A lower bound for 2-rainbow domination number of generalized Petersen graphs P(n,3)*

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Abstract

Assume we have a set of k colors and we assign an arbitrary subset of these colors to each vertex of a graph G. If we require that each vertex to which an empty set is assigned has in its neighborhood all k colors, then this assignment is called the k-rainbow dominating function of a graph G. The minimum sum of numbers of assigned colors over all vertices of G, denoted as $\gamma_{rk}(G)$, is called the k-rainbow domination number of G. In this paper, we prove that $\gamma_{r2}(P(n,3)) \geq \lceil \frac{7n}{8} \rceil$.
Keywords: Domination, 2-Rainbow domination, Generalized Petersen graph

1 Introduction

We only consider finite undirected graphs without loops or multiple edges.

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Let $G = (V(G), E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$. The *open neighborhood* of $v \in V(G)$ is denoted by $N(v) = \{u \in V(G) | uv \in E(G)\}$, and its *closed neighborhood* is denoted by $N[v] = N(v) \cup \{v\}$.

Let f be a function that assigns to each vertex a set of colors chosen from the set $\{1, \ldots, k\}$; that is, $f : V(G) \rightarrow \mathcal{P}(\{1, \ldots, k\})$. If $\bigcup_{u \in N(v)} f(u) = \{1, \ldots, k\}$ for each vertex $v \in V(G)$ with $f(v) = \emptyset$, then f is called a *k-rainbow dominating function* (kRDF) of G. The *weight*, $w(f)$, of a function f is defined as $w(f) = \sum_{v \in V(G)} |f(v)|$. The minimum weight of a kRDF is called the *k-rainbow domination number* of G, which is denoted by $\gamma_{rk}(G)$. For $k = 1$, this concept coincides with the ordinary domination.

Brešar, Henning and Rall [2, 4] introduced the concept of 2-rainbow domination of a graph G and connected this concept to usual domination in (products of) graphs. Brešar and Šumenjak [3] found the exact values of 2-rainbow domination number of paths, cycles and suns.

For $1 \leq k \leq n-1$, *the generalized Petersen graph* $P(n,k)$, defined by Watkins [10], is a graph on $2n$ ($n \geq 3$) vertices with $V(P(n,k)) = \{v_i, u_i : 0 \leq i \leq n-1\}$ and $E(P(n,k)) = \{v_i v_{i+1}, v_i u_i, u_i u_{i+k} : 0 \leq i \leq n-1$, subscripts modulo $n\}$.

Domination and its variations of the generalized Petersen graphs have been studied extensively in recent years[1, 5–8, 12, 13]. Brešar and Šumenjak [3] determined that $\lceil \frac{4n}{5} \rceil \leq \gamma_{r2}(P(n,k)) \leq n$ for relatively prime numbers n and k, and $\lceil \frac{4n}{5} \rceil \leq \gamma_{r2}(P(n,2)) \leq \lceil \frac{4n}{5} \rceil + \alpha$, where $\alpha = 0$ for n (mod 10)= 3,9 and $\alpha = 1$ for n (mod 10)=1, 5, 7. Tong et al. [9] further determined that $\gamma_{r2}(P(n,2)) = \lceil \frac{4n}{5} \rceil + \alpha$, where $\alpha = 0$ for n (mod 10)=0, 4 and $\alpha = 1$ for n (mod 10)= 1,2,5,6,7,8. Xu [11] showed that $\gamma_{r2}(P(n,3)) \leq \lceil \frac{7n}{8} \rceil + \alpha$, where $\alpha = 0$ for n (mod 16)=0,2,4,5,6,7,13,14,15 and $\alpha = 1$ for n (mod 16)=1,3,8,9,10,11,12. And he suspected that $\gamma_{r2}(P(n,3))$ is equal to the upper bound. In this paper, we will prove that $\gamma_{r2}(P(n,3)) \geq \lceil \frac{7n}{8} \rceil$. This is consistent with the suspicion of [11].

