Eigenpolynomials associated with subspaces in d-bounded distance-regular graphs*

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Abstract

Let Γ denote a d-bounded distance-regular graph with diameter $d \geq 2$. A regular strongly closed subgraph of Γ is said to be a subspace of Γ. Define the empty set $\emptyset$ to be the subspace with diameter -1 in Γ. For $0 \leq i \leq d-1$, let $\mathcal{L}(\leq i)$ (resp. $\mathcal{L}(\geq i)$) denote the set of all subspaces in Γ with diameters $\leq i$ (resp. $\geq i$) including Γ and $\emptyset$. If we define the partial order on $\mathcal{L}(\leq i)$ (resp. $\mathcal{L}(\geq i)$) by reverse inclusion (resp. ordinary inclusion), then $\mathcal{L}(\leq i)$ (resp. $\mathcal{L}(\geq i)$) is a poset, denoted by $\mathcal{L}_R(\leq i)$ (resp. $\mathcal{L}_O(\geq i)$). In the present paper we give the eigenpolynomials of $\mathcal{L}_R(\leq i)$ and $\mathcal{L}_O(\geq i)$.

Key words: Distance-regular graph, Subspaces, Eigenpolynomials.

1 Introduction

In this section we first recall some terminology and definitions about finite posets ([1, 3]), then introduce some concepts concerning d-bounded distance-regular graphs and our main results.

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Let P be a poset. For $a, b \in P$, we say a *covers* b, denoted by $b <\cdot a$, if $b < a$ and there exists no $c \in P$ such that $b < c < a$. If P has the minimum (resp. maximum) element, then we denote it by 0 (resp. 1) and say that P is a poset with 0 (resp. 1). Let P be a finite poset with 0. By a *rank function* on P, we mean a function r from P to the set of all the integers such that $r(0) = 0$ and $r(a) = r(b) + 1$ whenever $b <\cdot a$.

Let P be a locally finite poset and let R be a commutative ring with unit element. Assume that $\mu : P \longrightarrow R$ is a binary function on the poset P, then μ is called the Möbius function of P if the following (i) – (iii) hold.

(i) For any $a \in P$, $\mu(a, a) = 1$.

(ii) For $a, b \in P$, if $a \leq b$ dose not hold, then $\mu(a, b) = 0$.

(iii) For $a, b \in P$, if $a < b$, then $\sum_{a \leq c \leq b} \mu(a, c) = 0$.

Let P be a poset with minimal element 0 and maximal element 1. Assume that r is the rank function of P. The polynomial

$$\chi(P, x) = \sum_{a \in P} \mu(0, a) x^{r(1) - r(a)}$$

is said to be the *eigenpolynomial* on P, where μ is the Möbius function of P.

Now we shall introduce some concepts concerning d-bounded distance-regular graphs. Let $\Gamma = (X, R)$ be a connected regular graph. For vertices u and v in X, let $\partial(u, v)$ denote the *distance* between u and v. The maximum value of the distance function in Γ is called the *diameter* of Γ, denoted by $d = d(\Gamma)$. For vertices u and v at distance i, define

$$C(u, v) = C_i(u, v) = \{w \mid \partial(u, w) = i - 1, \partial(w, v) = 1\},$$

$$A(u, v) = A_i(u, v) = \{w \mid \partial(u, w) = i, \partial(w, v) = 1\}.$$

For the cardinalities of these sets we use lower case letters $c_i(u, v)$ and $a_i(u, v)$.

A connected regular graph Γ with diameter d is said to be *distance-regular* if $c_i(u, v)$ and $a_i(u, v)$ depend only on i for all $1 \leq i \leq d$. The reader is referred to [2, 4] for general theory of distance-regular graphs.

Recall that a subgraph induced on Δ of Γ is said to be *strongly closed* if $C(u,v) \cup A(u,v) \subseteq \Delta$ for every pair of vertices $u, v \in \Delta$. Suzuki ([9]) determined all the types of strongly closed subgraphs of a distance-regular graph.

A distance-regular graph Γ with diameter d is said to be *d-bounded*, if every strongly closed subgraph of Γ is regular, and any two vertices x and y are contained in a common strongly closed subgraph with diameter $\partial(x, y)$.

Weng ([10, 11]) used the term *weak-geodetically closed subgraphs* for strongly closed subgraphs, obtained many important results when a distance-regular graph is d-bounded. A regular strongly closed subgraph of Γ is said to be a *subspace* of Γ.

