots, u_{d+1}]$ and $[y, u_2, u_3, \ldots, u_{d+1}]$ are m-diametral paths in G. Moreover, $V(P) \cup \{x, y\} \subseteq J_G[\{x, y, v\}]$. Thus, a closed monophonic set S of G can be constructed starting with x, y, v. The preceding set inclusion implies that $|S| < n - d + 1$, which is impossible. Hence, $d_G(x, y) = 1$. This implies that $u \in Ext(G)$. Similarly, $v \in Ext(G)$.

Claim 2: $x \in Ext(G)$ for all $x \in V(G) \setminus V(P)$. Let $x \in V(G) \setminus V(P)$. If $|N(x)| = 1$, then $x \in Ext(G)$. Suppose that $|N(x)| \geq 2$, and suppose further that there exist $y, z \in N(x)$ with $d_G(y, z) \geq 2$. There are three cases:

Case1: Suppose that $y, z \in V(P)$. Then $y = u_i$ and $z = u_j$ for some $1 \leq i < j \leq d + 1$. Since the path $[u_1, u_2, \ldots, u_i, x, u_j, u_{j+1}, \ldots, u_{d+1}]$ is monophonic, $x \in J_G[u, v]$, a contradiction.

Case 2: Suppose that $y \in V(P)$ and $z \notin V(P)$. Let $y = u_i$ and z be adjacent to u_j. assume $i < j$. Then $[u_1, u_2, \ldots, u_i, x, z]$ is monophonic. Construct a closed monophonic set $S_k = \{v_1, v_2, \ldots, v_k\}$ with $v_1 = u$, $v_2 = v$ and $v_3 = z$. Then $x \in J_G[S_3]$ so that $k < n - d + 1$. This is impossible.

Case 3: Finally suppose that $y, z \notin V(P)$. If $y \in J_G[\{u, v, z\}]$, then a closed monophonic set $S_k = \{v_1, v_2, \ldots, v_k\}$ can be constructed with $v_1 = u$, $v_2 = v$ and $v_3 = z$. Certainly, $k < n - d + 1$, which is a contradiction. If $y \notin J_G[\{u, v, z\}]$, then a closed monophonic set $S_k = \{v_1, v_2, \ldots, v_k\}$ with $v_1 = u$, $v_2 = v$, $v_3 = z$ and $v_4 = y$. Then $x \in J_G[S_4]$ so that $k < n - d + 1$, a contradiction. This completes the proof of the theorem. ∎

Corollary 2.3 *Let G be a connected graph of order p. Then $m_c(G) = p - 1$ if and only if $m(G) = p - 1$.*

Proof: Suppose that $m_c(G) = p - 1$. Then G is not complete. We claim that $diam_m(G) = 2$. Suppose that, in the contrary, $diam_m(G) > 2$ and is determined by some vertices u and v. Construct a closed monophonic set $S = \{v_1, v_2, \ldots, v_k\}$ such that $v_1 = u$ and $v_2 = v$. Since $|J_G[u, v]| \geq 4$, $k < p - 1$, a contradiction. Thus $diam_m(G) = 2$. By Theorem 2.2, $G = \langle\{w\}\rangle + \cup_j^r K_{n_j}$, where $n_1 + n_2 + \cdots + n_r = p - 1$. It follows that $m(G) = p - 1$.

Conversely, suppose that $m(G) = p - 1$. Then $m_c(G) \geq p - 1$. Since G is not complete, $m_c(G) = p - 1$. ∎

Theorem 2.4 *For any positive integers m,n,k and d such that $4 \leq n \leq m$, $4 \leq d \leq k$ and $k > 2m - n + d + 3$, there exists a connected graph G such that $|V(G)| = k$, $diam_m(G) = d$, $m(G) = n$ and $m_c(G) = m$.*

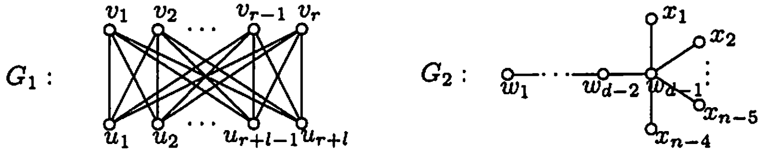

Figure 1: Graphs G_1 and G_2

Proof: Let $r = m-n+4$ and $l = k-2m+n-d-3$. Let $W = \{v_1, v_2, \ldots, v_r\}$ and $U = \{u_1, u_2, \ldots, u_{r+l}\}$ be the partite sets of the complete bipartite graph $G_1 = K_{r,(r+l)}$.

