Classification of eleven-point five-distance sets in the plane *

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Abstract

A point set X in the plane is called a k-distance set if there are exactly k different distances between two distinct points in X. We classify 11-point 5-distance sets.

1 Introduction

A point set X in the Euclidean plane is called a k-distance set if it determines exactly k different distances. For two planar point sets, we say that they are isomorphic if there exists a similar transformation from one to the other. Let $d(x, y)$ denote the distance between two planar points x and y. Let R_n denote the vertex set of a regular convex n-gon, R_n^+ be R_n plus its center point, $R_n - i$ denote a set of $n - i$ vertices of R_n. When $i \geq 2$, $R_n - i$ has dissimilar versions depending on which i points of R_n are absent. Let $g(k)$ be the largest possible cardinality of k-distance set. A k-distance set X is said to be maximum if X has $g(k)$ points. Clearly $g(k) \geq 2k + 1$ since R_{2k+1} is a k-distance set. Erdős-Fishburn [1] determined $g(k)$ for $k \leq 5$ and classified maximum k-distance sets for $k \leq 4$, and conjectured $g(6) = 13$. Shinohara [5] classified 3-distance sets with at least five points. Shinohara [6] proved the uniqueness of the 12-point 5-distance set and classified 8-point 4-distance sets. In this note we classify 11-point 5-distance sets, which play an important role in the proof of the conjecture $g(6) = 13$.

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2 Preliminaries and related Lemmas

Let $D = D(X)$ be the diameter of a finite set X, and let $X_D = \{x \in X : d(x,y) = D \text{ for some } y \in X\}$ and $m = m(X) = |X_D|$. The diameter graph $DG(X)$ of X is the graph with X as its vertices and where two vertices $x, y \in X$ are adjacent if $d(x,y) = D$. For $v \in X_D$, let $d(v)$ denote the number of D-length segments connected with vertex v in $DG(X_D)$. Clearly $DG(X_D)$ has no isolated vertex, and every two D-length segments in $DG(X_D)$ must cross if they do not share an end point. We denote a path and a cycle with n vertices by P_n and C_n, respectively. When indexing a set of t points, we identify indices modulo t.

As in [6] a subset H of $V(G)$ is an independent set of $V(G)$ if no two vertices in H are adjacent, and H is said to be maximal if no other independent sets contain H, the independence number $\alpha(G)$ of a graph G is the maximum cardinality among the independent sets of G. An independent set H of G is said to be maximum if $|H| = \alpha(G)$. We denote the set of all n-point k-distance sets and the set of all n-point convex k-distance sets by $E_n(k)$ and $M_n(k)$ respectively. In the following some proofs are omitted because of the restriction of the length of the paper.

Lemma 1. *[3] [4] Suppose S is the vertex set of a convex n-gon, $n \geq 3$, that determines exactly t different distances. Then $t \geq \lfloor n/2 \rfloor$. Moreover:*
(i) if n is odd and $t = (n-1)/2$, S is R_n;
(ii) if n is even, $t = n/2$, and $n \geq 8$, S is R_n or $R_{n+1} - 1$;
(iii) if $(n,t) = (7,4)$, S is $R_8 - 1$ or $R_9 - 2$;
(iv) if $(n,t) = (9,5)$, S is $R_{10} - 1$ or $R_{11} - 2$.

Lemma 2. *[1] Let D be the diameter of an n-point set X with $n \geq 3$ and $m = |X_D|$. Then*
(a) if $m \geq 3$, the points in X_D are the vertices of a convex m-gon;
(b) D can be eliminated as an interpoint distance by removing at most $\lceil \frac{m}{2} \rceil$ points from X, where $\lceil \frac{m}{2} \rceil$ is the smallest integer at least $m/2$.

Lemma 3. *[6] Let $G = DG(X)$ for X. Then*
(a) G contains no C_{2k} for any $k \geq 2$;
(b) if G contains C_{2k+1}, then any two vertices in $V(G) \setminus V(C_{2k+1})$ are not adjacent and every vertex not in the cycle is adjacent to at most one vertex of the cycle, where $V(G)$ denote the vertex set of the simple graph G. In particular, G contains at most one cycle.