The results on the lattices generated by subspaces in d-bounded distance-regular graphs can be found in Gao, Guo and Liu ([5]), Guo and Gao ([7]), Guo, Gao and Wang ([8]).

Let $\Gamma = (X, R)$ denote a d-bounded distance-regular graph with diameter $d \geq 2$. Define the empty set $\emptyset$ to be the subspace with diameter -1 in Γ. For $0 \leq i \leq d-1$, let $\mathcal{L}(\leq i)$ (resp. $\mathcal{L}(\geq i)$) denote the set of all subspaces in Γ with diameters $\leq i$ (resp. $\geq i$) including Γ and $\emptyset$. If we define the partial order on $\mathcal{L}(\leq i)$ (resp. $\mathcal{L}(\geq i)$) by reverse inclusion (resp. ordinary inclusion), then $\mathcal{L}(\leq i)$ (resp. $\mathcal{L}(\geq i)$) is a poset, denoted by $\mathcal{L}_R(\leq i)$ (resp. $\mathcal{L}_O(\geq i)$). Our main result is the following.

Theorem 1.1. *Let Γ be a d-bounded distance-regular graph with diameter $d \geq 2$ and let $0 \leq i \leq d-1$. Then*

$$\chi(\mathcal{L}_R(\leq i), t) = t^{i+2} - 1 - \sum_{m=0}^{i} N'(m,d) g_R(m,t),$$

$$\chi(\mathcal{L}_O(\geq i), t) = t^{d-i+1} - \sum_{l=i}^{d} N'(l,d) g_O(l;t),$$

where

$$g_R(m,t) = \sum_{j=0}^{m} (-1)^{m-j} \frac{(b_{j+1}-b_m)(b_{j+2}-b_m)\cdots(b_{m-1}-b_m)}{(b_j-b_{j+1})(b_j-b_{j+2})\cdots(b_j-b_{m-1})} \times N'(j,m)(t^{j+1}-1),$$

$$g_O(l,t) = \sum_{s=0}^{d-l} (-1)^s \frac{b_l b_{l+1} \cdots b_{l+s-1}}{(b_l-b_{l+1})(b_l-b_{l+2})\cdots(b_l-b_{l+s})} t^{d-l-s}$$

and

$$N'(h,u)=\frac{(b_0-b_u)(b_1-b_u)\cdots(b_{h-1}-b_u)\left(1+\sum_{j=1}^{u}\frac{(b_0-b_u)(b_1-b_u)\cdots(b_{j-1}-b_u)}{c_1c_2\cdots c_j}\right)}{(b_0-b_h)(b_1-b_h)\cdots(b_{h-1}-b_h)\left(1+\sum_{j=1}^{h}\frac{(b_0-b_h)(b_1-b_h)\cdots(b_{j-1}-b_h)}{c_1c_2\cdots c_j}\right)}.$$

2 Proof of Theorem 1.1

Proposition 2.1. *([11, Lemma 4.2]) Let Γ be a d-bounded distance-regular graph with diameter d, and let Δ be a subspace of Γ and $0\le i\le d(\Delta)$. Then Δ is distance-regular with intersection numbers $c_i(\Delta)=c_i$, $a_i(\Delta)=a_i$, $b_i(\Delta)=b_i-b_{d(\Delta)}$.*

Proposition 2.2. *([5, Lemma 2.1]) Let Γ be a d-bounded distance-regular graph with diameter $d\ge 2$. For $0\le i,s,t\le d$ and $i+1\le i+s\le i+s+t\le d$, suppose Δ and Δ' are strongly closed subgraphs with diameter i and $i+s+t$, respectively, and with $\Delta\subseteq\Delta'$. Then the number of the strongly closed subgraphs $\widetilde{\Delta}$ with diameter $i+s$ satisfying $\Delta\subseteq\widetilde{\Delta}\subseteq\Delta'$, denoted by $N(i,i+s;i+s+t)$, is determined by i,s and t, independent of the choice of Δ and Δ'; it is*

$$\frac{(b_i-b_{i+s+t})(b_{i+1}-b_{i+s+t})\cdots(b_{i+s-1}-b_{i+s+t})}{(b_i-b_{i+s})(b_{i+1}-b_{i+s})\cdots(b_{i+s-1}-b_{i+s})}.$$