Let P denote a path $[w_1, w_2, \ldots, w_{d-1}]$, and obtain the graph G_2, as in Figure 1, by adding to P distinct pendant edges $[x_i, w_{d-1}]$, $i = 1, 2, \ldots, n-4$. Using G_1 and G_2, form graph G, as in Figure 2, by joining the vertex w_1 of G_2 to the vertices v_j and u_i of G_1, $j = 1, 2, \ldots, r$; $i = 1, 2, \ldots, (r+l)$. Then $|V(G)| = r + (r+l) + (d-1) + (n-4) = k$, $diam_m(G) = d$ and $Ext(G) = \{x_1, x_2, \ldots, x_{n-4}\}$.

Since $J_G[Ext(G) \cup \{v_1, v_r, u_1, u_{r+l}\}] = V(G)$, $m(G) \leq n$. Let S be a minimum monophonic set in G. Since $Ext(G) \subseteq S$, $n-4 \leq |S|$. If S is a minimum monophonic set of G, then S contains the vertices of a m-diametral path of G, and thus contains a vertex v_j for some $j = 1, 2, \ldots, r$ or contains a vertex u_j for some $j = 1, 2, \ldots, r+l$. Since $Ext(G) \cup \{v', v'', u\}$ and $Ext(G) \cup \{u', u'', v\}$ are not monophonic sets of G for all vertices $v, v', v'' \in W$ and $u, u', u'' \in U$, $|S| \geq n$. Therefore, $m(G) = n$.

The set $Ext(G) \cup \{v_1, v_2, \ldots, v_r\}$ is a closed monophonic set of G. Thus $m_c(G) \leq m$. Let S be a minimum closed monophonic set of G. Then $|S| \geq n$. Thus S contains $Ext(G) \cup \{v_i, v_j\}$ for some $1 \leq i, j \leq r$ or S contains $Ext(G) \cup \{u_i, u_j\}$ for some $1 \leq i, j \leq r+l$. Since $l \geq 1$ and $U \subseteq J_G[Ext(G) \cup \{v_i, v_j\}]$, the definition of S implies that $v_1, v_2, \ldots, v_r \in S$. Thus $m_c(G) \geq m$. This proves that $m_c(G) = m$, and completes the proof of the theorem. ■

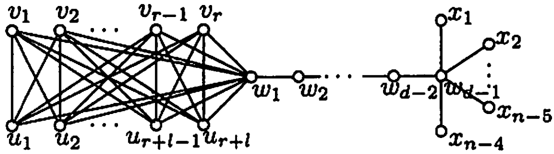

Figure 2: Graph G satisfying the conditions in Theorem 2.4

3 Minimal Closed Monophonic Number of a Graph

A *minimal closed monophonic set* of a connected graph G is any closed monophonic set S of G such that S contains no subset which is itself a closed monophonic set of G. The maximum cardinality among the minimal closed monophonic sets of G, denoted by $m_c^+(G)$, is the *minimal closed monophonic number* of G, i.e.,

$$m_c^+(G) = max\{|S| : S \textit{ is minimal closed monophonic set of } G\}.$$

Clearly, a minimum closed monophonic set of G is necessarily a minimal closed monophonic set of G. Thus we always have $m_c(G) \leq m_c^+(G)$.

Obviuosly, $m_c^+(G) = |V(G)|$ if and only if G is complete. We also have $G = \langle\{w\}\rangle + \cup_j^r K_{n_j}$, where $n_1 + n_2 + \cdots + n_r = |V(G)| - 1$ if and only if $m_c^+(G) = |V(G)| - 1$. In these cases, $m_c^+(G) = m_c(G)$.

