Binding Number and Fractional k-Factors of Graphs *

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Abstract

Let G be a graph with vertex set $V(G)$, and let $k \geq 2$ be an integer. A spanning subgraph F of G is called a fractional k-factor if $d_G^h(x) = k$ for all $x \in V(G)$, where $d_G^h(x) = \sum_{e \in E_x} h(e)$ is the fractional degree of $x \in V(F)$ with $E_x = \{e : e = xy \in E(G)\}$. The binding number $bind(G)$ is defined as follows,

$$bind(G) = min\{\frac{|N_G(X)|}{|X|} : \varnothing \neq X \subseteq V(G), N_G(X) \neq V(G)\}.$$

In this paper, a binding number condition for a graph to have fractional k-factors is given.

Keywords: graph, binding number, factor, fractional k-factor.

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1 Introduction

We consider only finite undirected simple graph G with vertex set $V(G)$ and edge set $E(G)$. For $x \in V(G)$, the degree of x in G is denoted by $d_G(x)$.

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The minimum vertex degree of G is denoted by $\delta(G)$. For any $S \subseteq V(G)$, we denote by $N_G(S)$ the neighborhood set of S in G, by $G[S]$ the subgraph of G induced by S, by $G - S$ the subgraph obtained from G by deleting vertices in S together with the edges incident to vertices in S. A vertex set $S \subseteq V(G)$ is called independent if $G[S]$ has no edges. The binding number of G is defined by Woodall [1] as

$$bind(G) = min\{\frac{|N_G(X)|}{|X|} : \varnothing \neq X \subseteq V(G), N_G(X) \neq V(G)\}.$$

Let k be an integer such that $k \geq 1$. Then a spanning subgraph F of G is called a k-factor if $d_F(x) = k$ for all $x \in V(G)$. If $k = 1$, then a k-factor is simply called a 1-factor. A fractional k–factor is a function h that assigns to each edge of a graph G a number in [0,1], so that for each vertex x we have $d_G^h(x) = k$, where $d_G^h(x) = \sum_{e \ni x} h(e)$ (the sum is taken over all edges incident to x) is a fractional degree of x in G. If $k = 1$, then a fractional k-factor is a fractional 1-factor. The other terminologies and notations not given in this paper can be found in [2,3].

Many authors have investigated k-factors [4-7], and fractional factors[8,9]. In [7], P. Katerinis and D. R. Woodall gave a binding number condition for a graph to have a k-factor. Recently, Sizhong Zhou obtained some sufficient conditions for graphs to have factors or fractional factors [10-12]. There is a necessary and sufficient condition for a graph to have a fractional k-factor which was given by Guizhen Liu [13].

Theorem 1 [13] *Let G be a graph. Then G has a fractional k-factor if and only if for every subset S of $V(G)$,*

$$\delta_G(S,T) = k|S| - k|T| + d_{G-S}(T) \geq 0,$$

where $T = \{x : x \in V(G) \setminus S, d_{G-S}(x) \leq k\}$.

In [1], D. R. Woodall gave the following result.

Theorem 2 [1] *Let G be a graph of order n with $bind(G) > c$. Then $\delta(G) > n - \frac{n-1}{c}$.*

In [7], P. Katerinis and D. R. Woodall gave a binding number condition for a graph to have a k-factor.

Theorem 3 [7] *Let k be an integer such that $k \geq 2$, and let G be a graph of order n such that $n \geq 4k - 6$, kn is even, and $bind(G) > \frac{(2k-1)(n-1)}{k(n-2)+3}$. Then G has a k-factor.*

In this paper, we give a binding number condition for a graph to have fractional k-factors. Our main result is a similar that of Theorem 3.

Theorem 4 *Let $k \geq 2$ be an integer, and let G be a graph of order n such that $n \geq 4k-6$. Then*
(1) If kn is even, and $bind(G) > \frac{(2k-1)(n-1)}{k(n-2)+3}$, then G has a fractional k-factor; and
(2) If kn is odd, and $bind(G) > \frac{(2k-1)(n-1)}{k(n-2)+2}$, then G has a fractional k-factor.

2 The Proof of Theorem 4

Proof. If kn is even, and $bind(G) > \frac{(2k-1)(n-1)}{k(n-2)+3}$. By Theorem 3, G has a k-factor. We have known that a k-factor is a special fractional k-factor. Thus, G has a fractional k-factor. In the following, we prove (2).