Lemma 2.3. *([6, Lemma 2.3]) Let Γ be a d-bounded distance-regular graph with diameter d. For $0\le i\le i+s\le d$, suppose that Δ is a fixed subspace with diameter i+s in the Γ. Then the number of the subspaces with diameter i in Δ, denoted by $N'(i,i+s)$, is determined by i and s, independent of the choice of Δ; it is*

$$\frac{(b_0-b_{i+s})(b_1-b_{i+s})\cdots(b_{i-1}-b_{i+s})\left(1+\sum_{l=1}^{i+s}\frac{(b_0-b_{i+s})(b_1-b_{i+s})\cdots(b_{l-1}-b_{i+s})}{c_1c_2\cdots c_l}\right)}{(b_0-b_i)(b_1-b_i)\cdots(b_{i-1}-b_i)\left(1+\sum_{l=1}^{i}\frac{(b_0-b_i)(b_1-b_i)\cdots(b_{l-1}-b_i)}{c_1c_2\cdots c_l}\right)}.$$

Proof. By Proposition 2.2, for each $x\in V(\Delta)$, there are $N(0,i;i+s)$ subspaces with diameter i in Δ. Thus there are total $|V(\Delta)|N(0,i;i+s)$ such subspaces. But each of these subspaces repeats α times, where α equals the number of vertices in a subspace with diameter i. So the number of the subspaces with diameter i in Δ is $|V(\Delta)|N(0,i;i+s)/\alpha$. By Proposition 2.1,

$$|V(\Delta)|=1+\sum_{l=1}^{i+s}\frac{(b_0-b_{i+s})(b_1-b_{i+s})\cdots(b_{l-1}-b_{i+s})}{c_1c_2\cdots c_l},$$

$$\alpha = 1 + \sum_{l=1}^{i} \frac{(b_0-b_i)(b_1-b_i)\cdots(b_{l-1}-b_i)}{c_1c_2\cdots c_l}.$$

So we have the desired result. □

Let Γ be a d-bounded distance-regular graph with diameter $d \geq 2$ and let Δ be the subspace with diameter m in Γ, where $0 \leq m \leq d$. Let $\mathcal{L}_m(\Delta)$ denoted the all subspaces in Δ including the empty set $\emptyset$. If we define the partial order on $\mathcal{L}_m(\Delta)$ by reverse inclusion, then $\mathcal{L}_m(\Delta)$ is a poset, which is also denoted by $\mathcal{L}_m(\Delta)$.

Lemma 2.4. *Let Γ be a d-bounded distance-regular graph with diameter $d \geq 2$. Then the Möbius function of $\mathcal{L}_m(\Delta)$ is*

$$\mu(\Delta',\Delta'') = \begin{cases} (-1)^s \frac{(b_{i+1}-b_{i+s})(b_{i+2}-b_{i+s})\cdots(b_{i+s-1}-b_{i+s})}{(b_i-b_{i+1})(b_i-b_{i+2})\cdots(b_i-b_{i+s-1})}, & \text{if } \Delta' \leq \Delta'' \neq \emptyset, \\ \sum_{j=0}^{s-1}(-1)^{s-j}\frac{(b_{j+1}-b_{s-1})}{(b_j-b_{j+2})} \\ \quad \times \frac{(b_{j+2}-b_{s-1})\cdots(b_{s-2}-b_{s-1})}{(b_j-b_{j+1})\cdots(b_j-b_{s-2})}N'(j,s-1), & \text{if } \Delta' \leq \Delta'' = \emptyset, \\ 0, & \text{otherwise}, \end{cases}$$

where Δ', $\Delta'' \in \mathcal{L}_m(\Delta)$, $d(\Delta') = i+s$ and $d(\Delta'') = i$. In particular, we set $\mu(\Delta',\Delta'') = 1$ if $s = 0$ and $\mu(\Delta',\Delta'') = -1$ if $s = 1$.

Proof. For any $\Delta' \in \mathcal{L}_m(\Delta)$, it is obvious that $\mu(\Delta',\Delta') = 1$. For any $\Delta',\Delta'' \in \mathcal{L}_m(\Delta)$ with $\Delta' < \Delta'' \neq \emptyset$, $d(\Delta') = i+s$ and $d(\Delta'') = i$, similar to the proof of Theorem 4.4 of [5], we have

$$\mu(\Delta',\Delta'') = (-1)^s \frac{(b_{i+1}-b_{i+s})(b_{i+2}-b_{i+s})\cdots(b_{i+s-1}-b_{i+s})}{(b_i-b_{i+1})(b_i-b_{i+2})\cdots(b_i-b_{i+s-1})}. \quad (1)$$