For any connected graph G, $|V(G)| \geq m_c^+(G) + d - 1$.

Theorem 3.1 *For every set of positive integers m, n, k and d with $2 \leq n \leq m$, $d \geq 3$ and $k \geq m + d - 1$, there exists a connected graph G such that $|V(G)| = k$, $m_c^+(G) = m$, $m_c(G) = n$ and $diam_m(G) = d$.*

Proof: Let $r = m - n + 1$ and $l = k - m - d + 1 \geq 0$. Let $\{v_1, v_2, \ldots, v_r\}$ and $\{x, y\}$ be the partite sets of $K_{r,2}$. If $l = 0$, put $G_1 = K_{r,2}$, but if $l \geq 1$, let G_1 be that graph as in Figure 3 obtained by adding l P_3s $[x, x_j, y]$, $1 \leq j \leq l$ to $K_{r,2}$. Let G_2 be the graph as in Figure 3 obtained by adding to $P_{d-3} = [w_1, w_2, \ldots, w_{d-3}]$ $n-1$ pendant edges $[w_{d-3}, u_j]$, $1 \leq j \leq n-1$. Using G_1 and G_2, form the graph G (see Figure 4) by joining the vertex w_1 to the vertices y and x_j, $1 \leq j \leq l$. Then $|V(G)| = k$ and $diam_m(G) = d$.

Now let S be a minimal closed monophonic set of G. By Lemma 2.1, $Ext(G) = \{u_1, u_2, \ldots, u_{n-1}\} \subseteq S$. Since $J_G[Ext(G)] = Ext(G) \cup \{w_{d-3}\} \neq V(G)$, $S \neq Ext(G)$. Suppose that $x \in S$. Since $Ext(G) \cup \{x\}$ is a closed monophonic set of G, the minimality of S implies that $S = Ext(G) \cup \{x\}$. Suppose that $x \notin S$. Since $v_j \notin J_G[S \setminus \{v_j\}]$, $v_j \in S$ for all $j = 1, 2, \ldots, r$. Since $Ext(G) \cup \{v_1, v_2, \ldots, v_r\}$ is a closed monophonic set of G, the minimality of S implies that $S = Ext(G) \cup \{v_1, v_2, \ldots, v_r\}$. Thus either $|S| = n$ or $|S| = r + (n-1) = m$. This means that $m_c(G) = n$ and $m_c^+(G) = m$. ■

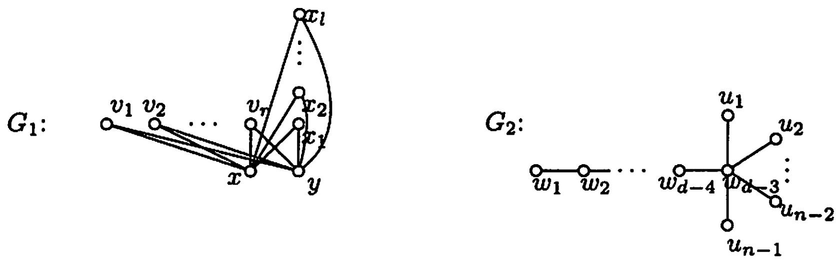

Figure 3: Graphs G_1 and G_2

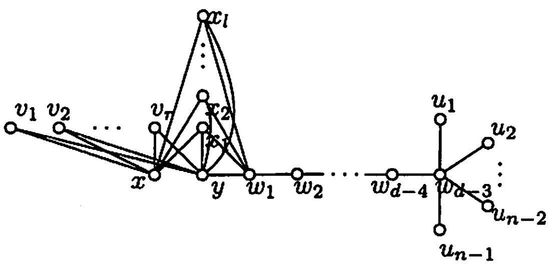

Figure 4: Graph G satisfying the conditions in Theorem 3.1

If $l = 0$ for the graph G in Figure 4, the resulting graph is the graph G given in Figure 5. With this graph, if $d = 3$, then $|V(G)| = m_c^+(G) + 2$. The removal of a vertex from G yields $m_c^+(G) = |V(G)| - 1$, in which case $m_c^+(G) = m_c(G)$. This means that if $k = m + d - 1$, then the graph G in Theorem 3.1 is of minimum order satisfying the desired properties. From this, the following corollary follows immediately.