2 The lower bound for 2-Rainbow Domination Number of $P(n,3)$

Let

$$\begin{aligned} V_0 &= \{v \in V(P(n,3)) : f(v) = \emptyset\}, \\ V_1 &= \{v \in V(P(n,3)) : f(v) \in \{\{1\}, \{2\}\}\}, \\ V_2 &= \{v \in V(P(n,3)) : f(v) = \{1,2\}\}, \end{aligned}$$

$$\begin{aligned} V_{i_1 i_2} &= \{v \in V_0 : |N(v) \cap V_t| = i_t, t = 1,2\}, \\ E_1 &= \{uv \in E(P(n,3)) : u, v \in V_1\}, \ , \\ E_2 &= \{uv \in E(P(n,3)) : u, v \in V_2\}, \\ E_{12} &= \{uv \in E(P(n,3)) : u \in V_1, v \in V_2\}. \end{aligned}$$

Then

$$\begin{aligned} V_0 &= \bigcup_{S \in \{V_{20}, V_{30}, V_{01}, V_{11}, V_{21}, V_{02}, V_{12}, V_{03}\}} S, \\ S_1 \bigcap S_2 &= \emptyset, S_1 \neq S_2 \text{ and } S_1, S_2 \in \{V_{20}, V_{30}, V_{01}, V_{11}, V_{21}, V_{02}, V_{12}, V_{03}\}, \\ V(P(n,3)) &= \bigcup_{S \in \{V_0, V_1, V_2\}} S, \\ S_1 \bigcap S_2 &= \emptyset, \quad S_1 \neq S_2 \text{ and } S_1, S_2 \in \{V_0, V_1, V_2\}, \end{aligned}$$

and

$$\begin{aligned} &(1) \quad 3|V_1| - 2|E_1| - |E_{12}| = 2|V_{20}| + 3|V_{30}| + |V_{11}| + 2|V_{21}| + |V_{12}|, \\ &(2) \quad 3|V_2| - 2|E_2| - |E_{12}| = |V_{01}| + |V_{11}| + |V_{21}| + 2|V_{02}| + 2|V_{12}| + 3|V_{03}|. \end{aligned}$$

By $(1) + 2 \times (2)$, we have

$$\begin{aligned} & 3|V_1| + 6|V_2| - 2|E_1| - 4|E_2| - 3|E_{12}| \\ = \ & 2|V_{20}| + 3|V_{30}| + 2|V_{01}| + 3|V_{11}| + 4|V_{21}| + 4|V_{02}| + 5|V_{12}| + 6|V_{03}|. \end{aligned}$$

It follows

$$\begin{aligned} 3|V_1| + 6|V_2| &= 2|V_{20}| + 3|V_{30}| + 2|V_{01}| + 3|V_{11}| + 4|V_{21}| + 4|V_{02}| + 5|V_{12}| \\ &\quad +6|V_{03}| + 2|E_1| + 4|E_2| + 3|E_{12}|, \\ 3|V_1| + 6|V_2| &= 2(|V_{20}| + |V_{30}| + |V_{01}| + |V_{11}| + |V_{21}| + |V_{02}| + |V_{12}| + |V_{03}|) \\ &\quad +|V_{30}| + |V_{11}| + 2|V_{21}| + 2|V_{02}| + 3|V_{12}| + 4|V_{03}| + 2|E_1| \\ &\quad +4|E_2| + 3|E_{12}|, \\ 3|V_1| + 6|V_2| &= 2(2n - |V_1| - |V_2|) + |V_{30}| + |V_{11}| + 2|V_{21}| + 2|V_{02}| + 3|V_{12}| \\ &\quad +4|V_{03}| + 2|E_1| + 4|E_2| + 3|E_{12}|, \\ 5|V_1| + 10|V_2| &= 4n + 2|V_2| + |V_{30}| + |V_{11}| + 2|V_{21}| + 2|V_{02}| + 3|V_{12}| + 4|V_{03}| \\ &\quad +2|E_1| + 4|E_2| + 3|E_{12}|, \\ 5w(f) &= 4n + 2|V_2| + |V_{30}| + |V_{11}| + 2|V_{21}| + 2|V_{02}| + 3|V_{12}| + 4|V_{03}| \\ &\quad +2|E_1| + 4|E_2| + 3|E_{12}|. \end{aligned}$$

Let

$$\beta = 2|V_2| + |V_{30}| + |V_{11}| + 2|V_{21}| + 2|V_{02}| + 3|V_{12}| + 4|V_{03}| + 2|E_1| + 4|E_2| + 3|E_{12}|.$$

Then we have Lemma 2.1.

