Self-inverse sequences related to Sheffer sets

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Abstract

In this paper, we define the self-inverse sequences related to Sheffer sets and give some interesting results of these sequences. Moreover, we study the self-inverse sequences related to the Laguerre polynomials of order α.

Keywords: self-inverse sequences; Sheffer sets; basic sequences; Laguerre polynomials

1. Introduction

The classical binomial inversion formulas states that

$$a_n = \sum_{k=0}^{n}(-1)^k \binom{n}{k} b_k \iff b_n = \sum_{k=0}^{n}(-1)^k \binom{n}{k} a_k.$$

We say that a sequence $\{a_n\}$ of complex numbers is self-inverse or invariant if

$$\sum_{k=0}^{n}(-1)^k \binom{n}{k} a_k = a_n \quad (n \geq 0).$$

Sun [7] and Wang [8] studied those self-inverse sequences and gave some results of self-inverse sequences. For general self-inverse pairs

$$a_n = \sum_{k=0}^{n} A(n,k) b_k \iff b_n = \sum_{k=0}^{n} A(n,k) a_k,$$

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we have the infinite lower triangle matrix $A = (A(n,k))_{n,k=0}^{\infty}$ satisfies $A^2 = I$. A sequence $\{a_n\}$ is called a general self-inverse sequence if it satisfies

$$a_n = \sum_{k=0}^{n} A(n,k) a_k \quad (n \geq 0),$$

where $A = (A(n,k))$ is an infinite lower triangle matrix and $A^2 = I$. We denote $A_m = (A(n,k))_{n,k=0}^{m}$, and we have $A_m^2 = I_m$. Hence, we get $(\det(A_m))^2 = (\prod_{i=0}^{m} A(i,i))^2 = 1$ for all $m \geq 0$. Thus we see that the diagonal entries of the matrix A are non-zero. Let

$$s_n(x) = \sum_{k=0}^{n} A(n,k) x^k \quad (n \geq 0).$$

Then $s_n(x)$ is exactly a polynomial of degree n for all $n \geq 0$. Following the inverse relation, we get

$$x^n = \sum_{k=0}^{n} A(n,k) s_k(x) \quad (n \geq 0).$$

If $q_m(x) = \sum_{k=0}^{m} q_{m,\,k} x^k$ is another polynomial, we denote $q_m(\mathbf{s}(x)) = \sum_{k=0}^{m} q_{m,k} s_k(x)$. With this notation, we have $s_n(\mathbf{s}(x)) = x^n$ for all $n \geq 0$.

The present authors studied the self-inverse sequences related to sequences of polynomials of binomial type in [1]. However, for the self-inverse pair

$$a_n = \sum_{k=0}^{n} \frac{n!}{k!} \binom{\alpha+n}{\alpha+k} (-1)^k b_k \iff b_n = \sum_{k=0}^{n} \frac{n!}{k!} \binom{\alpha+n}{\alpha+k} (-1)^k a_k,$$

we know that

$$L_n^{\alpha}(x) = \sum_{k=0}^{n} \frac{n!}{k!} \binom{\alpha+n}{\alpha+k} (-1)^k x^k \quad (n \geq 0)$$

is the Laguerre polynomials of order α, and $L_n^{\alpha}(\mathbf{L}^{\alpha}(x)) = x^n$. The Laguerre polynomials of order α is a Sheffer set, which will be defined in Section 2. Let

$$s_n(x) = \sum_{k=0}^{n} A(n,k) x^k \quad (n \geq 0)$$

be a Sheffer set and $s_n(\mathbf{s}(x)) = x^n$. A sequence $\{a_n\}$ is called a self-inverse sequence related to the Sheffer set $s_n(x)$ if

$$a_n = \sum_{k=0}^{n} A(n,k) a_k \quad (n \geq 0).$$

In this paper, we study the self-inverse sequences related to Sheffer sets. In order to render this work self-contained, we list some important results of sequences of Sheffer sets in Section 2, omitting the proofs which can be found in [6]. In Section 3, we give some general results of the self-inverse sequences related to Sheffer sets. We also study the self-inverse sequences related to the Laguerre polynomials of order α in Section 4.