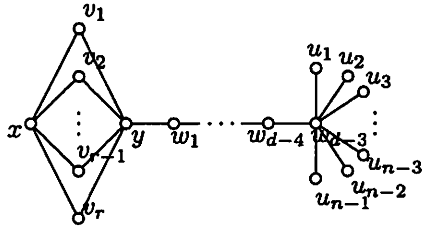

Figure 5: Graph G of minimum order satisfying the conditions in Theorem 3.1

Corollary 3.2 *For every set of positive integers m, n and d with $2 \leq n \leq m$ and $d \geq 3$, there is a connected graph G of minimum order such that $m_c(G) = n$, $m_c^+(G) = m$ and $diam_m(G) = d$.*

4 Composition of Graphs

The *composition* $G[H]$ of two graphs G and H is that graph with $V(G[H]) = V(G) \times V(H)$ and $(u,v)(u',v') \in E(G[H])$ if and only if either $uu' \in E(G)$ or $u = u'$ and$vv' \in V(H)$. It is known that $x \in Ext(G)$ if and only if $(x,y) \in Ext(G[K_m])$ for every $y \in V(K_m)$ [6], [4].

Given $S \subseteq V(G[H])$, the *G-projection* S_G (resp. H-projection S_H) of S is the set of all first (resp. second) components of S. That is,

$$S_G = \{x \in V(G) : (x,y) \in S \text{ } for \text{ } some \text{ } y \in V(H)\}.$$

Lemma 6.1 *Let G be a connected nocomplete graph, and let $m \geq 1$. If $[(u_1,v_1),(u_2,v_2),\ldots,(u_k,v_k)]$ is a monophonic path in $G[K_m]$, then $[u_1,u_2,\ldots,u_k]$ is a monophonic path in G. Conversely, if $[u_1,u_2,\ldots,u_k]$ is a monophonic path in G, then $[(u_1,v_1),(u_2,v_2),\ldots,(u_k,v_k)]$ is a monophonic path in $G[K_m]$ for any $v_1,v_2,\ldots,v_k \in V(K_m)$.*

Proof: Let $P = [(u_1,v_1),(u_2,v_2),\ldots,(u_k,v_k)]$ be a monophonic path in $G[K_m]$. First we claim that $u_1,u_2,\ldots,u_k$ are distinct vertices in G. For suppose that $u_i = u_j$ for some $i < j$. Then $d_{G[K_m]}((u_i,v_i),(u_{j+1},v_{j+1})) = 1$. That is, the edge $[(u_i,v_i),(u_{j+1},v_{j+1})]$ is a chord of P in $G[K_m]$, a contradiction. Suppose that there exist nonconsecutive vertices u_i and u_j in $[u_1,u_2,\ldots,u_k]$ such that $d_G(u_i,u_j) = 1$. Then $d_{G[K_m]}((u_i,v_i),(u_j,v_j)) = 1$, implying that the edge $[(u_i,v_i),(u_j,v_j)]$ is a chord of P in $G[K_m]$, a contradiction. Thus $[u_1,u_2,\ldots,u_k]$ is monophonic in G.

Now, let $[u_1, u_2, \ldots, u_k]$ be a monophonic path in G, and let $v_1, v_2, \ldots, v_k \in V(K_m)$. Since $d_G(u_i, u_{i+1}) = 1$, $d_{G[K_m]}((u_i, v_i), (u_{i+1}, v_{i+1})) = 1$. Thus, $P = [(u_1, v_1), (u_2, v_2), \ldots, (u_k, v_k)]$ is a (u_i, v_i)-(u_k, v_k) path in $G[K_m]$. Finally, suppose that (u_i, v_i) and (u_j, v_j) are nonconsecutive vertices in P with $d_{G[K_m]}((u_i, v_i), (u_j, v_j)) = 1$. Then either $u_i = u_j$ or $d_G(u_i, v_j) = 1$, which is impossible since $[u_1, u_2, \ldots, u_k]$ is monophonic. This implies that P is monophonic. ∎

Lemma 6.1 yields the following lemma.