Lemma 2.1. $5w(f) = 4n + \beta$. □

Let

$$V'(i,l) = \{v_j, u_j : i \leq j \leq i + l - 1, 0 \leq l \leq n - 1\}.$$

For $S \subseteq V$, let

$$
\begin{aligned}
n_1(S) &= |\{uv : uv \in E_1, u, v \in S\}|, \\
n_1'(S) &= |\{uv : uv \in E_1, |\{u,v\} \cap S| = 1\}|, \\
n_2(S) &= |\{uv : uv \in E_2, u, v \in S\}|, \\
n_2'(S) &= |\{uv : uv \in E_2, |\{u,v\} \cap S| = 1\}|, \\
n_{12}(S) &= |\{uv : uv \in E_{12}, u, v \in S\}|, \\
n_{12}'(S) &= |\{uv : uv \in E_{12}, |\{u,v\} \cap S| = 1\}|, \\
\beta(S) &= 2|V_2 \cap S| + |V_{30} \cap S| + |V_{11} \cap S| + 2|V_{21} \cap S| + 2|V_{02} \cap S| \\
&\quad +3|V_{12} \cap S| + 4|V_{03} \cap S| + 2n_1(S) + n_1'(S) + 4n_2(S) + 2n_2'(S) \\
&\quad +3n_{12}(S) + \tfrac{3}{2}n_{12}'(S).
\end{aligned}
$$

We have Lemma 2.2.

Lemma 2.2. If $\beta(V'(i,4)) \leq 1$, then $\beta(V'(i-4,8)) \geq 5$ or $\beta(V'(i,8)) \geq 3$ or $\beta(V'(i,8)) = 2 \wedge \beta(V'(i,12)) \geq 5$.

Proof. Suppose to the contrary that $\beta(V'(i-4,8)) \leq 4$, $\beta(V'(i,8)) \leq 2$ and $(\beta(V'(i,8)) \leq 1 \vee \beta(V'(i,12)) \leq 4)$. By symmetry, we may assume i=4. Since $\beta(V'(4,4)) \leq 1$, we have $v_j, u_j \notin V_2$ $(4 \leq j \leq 7)$, $v_j u_j \notin E_1$ $(4 \leq j \leq 7)$, $v_i v_{i+1} \notin E_1$ $(4 \leq j \leq 6)$ and $u_i u_{i+3} \notin E_1$ $(4 \leq j \leq 6)$. We need only consider six cases as follows:

Case 1. $v_4 \in V_0$, $u_4 \in V_0$, $v_5 \in V_0$ and $u_5 \in V_1$. Then $v_3 \in V_2$ and $v_6 \in V_1$. It follows $u_6, v_7 \in V_0$. Since $\beta(V'(0,8)) \leq 4$, $u_3 \in V_0$. Since $\beta(V'(4,8)) \leq 2$, $u_9 \in V_1$. If $u_7 \in V_0$, then $v_8 \in V_1$ and $u_{10} \in V_2$. It follows $u_8, v_9 \in V_0$. Since $f(v_8) \neq f(v_6)$ and $f(u_9) \neq f(v_6)$, we have $f(u_9) = f(v_8)$. It follows $v_{10} \notin V_0$, $\beta(V'(4,8)) \geq 5$ (see Fig.1(1), where white dot, black dot and odot(a small black dot inside a white dot) stand for the vertex of V_0, V_1, V_2 respectively), a contradiction. Hence $u_7 \in V_1$. It follows $v_8 \notin V_2$.
Case 1.1. $v_8 \in V_0$. Since $\beta(V'(4,8)) \leq 2$, $u_8, v_9 \notin V_2$. It follows $u_8, v_9 \in V_1$, $\beta(V'(4,8)) \geq 4$ (see Fig.1(2)), a contradiction.
Case 1.2. $v_8 \in V_1$. Since $\beta(V'(4,8)) \leq 2$, $u_8, v_9, u_{10} \in V_0$ and $v_{10}, v_{11} \notin V_2$. It forces $v_{10} \in V_1$. It follows $v_{11}, u_{11}, u_{12}, u_{13} \in V_0$, $v_{12} \in V_1$, $u_{14} \in V_2$ and at least one vertex of $\{v_{13}, v_{14}\}$ has to belong to $V_1 \cup V_2$, $\beta(V'(4,8)) = 2 \wedge \beta(V'(4,12)) \geq 7$(see Fig.1(3)), a contradiction.