Lemma 4. *If $d(v) = k \geq 2$ for $v \in X_D$ in the diameter graph $DG(X_D)$ of X_D, then the k vertices having D-length with v are consecutive.*

Proof. By Lemma 2, X_D is a convex set. If the k vertices having D-length with v are not consecutive, then there exists a point $p \in X_D$ between v_j

and v_k, where $d(v, v_j) = d(v, v_k) = D$, $d(v, p) \neq D$. Since $DG(X_D)$ has no isolated vertex, there exists a point $q \in X_D$ such that $d(p, q) = D$. Since every two D-length segments in $DG(X_D)$ must cross if they do not share an end point, the two segments $[v, v_j]$ and $[v, v_k]$ must cross with the segment $[p, q]$, this is impossible. □

Lemma 5. *For a planar point set X with $m = |X_D|$, let $X_D = \{1, 2, \ldots, m\}$, m points are consecutive with counter-clockwise order. If for a subset $S \subset X_D$, $S = \{k, k+1, k+2, \ldots, k+l-1\}$, the segment $[k, k+l-1]$ is the max-length segment of S and $d(k, k+i) < d(k, k+l-1)$ for any $i = 1, 2, 3, \ldots, l-2$, then $d(k, k+1) < d(k, k+2) < d(k, k+3) < \ldots < d(k, k+l-1) \leq D$.*

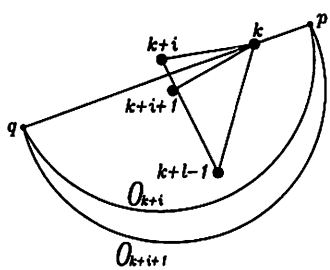

Figure 1: $S \bigcup \{p\}$ and $S \bigcup \{q\}$ are not convex sets

Proof. For a point $k+i$ of S, $i = 1, 2, 3, \ldots, l-2$, there exists at least one point $x \in (X_D \backslash S) \bigcup \{k+l-1\}$ such that $d(k+i, x) = D$ since $DG(X_D)$ has no isolated vertex. Let O_a denote the circle with the center a and the radius D, and denote p, q be two intersecting points about O_{k+i} and O_{k+i+1}. If $d(k, k+i) \geq d(k, k+i+1)$, then as shown in Figure 1, $S \bigcup \{p\}$ and $S \bigcup \{q\}$ are not convex sets, so p, q do not belong to X_D, since X_D is a convex set, that is to say, X_D contains no point which has D-length to points $k+i$ and $k+i+1$. Since $DG(X_D)$ has no isolated vertex, it must exist a point $x \in X_D \bigcap O_{k+i+1}$ such that $d(x, k+i+1) = D$. In fact, the line segment $[k+i+1, x]$ must intersect with the line segment $[k, k+l-1]$, since X_D is a convex set. But in this case we have $d(k+i, x) > D$, a contradiction. So we conclude that $d(k, k+i) < d(k, k+i+1)$ for $i = 1, 2, 3, \ldots, l-3$. Combining this with the condition $d(k, k+i) < d(k, k+l-1)$ for any $i = 1, 2, 3, \ldots, l-2$, we obtain the result. □

Lemma 6. *[2] Let X be the vertex set of convex n-gon with k intervertex distances $d_1 > d_2 > \cdots > d_k$. If a side of the convex n-gon has the d_1-length, then $k \geq n-2$.*

Lemma 7. *Let X be a 11-point 5-distance set and $m = |X_D| = 8$. Then X contains a subset $Y \in E_8(4)$ or for every point $v \in X_D$, $d(v) \leq 2$.*