Lemma 6.2 *Let G be a connected nocomplete graph, and let $m \geq 1$. Let $u, v, w \in V(G)$. If $w \notin J_G[u, v]$, then for any $x_1, x_2, x_3 \in V(K_m)$, $(w, x_1) \notin J_{G[K_m]}[(u, x_2), (v, x_3)]$. Conversely, if $x_1, x_2, x_3 \in V(K_m)$ and $(w, x_1) \notin J_{G[K_m]}[(u, x_2), (v, x_3)]$, then $w \notin J_G[u, v]$.*

Theorem 6.3 *Let G be a connected graph, $m \geq 1$, and let $S \subseteq V(G[K_m])$. If S is a closed monophonic set of $G[K_m]$, then S_G is a closed monophonic set of G.*

Proof: Let S be a monophonic set of $G[K_m]$, and let $x \in V(G) \setminus S_G$. Pick any $y \in V(K_m)$. Then there exist (u_1, v_1), $(u_2, v_2) \in S$ such that $(x, y) \in J_{G[K_m]}[(u_1, v_1), (u_2, v_2)]$. By Lemma 6.1, $x \in J_G[u_1, u_2]$. Thus, $V(G) = J_G[S_G]$. Futhermore, by Lemma 6.2, if S, in canonical form, is given by $S = \{(u_1, v_1), (u_2, v_2), \ldots, (u_k, v_k)\}$, then $\{u_1, u_2, \ldots, u_k\}$ defines a canonical representation for S_G. Therefore, S_G is a closed monophonic set of G. ∎

Given a connected graph G and vertices $u, v \in V(G)$, we define $J_G(u, v) = J_G[u, v] \setminus \{u, v\}$. For a nonempty $S \subseteq V(G)$, we define

$$S^\circ = \bigcup_{u,v \in S} [S \cap J_G(u, v)].$$

In view of Lemma 2.1, If S is a closed monophonic set of G, then $Ext(G) \subseteq S \setminus S^\circ$.

Theorem 6.4 *Let G be a connected graph, $m \geq 1$, and $S \subseteq V(G[K_m])$. Then S is a closed monophonic set of $G[K_m]$ if and only if*

$$S = [(A \setminus A^\circ) \times V(K_m)] \cup T$$

for some closed monophonic set A of G and some $T \subseteq V(G[K_m])$ with $T_G = A^\circ$.

Proof: Let S be a closed monophonic set of $G[K_m]$. Put $A = S_G$ and $T = \{(u,v) \in S : u \in A^\circ\}$. By Theorem 6.3, A is a closed monophonic set of G. Clearly, $S \subseteq [(A \setminus A^\circ) \times V(K_m)] \cup T$. Since $T \subseteq S$, to establish equality, we only show the inclusion $[(A \setminus A^\circ) \times V(K_m)] \subseteq S$. To this end, suppose that there exist $x \in A \setminus A^\circ$ and $y \in V(K_m)$ such that $(x,y) \notin S$. There exist $(u_1,v_1),(u_2,v_2) \in S$ such that $(x,y) \in J_{G[K_m]}[(u_1,v_1),(u_2,v_2)] \setminus \{(u_1,v_1),(u_2,v_2)\}$. By Lemma 6.1, $x \in J_G[u_1,u_2] \setminus \{u_1,u_2\}$. This means that $x \in A \cap A^\circ$, a contradiction. Thus, $[(A \setminus A^\circ) \times V(K_m)] \subseteq S$, and therefore, $S = [(A \setminus A^\circ) \times V(K_m)] \cup T$.