Suppose that G does not have a fractional k-factor. Then, according to Theorem 1, there exists some $S \subseteq V(G)$ such that

$$\delta_G(S,T) = k|S| - k|T| + d_{G-S}(T) \leq -1, \tag{1}$$

where $T = \{x : x \in V(G) \setminus S, d_{G-S}(x) \leq k\}$. We choose such subsets S and T so that $|T|$ is as small as possible.

Claim 1. $|T| \geq k+1$.

Proof. According to Theorem 2, we have

$$\begin{aligned}
|S| + d_{G-S}(x) &\geq d_G(x) \geq \delta(G) > n - \frac{n-1}{bind(G)} \\
&\geq \frac{n(k-1)+2k-3}{2k-1} \\
&\geq \frac{(4k-6)(k-1)+2k-3}{2k-1} \\
&\geq 2k-3 \geq k \qquad (\textit{since } k \geq 2 \textit{ is odd}).
\end{aligned}$$

If $|T| \leq k$, then by (1) we obtain

$$\begin{aligned}
-1 &\geq \delta_G(S,T) = k|S| + d_{G-S}(T) - k|T| \\
&\geq |T||S| + d_{G-S}(T) - k|T| \\
&= \sum_{x \in T}(|S| + d_{G-S}(x) - k) \geq 0,
\end{aligned}$$

which is a contradiction.

Claim 2. $d_{G-S}(x) \le k-1$ for all $x \in T$.

Proof. If $d_{G-S}(x) \ge k$ for some $x \in T$, then the subsets S and $T \setminus \{x\}$ satisfy (1). This contradicts the choice of S and T.

Define

$$h = \min\{d_{G-S}(x) | x \in T\}.$$

Then by Claim 2, we have

$$0 \le h \le k-1.$$

Choose $x_1 \in T$ such that $d_{G-S}(x_1) = h$. The proof splits into two cases.

Case 1 $1 \le h \le k-1$

Let $Y = (V(G) \setminus S) \setminus N_{G-S}(x_1)$. Then $x_1 \in Y \setminus N_G(Y)$, so $Y \ne \emptyset$ and $N_G(Y) \ne V(G)$, and $|N_G(Y)| \ge bind(G)|Y|$. Thus, we obtain

$$n-1 \ge |N_G(Y)| \ge bind(G)|Y| = bind(G)(n-|S|-h),$$

i.e.

$$|S| \ge n-h-\frac{n-1}{bind(G)} > n-h-\frac{k(n-2)+2}{2k-1}. \qquad (2)$$

Subcase 1.1 $3 \le h \le k-1$

By (1) and (2), and $|T| \le n-|S|$, we have

$$\begin{aligned}
-1 &\ge \delta_G(S,T) = k|S| + d_{G-S}(T) - k|T| \\
&\ge k|S| - k|T| + h|T| \\
&= k|S| - (k-h)|T| \\
&\ge k|S| - (k-h)(n-|S|) \\
&= (2k-h)|S| - kn + hn \\
&> (2k-h)(n-h-\frac{k(n-2)+2}{2k-1}) - kn + hn \\
&= kn - (2k-h)h - (2k-h)\frac{k(n-2)+2}{2k-1}.
\end{aligned}$$

Let $f(h) = kn - (2k-h)h - (2k-h)\frac{k(n-2)+2}{2k-1}$. Then

$$f'(h) = -2k + 2h + \frac{k(n-2)+2}{2k-1}.$$

Since $3 \le h \le k-1$, we have

$$\begin{aligned} f'(h) &\ge -2k+6+\frac{k(n-2)+2}{2k-1} \\ &= \frac{-4k^2+2k+12k-6+kn-2k+2}{2k-1} \\ &= \frac{kn-4k^2+12k-4}{2k-1} \\ &\ge \frac{k(4k-6)-4k^2+12k-4}{2k-1} \\ &= \frac{6k-4}{2k-1} > 0. \end{aligned}$$

Thus, we get

$$f(h) \ge f(3),$$

i.e.

$$\begin{aligned} -1 &> f(h) \ge f(3) = kn-3(2k-3)-(2k-3)\frac{k(n-2)+2}{2k-1} \\ &= \frac{2k^2n-kn-(6k-9)(2k-1)-(2k-3)(k(n-2)+2)}{2k-1} \\ &= \frac{2kn-8k^2+14k-3}{2k-1} \\ &\ge \frac{2k(4k-6)-8k^2+14k-3}{2k-1} = \frac{2k-3}{2k-1} > 0, \end{aligned}$$

which is a contradiction.