For any $\Delta',\Delta'' \in \mathcal{L}_m(\Delta)$ with $\Delta' < \Delta'' = \emptyset$, $d(\Delta') = s-1$, we have by Lemma 2.3 and (1),

$$\begin{aligned} \sum_{\Delta' \leq \widetilde{\Delta} \leq \Delta''} \mu(\Delta',\widetilde{\Delta}) = 1 &+ (-1)^1 N'(s-2,s-1) \\ &+(-1)^2 \tfrac{b_{s-2}-b_{s-1}}{b_{s-3}-b_{s-2}} N'(s-3,s-1) \\ &+(-1)^3 \tfrac{(b_{s-3}-b_{s-1})(b_{s-2}-b_{s-1})}{(b_{s-4}-b_{s-3})(b_{s-4}-b_{s-2})} N'(s-4,s-1) \\ &+\cdots \\ &+(-1)^{s-1} \tfrac{(b_1-b_{s-1})(b_2-b_{s-1})\cdots(b_{s-2}-b_{s-1})}{(b_0-b_1)(b_0-b_2)\cdots(b_0-b_{s-2})} N'(0,s-1) \end{aligned}$$

$$+\mu(\Delta', \emptyset)$$
$$= 0. \qquad \square$$

Lemma 2.5. *Let Γ be a d-bounded distance-regular graph with $d \geq 2$. Then the eigenpolynomial on $\mathcal{L}_m(\Delta)$ is*

$$\chi(\mathcal{L}_m(\Delta), t) = \sum_{j=0}^{m} (-1)^{m-j} \frac{(b_{j+1}-b_m)(b_{j+2}-b_m)\cdots(b_{m-1}-b_m)}{(b_j-b_{j+1})(b_j-b_{j+2})\cdots(b_j-b_{m-1})} \times N'(j,m)(t^{j+1}-1).$$

Proof. It is clear that for any $\Delta' \in \mathcal{L}_m(\Delta)$, $r(\Delta) = m - d(\Delta')$ is the rank function on $\mathcal{L}_m(\Delta)$. So,

$$\chi(\mathcal{L}_m(\Delta), t) = \sum_{\Delta' \in \mathcal{L}_m(\Delta)} \mu(\Delta, \Delta') t^{r(\emptyset)-r(\Delta')}.$$

For Δ', $\Delta'' \in \mathcal{L}_m(\Delta)$ with $d(\Delta') = d(\Delta'')$, we have

$$t^{r(\emptyset)-r(\Delta')} = t^{d(\Delta')+1} = t^{d(\Delta'')+1} = t^{r(\emptyset)-r(\Delta'')}.$$

It follows from Lemma 2.4 and Lemma 2.3 that

$$\begin{aligned}
\chi(\mathcal{L}_m(\Delta), t) &= t^{m+1} + (-1)^1 N'(m-1,m)t^m \\
&\quad + (-1)^2 \frac{b_{m-1}-b_m}{b_{m-2}-b_{m-1}} N'(m-2,m)t^{m-1} \\
&\quad + \cdots \\
&\quad + (-1)^m \frac{(b_1-b_m)(b_2-b_m)\cdots(b_{m-1}-b_m)}{(b_0-b_1)(b_0-b_2)\cdots(b_0-b_{m-1})} N'(0,m)t^1 \\
&\quad + \sum_{j=0}^{m} (-1)^{m+1-j} \frac{(b_{j+1}-b_m)(b_{j+2}-b_m)\cdots(b_{m-1}-b_m)}{(b_j-b_{j+1})(b_j-b_{j+2})\cdots(b_j-b_{m-1})} N'(j,m) \\
&= \sum_{j=0}^{m} (-1)^{m-j} \frac{(b_{j+1}-b_m)(b_{j+2}-b_m)\cdots(b_{m-1}-b_m)}{(b_j-b_{j+1})(b_j-b_{j+2})\cdots(b_j-b_{m-1})} \\
&\quad \times N'(j,m)(t^{j+1}-1).
\end{aligned} \qquad \square$$

It is obvious that $\chi(\mathcal{L}_m(\Delta), t)$, denoted by $g_R(d(\Delta), t)$, is uniquely determined by $d(\Delta) = m$.

Let Γ be a d-bounded distance-regular graph with diameter $d \geq 2$ and let Δ be a fixed strongly closed subgraph with diameter i in Γ, $0 \leq i \leq d-1$. Suppose that $P(\Delta)$ is a set of all strongly closed subgraphs containing Δ in Γ. If the partial order on $P(\Delta)$ is defined by ordinary inclusion, $P(\Delta)$ is denoted by $P_O(\Delta)$.