Conversely, suppose that $S = [(A \setminus A^\circ) \times V(K_m)] \cup T$, where A is a closed monophonic set of G and $T \subseteq V(G[K_m])$ such that $T_G = A^\circ$. Let $(x,y) \in V(G[K_m])$. If $x \in A \setminus A^\circ$, then $(x,y) \in S \subseteq I_{G[K_m]}[S]$. Suppose that $x \in A^\circ$. Then there exist $u,v \in A$ such that $x \in J_G[u,v] \setminus \{u,v\}$. Let $[u = x_1, x_2, \ldots, x_k = v]$ be a monophonic path in G containing x. Choose $v_0, v_0^* \in V(K_m)$ such that $(u,v_0),(v,v_0^*) \in S$. By Lemma 6.1, the path $[(x_1,v_0),(x_2,y),\ldots,(x_{k-1},y),(x_k,v_0^*)]$ is monophonic in $G[K_m]$ containing (x,y). Thus, $(x,y) \in J_{G[K_m]}[S]$. Suppose that $x \in V(G) \setminus A$. Since A is a monophonic set of G, there exist $u,v \in A$ such that $x \in J_G[u,v]$. Similarly as the preceding argument, if $(u,v_0),(v,v_0^*) \in S$, then $(x,y) \in J_{G[K_m]}[(u,v_0),(v,v_0^*)] \subseteq J_{G[K_m]}[S]$. Therefore, $V(G[K_m]) = J_{G[K_m]}[S]$.

Finally, we prove that S can be written in the required canonical form. Suppose that A, in canonical form, is given by $A = \{u_1, u_2, \ldots, u_k\}$. Put $S = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$, such that if $1 \le i < j \le n$ and $\alpha_i = (u_s, v)$ and $\alpha_j = (u_r, v^*)$, then $s \le r$. Note that for each $u \in A$, if $S(u) = \{(u,v) \in S : v \in V(K_m)\}$, then $\langle S(u)\rangle$ is a complete subgraph of $G[K_m]$. Thus, for distinct vertices x, y and z in $V(K_m)$, $(u,x) \notin J_G[(u,y),(u,z)]$. This and Lemma 6.2 imply that $\alpha_k \notin J_{G[K_m]}[\alpha_i, \alpha_j]$ whenever $i,j < k$. Therefore, S is a closed monophonic set of $G[K_m]$. ■

Theorem 6.5 *If A is a minimal closed monophonic set of a connected graph G and $v \in V(K_m)$, $m \ge 2$, and if*

$$S = [(A \setminus A^\circ) \times V(K_m)] \cup [A^\circ \times \{v\}],$$

then S is a minimal closed monophonic set of $G[K_m]$.

Proof: Let $A \subseteq V(G)$ be a minimal closed monophonic set of G, and let $v \in V(K_m)$. By Theorem 6.4, the set $S = [(A \setminus A^\circ) \times V(K_m)] \cup [A^\circ \times \{v\}]$ is a closed monophonic set of $G[K_m]$. We claim that S is minimal. Suppose that $S' \subseteq S$ and S' is a closed monophonic set of $G[K_m]$. By Theorem 6.4, $S' = [(B \setminus B^\circ) \times V(K_m)] \cup T$ for some closed monophonic set B of G and some $T \subseteq V(G[K_m])$ with $T_G = B^\circ$. If $x \in B \setminus B^\circ$, then $(x,y) \in [(B \setminus B^\circ) \times V(K_m)] \subseteq S' \subseteq S$ for all $y \in V(K_m)$, and thus $x \in A$. If

$x \in B^\circ = T_G$, then $(x,y) \in T \subseteq S$ for some $y \in V(K_m)$, showing that $x \in A$. Thus $B \subseteq A$. The minimality of A implies that $A = B$. Moreover, since $T \cap [(B \backslash B^\circ) \times V(K_m)] = \emptyset$ and $T_G = B^\circ = A^\circ$, we have $T = A^\circ \times \{v\}$. Therefore, $S = S'$, and S indeed is minimal. ∎

Theorem 6.6 *Let G be a connected graph, and let $m \geq 2$. Let $S \subseteq V(G[K_m])$. If S is a minimal closed monophonic set of $G[K_m]$, then*

$$S = [(A \setminus A^\circ) \times V(K_m)] \cup T,$$

where A is a closed monophonic set of G and $T \subseteq V(G[K_m])$ with $T_G = A^\circ$ and $|T| = |A^\circ|$.