2. Fundamentals

In this section, we list the main results of Sheffer sets which we shall use in the next two sections. These results were obtained by Rota, Kahaner and Odlyzko [6].

First, we give some definitions in the theory of binomial enumeration.

Definition 1. *A linear operator T which commutes with all shift operators is called a shift-invariant operator,i.e.,$TE^a = E^aT$.*

Definition 2. *A delta operator, usually denoted by the letter Q, is a shift-invariant operator for which Qx is a non-zero constant.*

Delta operators possess many of the properties of the derivative operator D.

Definition 3. *Let Q be a delta operator. A polynomial sequence $p_n(x)$ is called the sequence of basic polynomials for Q if:*
(1) $p_0(x) = 1$,
(2) $p_n(0) = 0$ whenever $n > 0$,
(3) $Qp_n(x) = np_{n-1}(x)$.

Without difficulty, we can show that every delta operator has a unique sequence of basic polynomials associated with it.

Definition 4. *A polynomial sequence $s_n(x)$ is called a Sheffer set for the delta operator Q if*
(1) $s_0(x) = c \neq 0$,
(2) $Qs_n(x) = ns_{n-1}(x)$.

Now, we can give some general consequences of Sheffer sets as lemmas.

Lemma 1 (Expansion theorem). *Let T be a shift-invariant operator, and let Q be a delta operator with basic set $p_n(x)$. Then*

$$T = \sum_{k\geq 0} \frac{a_k}{k!} Q^k$$

where $a_k = [Tp_k(x)]_{x=0}$.

Lemma 2. *Let* $\mathbf{P}$ *be a ring of polynomials over a field* $\mathbf{K}$ *and* Σ *be the ring of shift-invariant operators on* $\mathbf{P}$. *Suppose that* Q *is a delta operator and* F *is the ring of formal power series in the variable* t *over* $\mathbf{K}$. *Then there exists an isomorphism from* F *onto* Σ, *which carries*

$$f(t) = \sum_{k\geq 0} a_k \frac{t^k}{k!} \quad \text{into} \quad \sum_{k\geq 0} \frac{a_k}{k!} Q^k.$$

Lemma 3. *A shift-invariant operator* T *is invertible if and only if* $T1 \neq 0$.

Lemma 4. *Let* Q *be a delta operator with basic polynomial set* $q_n(x)$. *Then* $s_n(x)$ *is a Sheffer set relative to* Q *if and only if there exists an invertible shift invariant operator* S *such that* $s_n(x) = S^{-1} q_n(x)$.

Lemma 5 (Binomial Theorem). *Let* Q *be a delta operator with basic polynomials* $q_n(x)$, *and let* $s_n(x)$ *be a Sheffer set relative to* Q *and to some invertible shift invariant operator* S. *Then the following identity holds*

$$s_n(x+y) = \sum_{k\geq 0} \binom{n}{k} s_k(x) q_{n-k}(y).$$

Lemma 6. *Let* $Q = g(D)$ *be a delta operator, and let* $S = s(D)$ *be an invertible shift-invariant operator. Let* $q^{-1}(t)$ *be the formal power series inverse to* $q(t)$. *Then the generating function for the Sheffer set* $s_n(x)$ *relative to* Q *and* S *is given by*

$$\frac{1}{s(q^{-1}(t))} e^{xq^{-1}(t)} = \sum_{n\geq 0} \frac{s_n(x)}{n!} t^n .$$

Lemma 7. *Let* $s_n(x)$ *and* $t_n(x)$ *be Sheffer sets relative to the delta operators* $Q = q(D)$ *and* $P = p(D)$, *and to the invertible shift-invariant operators* $S = s(D)$ *and* $T = t(D)$, *respectively. Let* $q_n(x)$ *and* $p_n(x)$ *be the basic sets for* Q *and* P. *Denote* $s_n(\mathbf{t}(x))$ *by* $r_n(x)$, *then* $r_n(x)$ *is a Sheffer set relative to the invertible operator* $t(D)s(p(D))$ *and the delta operator* $q(p(D))$, *having as basic set the sequence* $q_n(\mathbf{p}(x))$.