Lemma 2.6. *([7] Lemma 2.9) Let Γ be a d-bounded distance-regular graph with diameter $d \geq 2$. Then the eigenpolynomial of $P_O(\Delta)$ is*

$$\chi(P_O(\Delta), t) = \sum_{s=0}^{d-i}(-1)^s \frac{b_i b_{i+1} \cdots b_{i+s-1}}{(b_i - b_{i+1})(b_i - b_{i+2}) \cdots (b_i - b_{i+s})} t^{d-i-s}.$$

It is obvious that $\chi(P_O(\Delta), t)$, denoted by $g_O(d(\Delta), t)$, is uniquely determined by $d(\Delta) = i$.

Proof of Theorem 1.1. For any $\Delta \in \mathcal{L}_R(\leq i)$, define

$$r_R(\Delta) = \begin{cases} i + 1 - d(\Delta), & \text{if} \Delta \neq \Gamma, \\ 0, & \text{if} \Delta = \Gamma. \end{cases}$$

It is clear that the function r_R is a rank function on $\mathcal{L}_R(\leq i)$. Write $\mathcal{L} = \mathcal{L}_R(\leq i)$. For $\Delta \in \mathcal{L}$, let

$$\mathcal{L}^\Delta = \{\Delta' \in \mathcal{L} | \Delta' \geq \Delta\}.$$

It is easy to see that $\mathcal{L}^\Gamma = \mathcal{L}$. So the eigenpolynomial on $\mathcal{L}$ be

$$\chi(\mathcal{L}, t) = \sum_{\Delta \in \mathcal{L}} \mu(\Gamma, \Delta) t^{r_R(\emptyset) - r_R(\Delta)} = \sum_{\Delta \in \mathcal{L}} \mu(\Gamma, \Delta) t^{i+2-r_R(\Delta)}.$$

By the Möbius inversion formula

$$t^{i+2} = \sum_{\Delta \in \mathcal{L}} \chi(\mathcal{L}^\Delta, t).$$

By Lemma 2.5, we can deduce that

$$\begin{aligned} \chi(\mathcal{L}, t) &= \chi(\mathcal{L}^\Gamma, t) \\ &= t^{i+2} - \sum_{\Delta \in \mathcal{L} \setminus \{\Gamma\}} \chi(\mathcal{L}^\Delta, t) \\ &= t^{i+2} - 1 - \sum_{\Delta \in \mathcal{L},\, 0 \leq d(\Delta) \leq i} \chi(\mathcal{L}^\Delta, t) \\ &= t^{i+2} - 1 - \sum_{m=0}^{i} N'(m, d) g_R(m, t). \end{aligned}$$

For any $\Delta \in \mathcal{L}_O(\geq i)$, define

$$r_O(\Delta) = \begin{cases} d(\Delta) - i + 1, & \text{if } \Delta \neq \emptyset, \\ 0, & \text{if } \Delta = \emptyset. \end{cases}$$

It is clear that the r_O is the rank function on $\mathcal{L}_O(\geq i)$. Write $P = \mathcal{L}_O(\geq i)$. For $\Delta \in P$, define the set P^Δ as follows:

$$P^{\Delta} = \{\widetilde{\Delta} \in P | \Delta \subseteq \widetilde{\Delta}\} = \{\widetilde{\Delta} \in P | \Delta \leq \widetilde{\Delta}\}.$$

It is clear that $P^{\emptyset} = P$. For any $\Delta \in P \backslash \{\emptyset\}$, P^{Δ} is the set of all subspaces containing Δ in Γ. It follows from Lemma 2.6 that

$$\chi(P^{\Delta}, t) = g_O(d(\Delta), t).$$

By the definition of eigenpolynomial on P,

$$\chi(P, t) = \chi(P^{\emptyset}, t) = \sum_{\Delta \in P^{\emptyset}} \mu(\emptyset, \Delta) t^{ro(\Gamma) - ro(\Delta)}.$$

By the Möbius inversion formula,

$$t^{d-i+1} = \sum_{\Delta \in P^{\emptyset}} \chi(P^{\Delta}, t).$$

It follows from Lemma 2.6 and Lemma 2.3 that

$$\begin{aligned} \chi(P, t) &= t^{d-i+1} - \sum_{\Delta \in P \backslash \{\emptyset\}} \chi(P^{\Delta}, t) \\ &= t^{d-i+1} - \sum_{\Delta \in P,\, d(\Delta) \geq i} \chi(P^{\Delta}, t) \\ &= t^{d-i+1} - \sum_{l=i}^{d} N'(l, d) g_O(l, t). \end{aligned}$$

□

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