Proof: Let $S \subseteq V(G[K_m])$ be a minimal closed monophonic set of $G[K_m]$. By Theorem 6.4, there exist a closed monophonic set A such that $S = [(A \setminus A^\circ) \times V(K_m)] \cup T$, where $T_G = A^\circ$. Suppose that $(x,y) \in T$. We claim that $(x,z) \notin T$ for all $z \in V(K_m) \backslash \{y\}$. Let $z \in V(K_m) \backslash \{y\}$. Since $x \in T_G = A^\circ$, $x \in J_G[A \setminus \{x\}]$. In view of Lemma 6.1, $(x,z) \in J_{G[K_m]}[S \setminus \{(x,z)\}]$. Moreover, for any $(u,v), (s,t) \in V(G[K_m])$, if $(u,v) \in J_{G[K_m]}[(x,z),(s,t)]$, then $(u,v) \in J_{G[K_m]}[(x,y),(s,t)]$. Thus the minimality of S implies that $(x,z) \notin T$. It means that for every $x \in A^\circ$ there is exactly one $y \in V(K_m)$ for which $(x,y) \in T$. This proves that $|T| = |A^\circ|$. ∎

Corollary 6.7 *Let G be a connected graph, and let $m \geq 2$. Then*

$$m_c(G[K_m]) = \min\{m\,|A| - (m-1)\,|A^\circ| : A \text{ is a closed monophonic set of } G\},$$

and

$$m_c^+(G[K_m]) = \max\{m\,|A| - (m-1)\,|A^\circ| : A \text{ is a minimal closed monophonic set of } G\}.$$

Proof: For convenience, let

$$L_1 = \min\{m\,|A| - (m-1)\,|A^\circ| : A \text{ is a closed monophonic set of } G\}$$

and

$$L_2 = \max\{m\,|A| - (m-1)\,|A^\circ| : A \text{ is a minimal closed monophonic set of } G\}.$$

Let A be a closed monophonic set of G. Pick any $v \in V(K_m)$, and put $S = [(A \setminus A^\circ) \times V(K_m)] \cup [A^\circ \times \{v\}]$. By Theorem 6.4, S is a closed monophonic set of $G[K_m]$. Thus

$$m_c(G[K_m]) \leq |S| = m\,|A| - (m-1)\,|A^\circ|.$$

Since A is arbitrary, $m_c(G[K_m]) \leq L_1$.

Conversely, let S be a closed monophonic set of $G[K_m]$ of minimum cardinality. Then S is a minimal closed monophonic set of $G[K_m]$. By Theorem 6.6, there exists a closed monophonic set A of G such that $S = [(A \setminus A^\circ) \times V(K_m)] \cup T$, where $T_G = A^\circ$ and $|T| = |A^\circ|$. Thus

$$m_c(G[K_m]) = |S| = m\,|A| - (m-1)\,|A^\circ| \geq L_1.$$

This establishes the first formula.

Let us turn to the second formula. Let A be a minimal closed monophonic set of G, and let $v \in V(K_m)$. By Theorem 6.5, the set $S = [(A \setminus A^\circ) \times V(K_m)] \cup [A^\circ \times \{v\}]$ is a minimal closed monophonic set of $G[K_m]$. Thus $m_c^+(G) \geq |S| = m\,|A| - (m-1)\,|A^\circ|$. Since A is arbitrary, $m_c^+(G[K_m]) \geq L_2$.