3. Main results

Let $p_n(x) = \sum_{k=0}^{n} B(n,k)x^k$ $(n \geq 0)$ be a sequence of polynomials of binomial type (studied in [3]) and $p_n(\mathbf{p}(x)) = x^n$. A sequence $\{a_n\}$ is called a self-inverse sequence related to the sequence of polynomials $p_n(x)$ of binomial type if $a_n = \sum_{k=0}^{n} B(n,k)a_k$ $(n \geq 0)$. The present authors have studied these sequences in [1].

Throughout this section, let

$$s_n(x) = \sum_{k=0}^{n} A(n,k)x^k \quad (n \geq 0), \tag{1}$$

which satisfies $s_n(\mathbf{s}(x)) = x^n$ for all n, be a Sheffer set relative to the invertible operator $S = s(D)$ and the delta operator $Q = q(D)$, where $s(t)$ and $q(t)$ are formal power series. Let $p_n(x)$ be the basic set for Q. By Lemma 7, we know that $s_n(\mathbf{s}(x)) = x^n$ is a Sheffer set relative to the invertible operator $s(D)s(q(D))$ and the delta operator $q(q(D))$ which has a basic set $p_n(\mathbf{p}(x))$. However, x^n is the basic sequence for the derivative operator D. Thus we have $s(t)s(q(t)) = 1$, $p_n(\mathbf{p}(x)) = x^n$ and $q(q(t)) = t$, i.e., $q(t) = q^{-1}(t)$. Using Lemma 6, we get that $s_n(x)$ $(n \geq 0)$ has the following exponential generating function:

$$\sum_{n\geq 0} s_n(x)\frac{t^n}{n!} = \frac{1}{s(q^{-1}(t))}e^{xq^{-1}(t)} = \frac{1}{s(q(t))}e^{xq(t)}. \tag{2}$$

The basic set $p_n(x)$ $(n \geq 0)$ has the following exponential generating function:

$$\sum_{n\geq 0} p_n(x)\frac{t^n}{n!} = e^{xq(t)} \tag{3}$$

From (1) and (2), we have

$$\sum_{n\geq 0}\frac{t^n}{n!}\sum_{k=0}^{n} A(n,k)x^k = \sum_{k\geq 0} x^k \sum_{n\geq k} A(n,k)\frac{t^n}{n!} = \frac{1}{s(q(t))}\sum_{k\geq 0} x^k\frac{(q(t))^k}{k!}\,.$$

Therefore, the entries $A(n,k)$ have a "vertical" generating function:

$$\varphi_k(t) = \sum_{n\geq k} A(n,k)\frac{t^n}{n!} = \frac{1}{s(q(t))}\frac{(q(t))^k}{k!}\,. \tag{4}$$

Now we can give the main results of the self-inverse sequences related to Sheffer sets.

For any sequence $\{a_n\}$, let the exponential generating function of $\{a_n\}$ be

$$A(x) = \sum_{n\geq 0} a_n\frac{x^n}{n!}\,.$$

We have:

Theorem 1. *$\{a_n\}$ is a self-inverse sequence related to the Sheffer set $s_n(x)$ if and only if $A(x) = \frac{1}{s(q(x))}A(q(x))$.*

Proof.

$$\begin{aligned} a_n &= \sum_{k=0}^{n} A(n,k)a_k \\ \iff \quad A(x) &= \sum_{n\geq 0} \frac{x^n}{n!} \sum_{k=0}^{n} A(n,k)a_k = \sum_{k\geq 0} a_k \sum_{n\geq k} A(n,k)\frac{x^n}{n!} \\ &= \sum_{k\geq 0} a_k \frac{1}{s(q(x))} \frac{(q(x))^k}{k!} = \frac{1}{s(q(x))} A(q(x)) \;. \square \end{aligned}$$

Obviously, if $\{a_n\}$ and $\{b_n\}$ are self-inverse sequences related to the Sheffer set $s_n(x)$, $\{\alpha a_n + \beta b_n\}$ is a self-inverse sequence related to the Sheffer set $s_n(x)$, where α and β are arbitrary constants.