Subcase 1.2 $h=2$

Claim 3. $(k-2)(2k-1)|T| \le k[(n-2)(2k-1)-(k(n-2)+2)]+(2k-1)$, that is, $|T| \le \frac{k}{k-2}(n-2-\frac{k(n-2)+2}{2k-1})+\frac{1}{k-2}$.

Proof. If $(k-2)(2k-1)|T| \ge k[(n-2)(2k-1)-(k(n-2)+2)]+(2k-1)+1$, that is, $|T| \ge \frac{k}{k-2}(n-2-\frac{k(n-2)+2}{2k-1})+\frac{1}{k-2}+\frac{1}{(2k-1)(k-2)}$. Then, by (2), we obtain

$$\begin{aligned} |S|+|T| &> n-2-\frac{k(n-2)+2}{2k-1}+\frac{k}{k-2}(n-2-\frac{k(n-2)+2}{2k-1}) \\ &\quad +\frac{1}{k-2}+\frac{1}{(2k-1)(k-2)} \\ &= \frac{2k-2}{k-2}(n-2-\frac{k(n-2)+2}{2k-1})+\frac{1}{k-2} \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{(2k-1)(k-2)} \\
= \;& n+\frac{kn-4k^2+6k-1}{(2k-1)(k-2)}+\frac{1}{(2k-1)(k-2)} \\
\geq \;& n+\frac{k(4k-6)-4k^2+6k-1}{(2k-1)(k-2)}+\frac{1}{(2k-1)(k-2)} \\
= \;& n-\frac{1}{(2k-1)(k-2)}+\frac{1}{(2k-1)(k-2)}=n.
\end{aligned}
$$

This contradicts $|S|+|T|\leq n$.

By combining Claim 3 with (1) and (2), we obtain

$$
\begin{aligned}
-1 \;\geq\;& \delta_G(S,T)=k|S|+d_{G-S}(T)-k|T| \\
\geq\;& k|S|-k|T|+2|T| \\
=\;& k|S|-(k-2)|T| \\
>\;& k(n-2-\frac{k(n-2)+2}{2k-1}) \\
&-(k-2)(\frac{k}{k-2}(n-2-\frac{k(n-2)+2}{2k-1})+\frac{1}{k-2}) \\
=\;& -1,
\end{aligned}
$$

a contradiction.

Subcase 1.3 $h=1$

Subcase 1.3.1 $|T|\leq \frac{k}{k-1}(n-1-\frac{k(n-2)+2}{2k-1})+\frac{1}{k-1}$

By combining this with (1) and (2), we have that

$$
\begin{aligned}
-1 \;\geq\;& \delta_G(S,T)=k|S|+d_{G-S}(T)-k|T| \\
\geq\;& k|S|-k|T|+|T| \\
=\;& k|S|-(k-1)|T| \\
>\;& k(n-1-\frac{k(n-2)+2}{2k-1}) \\
&-(k-1)(\frac{k}{k-1}(n-1-\frac{k(n-2)+2}{2k-1})+\frac{1}{k-1}) \\
=\;& -1,
\end{aligned}
$$

a contradiction.

Subcase 1.3.2 $|T|> \frac{k}{k-1}(n-1-\frac{k(n-2)+2}{2k-1})+\frac{1}{k-1}$

In view of (2), we obtain

$$
|S|+|T| \;>\; n-1-\frac{k(n-2)+2}{2k-1}
$$

$$
\begin{aligned}
& +\frac{k}{k-1}(n-1-\frac{k(n-2)+2}{2k-1})+\frac{1}{k-1} \\
> & \frac{2k-1}{k-1}(n-1-\frac{k(n-2)+2}{2k-1})+\frac{1}{k-1} \\
= & \frac{kn-n-1}{k-1}+\frac{1}{k-1}=n.
\end{aligned}
$$

This contradicts $|S|+|T|\leq n$.