Now, among the minimal closed monophonic sets of $G[K_m]$ of maximum cardinality, we choose S such that $|S_G|$ is minimum, i.e,

$$\begin{aligned} |S_G| \quad = \quad & min\{|S^*_G| : S^* \textit{ is a minimal closed monophonic set} \\ & \qquad\qquad \textit{of } G[K_m] \textit{ with } |S^*| = m_c^+(G[K_m])\}. \end{aligned}$$

By Theorem 6.6, $S = [(A \setminus A^\circ) \times V(K_m)] \cup T$, where $A = S_G$ and $T_G = A^\circ$ and $|T| = |A^\circ|$. In view of Theorem 6.5, we can assume that $T = A^\circ \times \{v\}$ for some $v \in V(K_m)$. We claim that A is a minimal closed monophonic set of G. Suppose that such claim is false, and let $x \in A$ be such that $B = A \setminus \{x\}$ is a closed monophonic set of G. By Theorem 6.5, the set

$$S' = [(B \setminus B^\circ) \times V(K_m)] \cup [B^\circ \times \{v\}]$$

is a closed monophonic set of $G[K_m]$. By the definition of S, $|S'| \leq |S|$. Clearly, $B^\circ \subseteq A^\circ \setminus \{x\}$. If $B^\circ = A^\circ \setminus \{x\}$, then S' is a proper subset of S, a contradiction. Now, suppose that $D = (A^\circ \setminus \{x\}) \setminus B^\circ \neq \emptyset$. Then $|A^\circ| - |B^\circ| \geq 2$ and $B \setminus B^\circ = (A \setminus A^\circ) \cup D$. Thus

$$S' = \{[(A \setminus A^\circ) \cup D] \times V(K_m)\} \cup [B^\circ \times \{v\}]$$

so that

$$\begin{aligned} |S'| &= |S| + (m-1)(|A^\circ| - |B^\circ|) - m \\ &\geq |S| + 2(m-1) - m \\ &= |S| + m - 2 \\ &\geq |S|. \end{aligned}$$

By the definition of S, S' is not minimal. Let T be a proper subset of S' which is itself a closed monophonic set of $G[K_m]$, and put $M = T \setminus [D \times$

$V(K_m)]$. Then M is a proper subset of S. The minimality of S implies that $J_{G[K_m]}[M] \neq V(G[K_m])$. And since $\langle \{u\} \times V(K_m) \rangle$ is a complete subgraph of $G[K_m]$ for every $u \in D$, we have $T = M \cup (D_0 \times V(K_m))$ for some nonempty proper subset D_0 of D. Let $a \in D \setminus D_0$. Then $a \notin B^\circ$. On the other hand, there exist $u, w \in T_G$ such that $a \in J_G[u.w] \setminus \{u, w\}$. This means that $a \in B^\circ$. This contradiction establishes the above claim. Therefore $m_c^+(G[K_m]) = |S| \leq L_2$. ∎

Corollary 6.8 *If G is an extreme geodesic graph (i.e., $Ext(G)$ is a minimum geodetic set of G), then $m_c(G[K_m]) = m \cdot m_c(G)$ and $m_c^+(G[K_m]) = m \cdot m_c^+(G)$ for all $m \geq 2$.*

Acknowledgment The authors would like to thank the referee for reviewing the paper and for giving comments which led to the improvement of the paper.

References

[1] Aniversario, Imelda S., Jamil, Ferdinand P., and Canoy, Sergio, Jr.R. *The closed geodetic numbers of graphs.* Utilitas Mathematica. Vol.74 pp. 3-18. 2007.

[2] Buckley, Fred and Frank Harary. *Distance in Graphs.* Redwood City. CA: Addison-Wesley. 1990.

[3] Canoy, Sergio R. Jr, and Paluga, Esamel M. *Monophonic numbers of the join and composition of connected graphs.* Discrete Mathematics. Vol. 307(9-10). pp. 1146-1154. May 6, 2007.

[4] Canoy, Sergio R. Jr., Cagaanan, Gilbert, and Gervacio, Severino, *Convexity, geodetic, and hull numbers of the join of graphs*, Utilitas Mathematica, 71(2006) 143-159.

[5] Canoy, Sergio R. Jr., Paluga, Esamel M., *Minimal path convexity under some graph operations*, Utilitas Mathematica, Vol 72, 2007, pp 77-88.

[6] Chartrand, G., Haray, F., and Zhang, P. *Geodetic sets in graphs.* Discuss. Math. Graph Theory. Vol. 20. pp. 129-139. 2000.

[7] Necaskova, M., *A note on the Achievement Geodetic Games*, Questiones Math., Vol 12, pp115-119, 1988.