Case 2. $v_4 \in V_0$, $u_4 \in V_0$, $v_5 \in V_1$ and $u_5 \in V_0$. Then $v_6 \in V_0$ and $v_3 \notin V_0$. Since $\beta(V'(i-4,8)) \leq 4$, we have $u_3 \notin V_2$.
Case 2.1. $u_6 \in V_0$. Then $v_7 \in V_1$. It follows $u_7 \in V_0$, $u_1 \in V_2$. Since $\beta(V'(0,8)) \leq 4$, $v_1, v_2 \in V_0$ and $u_2, v_3 \in V_1$. Since $f(u_2) \neq f(v_3)$ and $f(v_5) \neq f(v_3)$, we have $f(u_2) = f(v_5)$. It follows $u_8 \in V_1$. Since $\beta(V'(0,8)) \leq 4$, $u_3 \in V_0$. It follows $u_9 \in V_2$, $\beta(V'(4,8)) \geq 3$ (see Fig.2(1)), a contradiction.
Case 2.2. $u_6 \in V_1$. Since $\beta(V'(4,8)) \leq 2$, $u_9 \notin V_2$.

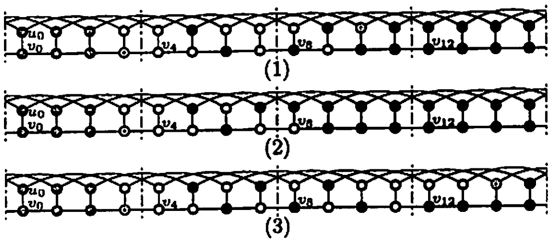

Fig. 1. The case for $v_4 \in V_0$, $u_4 \in V_0$, $v_5 \in V_0$ and $u_5 \in V_1$

Case 2.2.1. $v_7 \in V_0$. Since $\beta(V'(4,8)) \le 2$, $|\{v_8, u_{10}\} \cap V_2| \ne 2$. It follows $u_7, v_8 \in V_1$.

If $u_8 \in V_1$, then $u_9, v_9, u_{10} \in V_0$ and $v_{10} \in V_1$. since $f(u_7) \ne f(v_8)$ and $f(v_{10}) \ne f(v_8)$, we have $f(u_7) = f(v_{10})$. It follows $u_{13} \in V_1$, $\beta(V'(4,8)) \ge 3$ (see Fig.2(2)), a contradiction. Hence $u_8 \in V_0$. Since $\beta(V'(4,8)) \le 2$, $u_{11} \in V_1$ and $v_9 \notin V_2$.

If $v_9 \in V_1$, then $u_9, u_{10}, v_{10} \in V_0$. It follows $v_{11} \in V_1$, $\beta(V'(4,8)) \ge 4$ (see Fig.2(3)), a contradiction. Hence, $v_9 \in V_0$.

If $u_9 \in V_1$, then $v_{10}, u_{10} \in V_0$. It follows $v_{11} \in V_2$, $\beta(V'(4,8)) \ge 4$ (see Fig.2(4)), a contradiction. Hence, $u_9 \in V_0$. It follows $v_{10}, u_{11} \in V_1$, $v_{11} \in V_0$ and $u_{12} \notin V_0$. Since $f(u_7) \ne f(v_8)$ and $f(v_{10}) \ne f(v_8)$, we have $f(u_7) = f(v_{10})$. It forces $u_{13} \in V_1$. Since $f(v_8) \ne f(v_{10})$ and $f(v_8) \ne f(u_{11})$, we have $f(v_{10}) = f(u_{11})$. It forces $v_{12} \in V_1$, $v_{13} \in V_0$ and $u_{12} \in V_1$. Since $f(v_{10}) \ne f(v_{12})$ and $f(v_{10}) \ne f(u_{13})$, we have $f(v_{12}) = f(u_{13})$. It forces $v_{14} \in V_1$, $\beta(V'(4,8)) = 2 \wedge \beta(V'(4,12)) \ge 5$ (see Fig.2(5)), a contradiction.

Case 2.2.2. $v_7 \in V_1$. Then $v_3 \in V_1$ and $u_7, v_8 \in V_0$ and $u_1 \in V_2$. Since $\beta(V'(4,8)) \le 2$, $u_{11} \notin V_2$. It follows $u_8 \in V_1$. Since $\beta(V'(0,8)) \le 4$, $v_1, v_2 \in V_0$. It follows $u_2 \in V_1$, $\beta(V'(4,4)) \ge 2$ (see Fig.2(6)), a contradiction.