Proof. Let X be the 11-point 5-distance set, and 5 distances are $D = d_1 > d_2 > d_3 > d_4 > d_5$. By lemma 2, we know X_D is a convex set. Denote $X_D = \{1, 2, 3, \ldots, 8\}$, eight points are consecutive with counter-clockwise order. If X_D is a 4-distance set, by lemma 1, $X_D = R_8, R_9 - 1 \in E_8(4)$. So in the following assume X_D is a 5-distance set. If $d(i, i+1) = D$, then by lemma 6, X_D determines at least 6 distances, a contradiction to 5-distance. So for any $i \in X_D$, $d(i) \leq 5$. If $d(i, i+2) = D$, then we know X contains a subset $Y \in E_8(4)$ by removing three points $i, i+1, i+2$ from X_D with the diameter D been eliminated, since two D-length segments in X must cross if they do not share an end point. So in the following we may assume for any $i \in X_D$, $d(i) \leq 3$, $d(i, i+1) \neq D$ and $d(i, i+2) \neq D$. If for $d(i) = 3$ we obtain a contradiction to 5-distance or prove that X contains a subset $Y \in E_8(4)$, then the result is correct. Now without loss of generality we may assume $d(1) = 3$, and $d(1,4) = d(1,5) = d(1,6) = D$. Clearly for point 2, $d(2,6) = D$ or $d(2,7) = D$, and for point 8, $d(3,8) = D$ or $d(4,8) = D$.

Case 1: $d(2,6) \neq D$. Then $d(2,7) = D$, and by Lemma 5, $d_5 = d(2,3) < d(2,4) < d(2,5) < d(2,6) < d(2,7) = D$, $d_5 = d(2,3) = d(3,4) = d(4,5) = d(5,6)$, $d_4 = d(4,6) = d(3,5) = d(2,4)$. So $\triangle 345 \cong \triangle 456$. But $\angle 345 \neq \angle 456$, a contradiction.

Case 2: $d(2,6) = D$. If $d(4,8) \neq D$, then $d(3,8) = D$, similar as the proof for Case 1, we can obtain a contradiction. So we may assume $d(4,8) = D$.

Case 2.1: $d(2,7) = d(3,8) = D$. Clearly $d(4,7) \neq D$ and $d(3,6) \neq D$. If $d(3,7) \neq D$, then we know X contains a subset $Y \in E_8(4)$ by removing three points 1, 2, 8 from X_D with the diameter D been eliminated. So assume $d(3,7) = D$, then $DG(\{3,8,4,1,6,2,7\}) = C_7$. When $\{3,8,4,1,6,2,7\}$ is a 3-distance set, $\{3,8,4,1,6,2,7\} = R_7$ since $g(3) = 7$ and the only convex 7-point set that determines three distances is R_7 [1], but $R_7 \bigcup \{5\}$ is not a 5-distance, a contradiction. When $\{3,8,4,1,6,2,7\}$ is a 4-distance set, $\{3,8,4,1,6,2,7\} = R_8 - 1$ or $R_9 - 2$ by Lemma 1, but $DG(R_8 - 1) \neq C_7$ and $DG(R_9 - 2) \neq C_7$. Now assume $\{3,8,4,1,6,2,7\}$ is 5-distance. By Lemma 5, $d(1,2) \leq d_3$, $d(1,8) \leq d_3$ and the other $d(x, x+1) \leq d_4$ for $x = 2,3,4,5,6,7 \in X_D$. Hence in the following proof we first consider the case $d(1,2) = d_3$ or $d(1,8) = d_3$ and give the complete proof, secondly we consider the case $d(x, x+1) \leq d_4$ for every $x \in X_D$ and the proof is omitted since we use the similar way.

We know that there exists no point on a circle which has the same distance to other three points on the circle, so in the following if we obtain all points of X_D lie on a circle, then we can conclude a contradiction, and hence X_D is not a 5-distance set. Now for the case $d(1,2) = d_3$ or $d(1,8) = d_3$, we depart two parts to proof.

Part I: $d(1,2) = d_3$ and $d(1,8) = d_3$.

By Lemma 5, $d_3 = d(1,2) < d(2,8) < d(2,7) = d_1$, $d_3 = d(1,8) < d(1,7) < d(1,6) = d_1$, $d_3 = d(1,2) < d(1,3) < d(1,4) = d_1$. Since $\triangle 138 \cong \triangle 172$, we know $\angle 318 = \angle 712$, and hence $\angle 312 = \angle 718$, $\triangle 312 \cong \triangle 718$, conclude $d(2,3) = d(7,8)$ and $28 \| 37$. Since $\angle 846 = \angle 146 - \angle 148 = \angle 164 - \angle 162 = \angle 264$, we conclude $\triangle 264 \cong \triangle 846$, and hence $d(2,4) = d(6,8)$. Clearly now $\triangle 124 \cong \triangle 186$ and $\angle 214 = \angle 816$. Since $\angle 213 = \angle 817$, we know that $\angle 314 = \angle 716$. Combining this with $d(1,3) = d(1,7)$ and $d(1,4) = d(1,6)$, we conclude $\triangle 314 \cong \triangle 716$ and hence $d(3,4) = d(6,7)$.