Case 2 $h=0$

Let m be the number of vertices x in T such that $d_{G-S}(x)=0$, and let $Y=V(G)\setminus S$. Then $N_G(Y)\neq V(G)$ since $h=0$, and $Y\neq\varnothing$ by Claim 1, and so $|N_G(Y)|\geq bind(G)|Y|$. Thus

$$n-m\geq |N_G(Y)|\geq bind(G)|Y|=bind(G)(n-|S|).$$

So

$$|S|\geq n-\frac{n-m}{bind(G)}. \tag{3}$$

In view of (1) and (3), and $|T|\leq n-|S|$, we get that

$$
\begin{aligned}
-1 &\geq \delta_G(S,T)=k|S|+d_{G-S}(T)-k|T| \\
&\geq k|S|-k|T|+|T|-m \\
&\geq k|S|-(k-1)(n-|S|)-m \\
&= (2k-1)|S|-kn+n-m \\
&\geq (2k-1)(n-\frac{n-m}{bind(G)})-kn+n-m \\
&= 2kn-n-\frac{n(2k-1)}{bind(G)}+\frac{m(2k-1)}{bind(G)}-kn+n-m \\
&= kn-\frac{n(2k-1)}{bind(G)}+\frac{m(2k-1)}{bind(G)}-m \\
&\geq kn-\frac{n(2k-1)}{bind(G)}+\frac{2k-1}{bind(G)}-1 \\
&= kn-\frac{(n-1)(2k-1)}{bind(G)}-1 \\
&> kn-(k(n-2)+2)-1=2k-3>0.
\end{aligned}
$$

This is a contradiction.

From all the cases above, we deduced the contradiction. Hence, G has a fractional k-factor.
Completing the proof of Theorem 4.

Remark. Let kn be even. Then, let us show that the condition $bind(G) > \frac{(2k-1)(n-1)}{k(n-2)+3}$ in Theorem 4 can not be replaced by $bind(G) \geq \frac{(2k-1)(n-1)}{k(n-2)+3}$. Let r be a positive integer and let $l = rk-1$ and $m = 2l-2r$, so that $km = 2l(k-1)-2$ and $n = m+2l = (4k-2)r-4$. Let $H = K_m \bigvee lK_2$. Let $X = V(lK_2)$. Then for any $x \in X$, $|N_H(X \setminus x)| = n-1$. By the definition of $bind(H)$, $bind(H) = \frac{|N_H(X\setminus x)|}{|X\setminus x|} = \frac{n-1}{2l-1} = \frac{n-1}{2rk-3} = \frac{(2k-1)(n-1)}{k(n-2)+3}$. Let $S = V(K_m) \subseteq V(H)$, $T = V(lK_2) \subseteq V(H)$. Then $|S| = m$, $|T| = 2l$. Thus, we get

$$\begin{aligned}\delta_H(S,T) &= k|S| - k|T| + d_{H-S}(T)\\ &= k|S| - k|T| + |T| = k|S| - (k-1)|T|\\ &= km - 2(k-1)l = -2 < 0.\end{aligned}$$

By Theorem 1, there are not any fractional k-factors in H. In the above sense, the result in Theorem 4 is best possible.

Let kn be odd. Then, let us show that the condition $bind(G) > \frac{(2k-1)(n-1)}{k(n-2)+2}$ in Theorem 4 can not be replaced by $bind(G) \geq \frac{(2k-1)(n-1)}{k(n-2)+2}$. Let $r \geq 1, k \geq 3$ be two odd positive integer and let $l = \frac{5kr-1}{2}$ and $m = 5kr-5r-1$, so that $n = m+2l = (10k-5)r-2$. Clearly, n is odd. Let $H = K_m \bigvee lK_2$. Let $X = V(lK_2)$. Then for any $x \in X$, $|N_H(X\setminus x)| = n-1$. By the definition of $bind(H)$, $bind(H) = \frac{|N_H(X\setminus x)|}{|X\setminus x|} = \frac{n-1}{2l-1} = \frac{n-1}{5kr-2} = \frac{(2k-1)(n-1)}{k(n-2)+2}$. Let $S = V(K_m) \subseteq V(H)$, $T = V(lK_2) \subseteq V(H)$. Then $|S| = m$, $|T| = 2l$. Thus, we get

$$\begin{aligned}\delta_H(S,T) &= k|S| - k|T| + d_{H-S}(T)\\ &= k|S| - k|T| + |T| = k|S| - (k-1)|T|\\ &= km - 2(k-1)l = k(5kr-5r-1) - (k-1)(5kr-1)\\ &= -1 < 0.\end{aligned}$$

By Theorem 1, there are not any fractional k-factors in H. In the above sense, the result in Theorem 4 is best possible.

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