Case 3. $v_4 \in V_0$, $u_4 \in V_1$, $v_5 \in V_0$ and $u_5 \in V_1$. Then $v_6 \in V_1$, $u_6, v_7, u_7 \in V_0$. Since $\beta(V'(4,8)) \le 2$, $v_8, u_{10} \notin V_2$. It follows $v_8, u_{10} \in V_1$ and $u_8 \in V_0$. Since $f(u_5) \ne f(v_6)$ and $f(v_8) \ne f(v_6)$, we have $f(u_5) = f(v_8)$. It forces $u_{11} \in V_1$ and at least one vertex of $\{v_9, v_{10}, v_{11}\}$ has to belong to $V_1 \cup V_2$, $\beta(V'(4,8)) \ge 3$ (see Fig.3), a contradiction.

Case 4. $v_4 \in V_0$, $u_4 \in V_1$, $v_5 \in V_1$ and $u_5 \in V_0$. Then $v_6, u_7 \in V_0$.
Case 4.1. $u_6 \in V_0$. Then $v_7 \in V_1$.
Case 4.1.1. $v_3 \in V_0$. Since $f(u_4) \ne f(v_5)$ and $f(v_7) \ne f(v_5)$, we have $f(u_4) = f(v_7)$. It follows $u_{10} \in V_1$. Since $\beta(V'(4,8)) \le 2$, $v_8, v_{10} \in V_0$, $u_9, u_{11} \notin V_2$. It forces $u_8, v_9 \in V_1$, $u_9, v_{11}, u_{11} \in V_0$, $v_{12} \in V_2$ and $u_{12} \in V_1$,

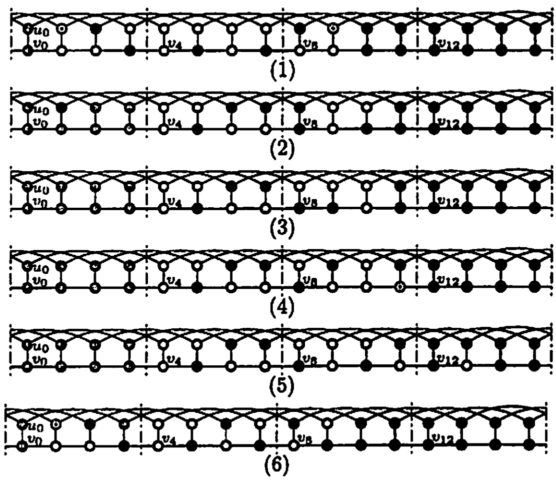

Fig. 2. The case for $v_4 \in V_0$, $u_4 \in V_0$, $v_5 \in V_1$ and $u_5 \in V_0$

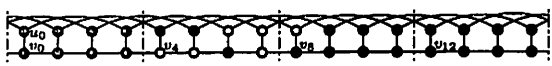
Fig. 3. The case for $v_4 \in V_0$, $u_4 \in V_1$, $v_5 \in V_0$ and $u_5 \in V_1$

$\beta(V'(4,8)) = 2 \wedge \beta(V'(4,12)) \geq 7$ (see Fig.4(1)), a contradiction.
Case 4.1.2. $v_3 \in V_1$. Then $v_8 \in V_0$. Since $\beta(V'(0,8)) \leq 4$, we have $u_3 \notin V_2$. Since $\beta(V'(4,8)) \leq 2$, $u_9 \in V_1$ and $u_{11} \notin V_2$. It follows $u_8 \in V_1$ and $u_2 \in V_0$. Since $f(v_7) \neq f(v_5)$ and $f(u_8) \neq f(v_5)$, we have $f(v_7) = f(u_8)$. It follows $v_9 \in V_1$, $\beta(V'(4,8)) \geq 4$ (see Fig.4(2)), a contradiction.
Case 4.2. $u_6 \in V_1$.
Case 4.2.1. $v_7 \in V_0$. Then $v_8 \in V_2$. Since $\beta(V'(4,8)) \leq 2$, we have $u_8, v_3, u_3 \in V_0$. It forces $v_2 \in V_2$ and $u_2 \in V_1$, $\beta(V'(0,8)) \geq 5$ (see Fig.4(3)), a contradiction.
Case 4.2.2. $v_7 \in V_1$. Then $v_8, u_9, u_{10} \in V_0$. Since $\beta(V'(4,8)) \leq 2$, $v_{10}, u_{11} \notin V_2$. It follows $u_8, v_9 \in V_1$, $v_{10}, u_{11}, u_{12} \in V_0$, $v_{11} \in V_1$, $v_{12} \in V_0$ and $u_{13}, u_{15} \in V_2$, $\beta(V'(4,8)) = 2 \wedge \beta(V'(4,12)) \geq 6$ (see Fig.4(4)), a contradiction.