Suppose $d(2,3) = d(7,8) = d_4$. By Lemma 5, $d_4 = d(2,3) < d(2,4) < d(2,5) < d(2,6) = d_1$, $d_4 = d(7,8) < d(6,8) < d(5,8) < d(4,8) = d_1$. Now $\triangle 428 \cong \triangle 682$, $\angle 482 = \angle 628$, $\angle 528 = \angle 582$, and hence $\angle 526 = \angle 485$. In this case $\triangle 526 \cong \triangle 584$, and $d(4,5) = d(5,6)$. It follows that $28 \| 37 \| 46$ and $15 = \perp 46 = \perp 37 = \perp 28$. By Lemma 5 we know that $d_4 \leq d(4,6) \leq d_3$. Take $d(4,6) = d_3$, then the five points 1, 2, 4, 6, 8 lie on a circle and $d(2,8) = d_1$, this contradicts the fact $d(2,8) = d_2$. So $d(4,6) = d_4$, and hence $d(4,5) = d(5,6) = d_5$, by Lemma 5 we know that $d_4 \leq d(3,5) \leq d_3$. Take $d(3,5) = d_4$, then $d(3,4) = d_5$ and $\triangle 345 \cong \triangle 456$, this contradicts the fact that $\angle 345 \neq \angle 456$. So $d(3,5) = d(5,7) = d_3$, and hence $d(3,6) = d(4,7) = d_2$. Now we conclude that points 1, 2, 3, 5 lie on a circle, points 1, 3, 5, 8 lie on a circle, points 1, 2, 5, 7 lie on a circle, points 2, 4, 7, 8 lie on a circle, points 2, 4, 6, 8 lie on a circle, clearly at last all points of X_D lie on the circle, this contradicts the fact $d(1,4) = d(1,5) = d(1,6) = d_1$.

Suppose $d(3,4) = d(6,7) = d_4$. By Lemma 5, $d_4 = d(3,4) < d(3,5) < d(3,6) < d(3,7) = d_1$, $d_3 = d(3,5) < d(2,5) < d(1,5) = d_1$, $d_4 = d(6,7) < d(5,7) < d(4,7) < d(3,7) = d_1$, $d_3 = d(5,7) < d(5,8) < d(1,5) = d_1$, $d_4 = d(3,4) < d(2,4) < d(2,5) = d_2$, $d_4 = d(6,7) < d(6,8) < d(5,8) = d_2$. Now we conclude that points 1, 2, 3, 5 lie on a circle, points 1, 5, 7, 8 lie on a circle, and hence $15 || 23 || 78$. Combining this with $28 \| 37$, conclude that $d_2 = d(2,8) = d(3,7) = d_1$, a contradiction.

Suppose $d(2,3) = d(7,8) = d_5$ and $d(3,4) = d(6,7) = d_5$. At first assume $d(5,6) = d(4,5) = d_4$. Then $d_4 = d(4,5) < d(3,5) < d(2,5) < d(1,5) = d_1$ and $d_4 = d(5,6) < d(5,7) < d(5,8) < d(1,5) = d_1$, and obtain $23 || 15 || 78$. Combining this with $28 \| 37$, we can see that $d_2 = d(2,8) = d(3,7) = d_1$, a contradiction. Secondly assume $d(5,6) = d(4,5) = d_5$. Then $\angle 345 = \angle 567 < \angle 456$, and it must have $d(3,5) = d(5,7) = d_4$, $d(4,6) = d_3$ and $d(2,5) = d(5,8) = d(3,6) = d(1,3) = d_2$. Until now all points of X_D lie on the circle, a contradiction. At last we may assume $d(4,5) = d_5$ and $d(5,6) = d_4$. Then $d_4 = d(5,6) < d(5,7) < d(5,8) < d(1,5) = d_1$ and $d_4 = d(5,6) < d(4,6) < d(3,6) < d(2,6) = d_1$, all points of X_D lie on the circle, a contradiction.