Sun[7] gave some transformation formulas for self-inverse sequences. Similarly, we have the following theorems.

Theorem 2. *Let $\{a_n\}$ be a self-inverse sequence related to the sequence of polynomials $p_n(x)$ of binomial type, and $c_n = \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}$. Then $\{c_n\}$ is a self-inverse sequence related to the Sheffer set $s_n(x)$ if and only if $\{b_n\}$ is a self-inverse sequence related to the Sheffer set $s_n(x)$.*

Proof. By Theorem 1 in [1], we have $A(x) = A(q(x))$. Let $C(x) = \sum_{n\geq 0} c_n \frac{x^n}{n!}$ and $B(x) = \sum_{n\geq 0} b_n \frac{x^n}{n!}$. Then

$$C(x) = \sum_{n\geq 0} \frac{x^n}{n!} \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k} = \sum_{k\geq 0} a_k \frac{x^k}{k!} \sum_{n\geq k} b_{n-k} \frac{x^{n-k}}{(n-k)!} = A(x)B(x) \,.$$

Hence, we have

$$\begin{aligned} C(x) = \frac{1}{s(q(x))} C(q(x)) \quad &\iff \quad A(x)B(x) = \frac{1}{s(q(x))} A(q(x))B(q(x)) \\ &\iff \quad B(x) = \frac{1}{s(q(x))} B(q(x)) \,. \end{aligned}$$

The result follows Theorem 1. $\square$

Theorem 3. *Let $u_n(x)$ be a Sheffer set relative to the invertible operator $U = u(D)$ and the delta operator Q, and let $v_n(x)$ be a Sheffer set relative to the invertible operator $u(D)s(D)$ and the delta operator Q. Suppose* *that $\{a_n\}$ is a self-inverse sequence related to the Sheffer set $s_n(x)$ and $\{b_n\}$ is a self-inverse sequence related to the Sheffer set $u_n(x)$. Let $c_n = \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}$. Then $\{c_n\}$ is a self-inverse sequence related to the Sheffer set $v_n(x)$.*

Proof. Let $C(x) = \sum_{n\geq 0} c_n \frac{x^n}{n!}$ and $B(x) = \sum_{n\geq 0} b_n \frac{x^n}{n!}$. Then

$$C(x) = \sum_{n\geq 0} \frac{x^n}{n!} \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k} = \sum_{k\geq 0} a_k \frac{x^k}{k!} \sum_{n\geq k} b_{n-k} \frac{x^{n-k}}{(n-k)!} = A(x)B(x) .$$

By Theorem 1,we have:

$$A(x) = \frac{1}{s(q(x))} A(q(x)) \quad \textit{and} \quad B(x) = \frac{1}{u(q(x))} B(q(x)) .$$

Hence,

$$C(x) = A(x)B(x) = \frac{1}{s(q(x))u(q(x))} A(q(x))B(q(x)) = \frac{1}{s(q(x))u(q(x))} C(q(x)) .$$

From Theorem 1, we have $\{c_n\}$ is a self-inverse sequence related to the Sheffer set $v_n(x)$. □

Now we give some methods by which we can create the self-inverse sequences related to the Sheffer set $s_n(x)$.