Case 5. $v_4 \in V_1$, $u_4 \in V_0$, $v_5 \in V_0$ and $u_5 \in V_0$. Then $v_6 \in V_1$. It follows $u_6, v_7 \in V_0$.
Case 5.1. $u_7 \in V_0$. Then $u_{10} \in V_2$. Since $\beta(V'(4,8)) \leq 2$, we have $v_8 \in V_1$, $v_3, u_8, v_9, v_{10} \in V_0$. It follows $u_2 \in V_2$, $u_9 \in V_1$, $u_3 \in V_0$, $v_2 \notin V_0$, $\beta(V'(0,8)) \geq 5$ (see Fig.5(1)), a contradiction.
Case 5.2. $u_7 \in V_1$. Then $v_8 \notin V_2$.
Case 5.2.1. $v_8 \in V_0$. Since $f(v_4) \neq f(v_6)$ and $f(u_7) \neq f(v_6)$, we have

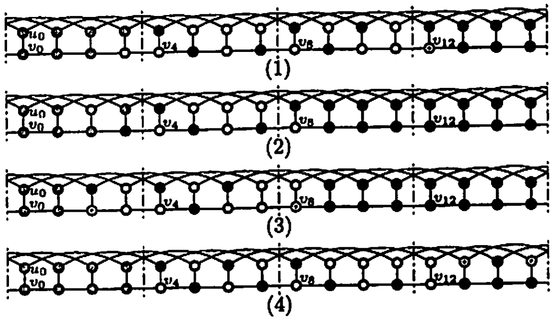

Fig. 4. The case for $v_4 \in V_0$, $u_4 \in V_1$, $v_5 \in V_1$ and $u_5 \in V_0$

$f(v_4) = f(u_7)$. It follows $u_1 \in V_1$. Since $\beta(V'(4,8)) \leq 2$, we have $u_8, v_9 \in V_1$. It follows $u_9, v_{10}, u_{10}, u_{11} \in V_0$ and $v_{11} \in V_1$. Since $f(u_8) \neq f(v_9)$ and $f(v_{11}) \neq f(v_9)$, we have $f(u_8) = f(v_{11})$. It follows $u_{14} \in V_1$, $v_{12} \in V_0$ and $u_{12}, u_{13} \in V_1$. Since $f(v_{11}) \neq f(v_9)$ and $f(u_{12}) \neq f(v_9)$, we have $f(v_{11}) = f(u_{12})$. It follows $v_{13} \in V_1$, $\beta(V'(4,8)) = 2 \wedge \beta(V'(4,12)) \geq 5$ (see Fig.5(2)), a contradiction.
Case 5.2.2. $v_8 \in V_1$. Since $\beta(V'(4,8)) \leq 2$, we have $u_8 \in V_0$. It follows $u_2 \in V_2$. Since $\beta(V'(0,8)) \leq 4$, we have $v_2, v_3 \in V_0$. It follows $u_3 \in V_1$. Since $f(u_3) \neq f(v_4)$ and $f(v_6) \neq f(v_4)$, we have $f(u_3) = f(v_6)$. It follows $u_9 \in V_1$, $\beta(V'(4,4)) \geq 2$ (see Fig.5(3)), a contradiction.

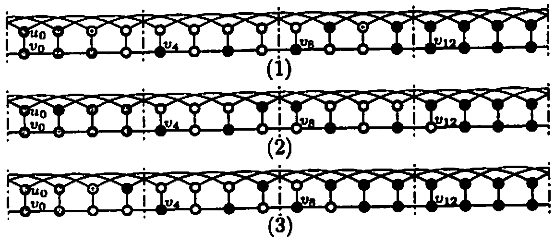

Fig. 5. The case for $v_4 \in V_1$, $u_4 \in V_0$, $v_5 \in V_0$ and $u_5 \in V_0$

Case 6. $v_4 \in V_1$, $u_4 \in V_0$, $v_5 \in V_0$ and $u_5 \in V_1$.
Case 6.1. $v_6 \in V_0$. Then $u_6, v_7 \in V_1$. It follows $u_7 \in V_0$. Since $\beta(V'(4,8)) \leq 2$, $u_{10} \in V_1$ and $v_8, u_8, u_9 \notin V_2$. If $v_8 \in V_1$, then $u_8, v_9, u_9 \in V_0$. It follows $v_{10} \in V_1$, $\beta(V'(4,8)) \geq 4$ (see Fig.6(1)), a contradiction. Hence $v_8 \in V_0$. If $u_8 \in V_1$, then $v_9, u_9 \in V_0$. It follows $v_{10} \in V_2$, $\beta(V'(4,8)) \geq 4$ (see Fig.6(2)), a contradiction. Hence $u_8 \in V_0$. It follows $v_9, u_{11} \in V_1$, $u_9, v_{10} \in V_0$. Since $f(v_9) \neq f(v_7)$ and $f(u_{10}) \neq f(v_7)$, we have $f(v_9) = f(u_{10})$. It forces $v_{11} \in V_1$, $\beta(V'(4,8)) \geq 3$ (see Fig.6(3)), a contradiction.
Case 6.2. $v_6 \in V_1$. Then $u_6, v_7, u_8 \in V_0$. Since $\beta(V'(4,8)) \leq 2$, $u_{10} \notin V_2$.

It follows $u_7 \in V_1$, $v_8 \in V_0$ and $v_9 \in V_2$, $\beta(V'(4,8)) \geq 3$(see Fig.6(4)), a contradiction.

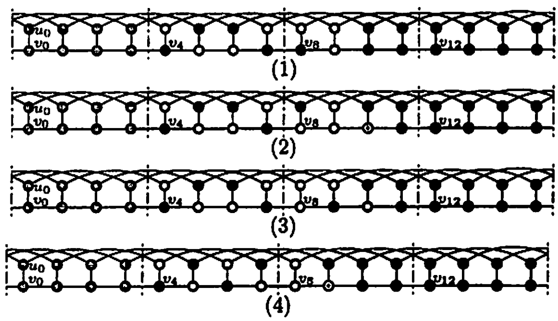

Fig. 6. The case for $v_4 \in V_1$, $u_4 \in V_0$, $v_5 \in V_0$ and $u_5 \in V_1$

From cases 1-6, we conclude the proof. □

Theorem 2.3. $\gamma_{r2}(P(n,3)) \geq \lceil \frac{7n}{8} \rceil$.

Proof. By Lemma 2.2, if $\beta(V'(4i,4)) \leq 1$, then $\beta(V'(4(i-1),8)) \geq 5$ or $\beta(V'(4i,8)) \geq 3$ or $\beta(V'(4i,8)) = 2 \wedge \beta(V'(4i,12)) \geq 5$. If $\beta(V'(4(i-1),8)) \geq 5$, then $\beta(V'(4(i-2),4)) \leq 1$ or $\beta(V'(4(i-2),4)) \geq 2$. If $\beta(V'(4(i-2),4)) \leq 1$, then $\beta(V'(4(i-3),4)) \leq 1$ or $\beta(V'(4(i-3),4)) \geq 2$.

Let

$$\begin{aligned}
S_{1151} = &\{V'(4(i-3),4), V'(4(i-2),4), V'(4(i-1),4), V'(4i,4) : \\
&\beta(V'(4i,4)) \leq 1, \beta(V'(4(i-1),8)) \geq 5, \beta(V'(4(i-2),4)) \leq 1, \\
&\beta(V'(4(i-3),4)) \leq 1\}, \\
S_{151} = &\{V'(4(i-2),4), V'(4(i-1),4), V'(4i,4) : \beta(V'(4i,4)) \leq 1, \\
&\beta(V'(4(i-1),8)) \geq 5, \beta(V'(4(i-2),4)) \leq 1, \\
&\beta(V'(4(i-3),4)) \geq 2\}, \\
S_{51} = &\{V'(4(i-1),4), V'(4i,4) : \beta(V'(4i,4)) \leq 1, \beta(V'(4(i-1),8)) \geq 5, \\
&\beta(V'(4(i-2),4)) \geq 2\}, \\
S_{13} = &\{V'(4i,4), V'(4(i+1),4) : V'(4i,4) \notin S_{1151} \cup S_{151} \cup S_{51}, \\
&\beta(V'(4i,4)) \leq 1, \beta(V'(4i,8)) \geq 3\}, \\
S_{115} = &\{V'(4i,4), V'(4(i+1),4), V'(4(i+2),4) : V'(4i,4) \notin S_{1151} \cup S_{151} \\
&\cup S_{51}, \beta(V'(4i,4)) \leq 1, \beta(V'(4i,8)) = 2, \beta(V'(4i,12)) \geq 5\}, \\
S_2 = &\{V'(4i,4) : V'(4i,4) \notin S_{1151} \cup S_{151} \cup S_{51} \cup S_{13} \cup S_{115}\},
\end{aligned}$$

where $4x$ of $V'(4x,4y)$ is modulo n.