Theorem 4. *$\{a_n\}$ is a self-inverse sequence related to the Sheffer set $s_n(x)$ if and only if there exists a self-inverse sequence $\{v_n\}$ related to the Sheffer set $s_n(x)$ such that*

$$a_n = \sum_{k=0}^{n} A(n,k) v_k + v_n \qquad (n \geq 0).$$

Proof. Suppose that $\{a_n\}$ is a self-inverse sequence related to the Sheffer set $s_n(x)$. Then $a_n = \sum_{k=0}^{n} A(n,k) a_k$. Let $v_n = \frac{a_n}{2}$. We have

$$\sum_{k=0}^{n} A(n,k) \frac{a_k}{2} + \frac{a_n}{2} = \frac{a_n}{2} + \frac{a_n}{2} = a_n .$$

Conversely, let $V(x) = \sum_{k\geq 0} v_k \frac{x^k}{k!}$. Then

$$\begin{aligned} A(x) &= \sum_{n\geq 0} \left(\sum_{k=0}^{n} A(n,k) v_k + v_n \right) \frac{x^n}{n!} = \sum_{k\geq 0} v_k \sum_{n\geq k} A(n,k) \frac{x^n}{n!} + \sum_{n\geq 0} v_n \frac{x^n}{n!} \\ &= \sum_{k\geq 0} v_k \frac{1}{s(q(x))} \frac{(q(x))^k}{k!} + \sum_{n\geq 0} v_n \frac{x^n}{n!} = \frac{1}{s(q(x))} V(q(x)) + V(x) . \end{aligned}$$

Thus we have

$$\frac{1}{s(q(x))}A(q(x)) = \frac{1}{s(q(x))}\left(\frac{1}{s(x)}V(x) + V(q(x))\right) = V(x)+\frac{1}{s(q(x))}V(q(x))$$

$$= A(x)\ . \ \Box$$

Corollary 1. *Let $a_0 = 0$, $a_1 = A(1,1) + 1$, and $a_n = A(n,1)$ $(n \geq 2)$. Then $\{a_n\}$ is a self-inverse sequence related to the Sheffer set $s_n(x)$.*

Proof. Let $v_0 = 0$, $v_1 = 1$ and $v_n = 0$ $(n \geq 2)$ in Theorem 4. □

Corollary 2. *Let $a_n = s_n(a) + a^n$, where a is an arbitrary constant. Then $\{a_n\}$ is a self-inverse sequence related to the Sheffer set $s_n(x)$.*

Proof. Let $v_n = a^n$ $(n \geq 0)$ in Theorem 4. □

Theorem 4 has an equivalent form:

Theorem 5. *$\{a_n\}$ is a self-inverse sequence related to the Sheffer set $s_n(x)$ if and only if there exists a function $f : \mathbb{Z} \to \mathbb{C}$ such that*

$$a_n = \sum_{k=0}^{n}\binom{n}{k}p_{n-k}(1)s_k(\Delta)f(0) + f(n)\ .$$

Proof. Following the string of identities:

$$\sum_{k=0}^{n}\binom{n}{k}p_{n-k}(1)s_k(\Delta)f(0) = \left[\left(\sum_{k=0}^{n}\binom{n}{k}p_{n-k}(1)s_k(\Delta)\right)f(x)\right]_{x=0}$$

$$= \quad [s_n(\Delta + I)f(x)]_{x=0} = [s_n(E)f(x)]_{x=0} = [\sum_{k=0}^{n} A(n,k)E^k f(x)]_{x=0}$$

$$= \sum_{k=0}^{n} A(n,k)f(k)\ ,$$

we get the result. □

The following theorem is shown by the operator method.

Theorem 6. *Suppose that $r_n(x) = \sum_{k=0}^{n} r_{n,\,k}x^k$ is a Sheffer set relative to the invertible operator $R = r(D)$ and the delta operator $P = p(D)$. Let $u_n(x) = r_n(\mathrm{s}(x)) = \sum_{k=0}^{n} u_{n,\,k}x^k$. If $\{a_n\}$ is a self-inverse sequence related to the Sheffer set $s_n(x)$, we have the following identity:*

$$\sum_{k=0}^{n} r_{n,\,k} a_k = \sum_{k=0}^{n} u_{n,\,k} a_k \,.$$

Proof. By Lemma 7, we know that $u_n(x)$ is a Sheffer set relative to the invertible operator $s(D)r(q(D))$ and the delta operator $p(q(D))$. Let T be a linear operator such that $Tx^n = a_n$. Then

$$Ts_n(x) = T\sum_{k=0}^{n} A(n,k)x^k = \sum_{k=0}^{n} A(n,k)Tx^k = \sum_{k=0}^{n} A(n,k)a_k = a_n \,.$$