By Lemma 2.1, we have

$$\begin{aligned}20w(f) &= 16n + 4\beta \\ &\geq 16n + \tfrac{|S_{1151}|}{4} \times 6 + \tfrac{|S_{151}|}{3} \times 5 + \tfrac{|S_{51}|}{2} \times 5 + \tfrac{|S_{13}|}{2} \times 3 + \tfrac{|S_{115}|}{3} \times 5 \\ &\quad + |S_2| \times 2 \\ &= 16n + |S_{1151}| \times \tfrac{3}{2} + |S_{151}| \times \tfrac{3}{2} + |S_{151}| \times \tfrac{1}{6} + |S_{51}| \times \tfrac{3}{2} + |S_{51}| \\ &\quad + |S_{13}| \times \tfrac{3}{2} + |S_{115}| \times \tfrac{3}{2} + |S_{115}| \times \tfrac{1}{6} + |S_2| \times \tfrac{3}{2} + |S_2| \times \tfrac{1}{2} \\ &\geq 16n + (|S_{1151}| + |S_{151}| + |S_{51}| + |S_{13}| + |S_{115}| + |S_2|) \times \tfrac{3}{2} \\ &= 16n + n \times \tfrac{3}{2} = \tfrac{35n}{2}.\end{aligned}$$

i.e. $\gamma_{r2}(P(n,3)) \geq \lceil \frac{7n}{8} \rceil$. □

Corollary 2.4. $\gamma_{r2}(P(n,3)) = \lceil \frac{7n}{8} \rceil$, for n (mod 16)=0,2,4,5,6,7,13,14,15.

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References

[1] A. Behzad, M. Behzad and C. E. Praeger, On the domination number of the generalized Petersen graphs, *Discret Appl. Math.*, 308(2008), 603–610.

[2] B. Brešar, M. A. Henning and D. F. Rall, Paired-domination of Cartesian products of graphs and rainbow domination, *Discret. Math.*, 22(2005), Note, 233–237 (electronic).

[3] B. Brešar and T. K. Šumenjak, On the 2-rainbow domination in graphs, *Discret Appl. Math.*, 155(2007), 2394–2400.

[4] B. Brešar, M. A. Henning and D. F. Rall, Rainbow domination in graphs, *Taiwan. J. Math.*, 12(1)(2008), 213–225.

[5] Fu Xueliang, Yang Yuansheng and Jiang Baoqi, On the domination number of generalized Petersen graph P(n,3), *Ars comb.*, 84(2007), 373–383.

[6] Fu Xueliang, Yang Yuansheng and Jiang Baoqi, Roman domination in regular graphs *Discret. Math.*, 309(2009), 1528–1537.

[7] Fu Xueliang, Yang Yuansheng and Jiang Baoqi, On the packing number of generalized Petersen graph P(n,2), *Ars comb.*, to appear.

[8] Fu Xueliang, Yang Yuansheng and Jiang Baoqi, On the domination number of generalized Petersen graph P(n,2), *Discret. Math.*, 309(2009), 2445–2451.

[9] Tong Chunling, Lin Xiaohui, Yang Yuansheng and Luo Meiqin, 2-rainbow domination of generalized Petersen graphs P(n,2), *Discret Appl. Math.*, 157(2009), 1932–1937.

[10] M. Watkins, A theorem on Tait colorings with an application to the generalized Petersen graph, *J. Comb. Theory,* 6(1969), 152–164.
[11] Xu Guangjun, 2-rainbow domination in generalized Petersen graphs P(n,3), *Discret Appl. Math.,* 157(2009), 2570–2573.
[12] Yan Hong, Kang Liying and Xu Guangjun, The exact domination number of the generalized Petersen graphs, *Discret. Math.,* 309(2009), 2596–2607.
[13] B. Zelinka, Domination in generalized Petersen graphs, *Czech. Math. J.,* 52 (127) (2002), 11–16.