Therefore, we have $Tx^n = Ts_n(x)$. If $f(x)$ is a polynomial, we get $Tf(x) = Tf(\mathrm{s}(x))$ by linearity. Let $f(x) = r_n(x)$. Immediately, we have

$$Tr_n(x) = Tr_n(\mathrm{s}(x)) = Tu_n(x) \;\; ,$$

i.e.,

$$\sum_{k=0}^{n} r_{n,\,k} Tx^k = \sum_{k=0}^{n} u_{n,\,k} Tx^k \;\; .$$

The result follows $Tx^k = a_k$. □

4. Self-inverse sequences related to the Laguerre polynomials of order α

The Laguerre polynomials of order α,

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n} \frac{n!}{k!}\binom{\alpha+n}{\alpha+k}(-1)^k x^k,$$

is a Sheffer set relative to the invertible operator $S = s(D) = (I-D)^{-\alpha-1}$ and the delta operator $Q = q(D) = \frac{D}{D-I}$. The basic sequence for the delta operator Q is the basic Laguerre polynomials

$$L_n^{(-1)}(x) = L_n(x) = \sum_{k=1}^{n} \frac{n!}{k!}\binom{n-1}{k-1}(-1)^k x^k \,.$$

We know that $L_n^{(\alpha)}(\mathrm{L}^{(\alpha)}(x)) = x^n$. If $a_n = \sum_{k=0}^{n} \frac{n!}{k!}\binom{\alpha+n}{\alpha+k}(-1)^k a_k$ $(n \geq 0)$, we say that $\{a_n\}$ is a self-inverse sequence related to the Sheffer set $L_n^{(\alpha)}(x)$. From Theorem 1, we have:

Proposition 1. *$\{a_n\}$ is a self-inverse sequence related to the Sheffer set $L_n^{(\alpha)}(x)$ if and only if the exponential generating function $A(x) = \sum_{n\geq 0} a_n \frac{x^n}{n!}$ satisfies $A(x) = (1-x)^{-\alpha-1}A(\frac{x}{x-1})$.*

Obviously, using Corollary 1, we have the sequence $\{a_n\}$, which satisfies $a_0 = a_1 = 0$ and $a_n = -n!\binom{\alpha+n}{\alpha+1}$ $(n \geq 2)$, is a self-inverse sequence related to the Sheffer set $L_n^{(\alpha)}(x)$. Moreover, by Corollary 2, the sequence $\{a_n\}$, which satisfies

$$a_n = \sum_{k=0}^{n} \frac{n!}{k!}\binom{\alpha+n}{\alpha+k}(-a)^k + a^n$$

where a is an arbitrary constant, is a self-inverse sequence related to the Sheffer set $L_n^{(\alpha)}(x)$.

Applying Theorem 2, we have the following proposition:

Proposition 2. *Let $\{a_n\}$ be a self-inverse sequence related to the Laguerre polynomials $L_n(x)$, and $c_n = \sum_{k=0}^{n}\binom{n}{k}a_k b_{n-k}$. Then $\{c_n\}$ is a self-inverse sequence related to the Sheffer set $L_n^{(\alpha)}(x)$ if and only if $\{b_n\}$ is a self-inverse sequence related to the Sheffer set $L_n^{(\alpha)}(x)$.*

From [1], we know that the sequence $\{a_n\}$, which satisfies $a_0 = a_1 = 0$ and $a_n = -n!$ $(n \geq 2)$, is a self-inverse sequence related to $L_n(x)$. Then we have:

Corollary 3. *Let $c_0 = c_1 = 0$ and $c_n = -\sum_{k=2}^{n}\frac{n!}{(n-k)!}b_{n-k}$ $(n \geq 2)$. Then $\{c_n\}$ is a self-inverse sequence related to the Sheffer set $L_n^{(\alpha)}(x)$ if and only if $\{b_n\}$ is a self-inverse sequence related to the Sheffer set $L_n^{(\alpha)}(x)$.*

For the Laguerre polynomials $L_n^{(\alpha)}(x)$ of order α, Theorem 4 and Theorem 5 can be restated as follows.

Proposition 3. *$\{a_n\}$ is a self-inverse sequence related to the Sheffer set $L_n^{(\alpha)}(x)$ if and only if there exists a sequence $\{v_n\}$ such that*

$$a_n = \sum_{k=0}^{n} \frac{n!}{k!}\binom{\alpha+n}{\alpha+k}(-1)^k v_k + v_n \quad (n \geq 0)\,.$$

Proposition 4. *$\{a_n\}$ is a self-inverse sequence related to the Sheffer set $L_n^{(\alpha)}(x)$ if and only if there exists a function $f : \mathbb{Z} \to \mathbb{C}$ such that*

$$a_n = \sum_{k=0}^{n}\binom{n}{k}L_{n-k}(1)L_k^{(\alpha)}(\Delta)f(0) + f(n)\,.$$

The Laguerre polynomials $L_n^{(\alpha)}(x)$ of order α is a Sheffer set relative to the invertible operator $s(D) = (I - D)^{-\alpha-1}$ and the delta operator $q(D) = \frac{D}{D-I}$, and $L_n^{(\beta)}(x)$ is a Sheffer set relative to the invertible operator $(I-D)^{-\beta-1}$ and the delta operator $\frac{D}{D-I}$. However, $L_n^{(\alpha+\beta+1)}(x)$ is a Sheffer set relative to the invertible operator $(I - D)^{-\alpha-1}(I - D)^{-\beta-1} = (I - D)^{-(\alpha+\beta+1)-1}$ and the delta operator $\frac{D}{D-I}$. Hence, by Theorem 3, we have:

Proposition 5. *Suppose that $\{a_n\}$ is a self-inverse sequence related to the Sheffer set $L_n^{(\alpha)}(x)$ and $\{b_n\}$ is a self-inverse sequence related to the Sheffer set $L_n^{(\beta)}(x)$. Let $c_n = \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}$. Then $\{c_n\}$ is a self-inverse sequence related to the Sheffer set $L_n^{(\alpha+\beta+1)}(x)$.*

Proposition 6. *If $\{a_n\}$ is a self-inverse sequence related to the Sheffer set $L_n^{(\alpha)}(x)$, we have the following identity:*

$$\sum_{k=0}^{n} \binom{n}{k} (\beta + k + 1)_k a_{n-k} = \sum_{k=0}^{n} \binom{n}{k} \binom{\alpha+\beta+n}{\alpha+\beta+k} (-1)^k a_k \ . \qquad (5)$$

Proof. From [6], we know that

$$M_n^{(\beta)}(x) = \sum_{k=0}^{n} \binom{n}{k} (\beta + k + 1)_k x^{n-k}$$

is a Sheffer set relative to the invertible operator $(I - D)^{\beta}$ and the delta operator D. By Lemma 7,

$$M_n^{(\beta)}(\mathbf{L}^{(\alpha)}(x)) = L_n^{(\alpha+\beta)}(x) = \sum_{k=0}^{n} \frac{n!}{k!} \binom{\alpha+\beta+n}{\alpha+\beta+k} (-1)^k x^k$$

is a Sheffer set relative to the invertible operator $(I - D)^{-\alpha-\beta-1}$ and the delta operator $\frac{D}{D-I}$. Let T be a linear operator such that $Tx^n = a_n$. From Theorem 6, we have

$$TM_n^{(\beta)}(x) = TM_n^{(\beta)}(\mathbf{L}^{(\alpha)}(x)) = TL_n^{(\alpha+\beta)}(x) \ ,$$

i.e.,

$$\sum_{k=0}^{n} \binom{n}{k} (\beta + k + 1)_k Tx^{n-k} = \sum_{k=0}^{n} \frac{n!}{k!} \binom{\alpha+\beta+n}{\alpha+\beta+k} (-1)^k Tx^k \ .$$

The result follows immediately. □

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