Part II: $d(1,2) = d_3$ and $d(1,8) < d_3$.

By Lemma 5, $d_3 = d(1,2) < d(2,8) < d(2,7) = d_1$, $d_3 = d(1,2) < d(1,3) < d(1,4) = d_1$.

Suppose $d(5,6) = d_4$. By Lemma 5, $d_4 = d(5,6) < d(4,6) < d(3,6) < d(2,6) = d_1$, $d_4 = d(5,6) < d(5,7) < d(5,8) < d(1,5) = d_1$, $d_3 = d(5,7) < d(4,7) < d(3,7) = d_1$, $d(3,5) \le d_3$, $d(2,4) \le d_3$. Since $d(5,7) = d(4,6) = d_3$, clearly it must have $d(4,5) \neq d(6,7)$. Now points 1, 2, 4, 6 lie on a circle, points 2, 3, 6, 8 lie on a circle, points 1, 3, 5, 8 lie on a circle, points 1, 2, 5, 7 lie on a circle, points 1, 3, 4, 7 lie on a circle, points 2, 4, 7, 8 lie on a circle. If $d(2,4) = d_3$, then $d(2,4) < d(2,5) = d_2$ and points 2, 4, 5, 7 lie on a circle, and hence all points of X_D lie on the circle, a contradiction. So $d(2,4) = d_4$, then $d_3 \le d(2,5) \le d_2$. We can prove that $d(2,5) = d_2$, since otherwise $d(2,5) = d_3$ and points 2, 4, 5, 6 lie on a circle, combining this with the former results we know that all points of X_D lie on the circle, a contradiction. Similarly $d(7,8) = d_4$, since otherwise $d(7,8) = d_5$ and points 2, 3, 7, 8 lie on a circle, combining this with the former results we know that all points of X_D lie on the circle, a contradiction. Since $d(7,8) = d_4$, we know that $d(6,8) = d_3$ and points 2, 4, 7, 8 lie on a circle with $d_2^2 + d_4^2 = d_1^2$. When $d(3,5) = d_4$, $d(4,5) = d_5$, points 1, 3, 5, 8 lie on a circle with $d(1,8) = d_4$ since $d_2^2 + d_4^2 = d_1^2$. Then $d(6,7) = d_4$, since otherwise $d(6,7) = d_5$, $\triangle 456 \cong \triangle 765$ and $\angle 456 \cong \angle 765$, this contradicts the fact $\angle 456 > \angle 765$. Now $d(1,7) = d(6,8) = d(5,7) = d_3$, and points 1, 5, 6, 7, 8 lie on a circle, and hence all points of X_D lie on the circle, a contradiction. When $d(3,5) = d_3$, points 1, 2, 3, 5 lie on a circle, finally all points of X_D lie on the circle, a contradiction. Therefore $d(5,6) = d_5$.

Suppose $d(4,5) = d_4$. By Lemma 5, $d_4 = d(4,5) < d(4,6) < d(4,7) < d(4,8) = d_1$, $d_4 = d(4,5) < d(3,5) < d(2,5) < d(1,5) = d_1$, $d_3 = d(3,5) < d(3,6) < d(3,7) = d_1$, $d_4 \le d(5,7) \le d_3$. Now points 2, 4, 7, 8 lie on a circle, points 1, 3, 4, 7 lie on a circle, points 1, 2, 3, 5 lie on a circle, points 1, 2, 4, 6 lie on a circle. If $d(3,4) = d_5$, then $\triangle 345 \cong \triangle 654$ and $\angle 345 = \angle 654$, this contradicts the fact $\angle 345 = \angle 143 + \angle 145 = \angle 143 + \angle 154 < \angle 154 + \angle 156 = \angle 654$, a contradiction. So $d(3,4) = d_4$, by Lemma 5, $d_4 = d(3,4) < d(2,4) < d(2,5) = d_2$. By the same reason, $d(2,3) = d_4$. Now points 2, 3, 4, 5 lie on a circle, and hence all points of X_D lie on the circle, a contradiction. Therefore $d(4,5) = d_5$.

Suppose $d(6,7) = d_4$. By Lemma 5, $d_4 = d(6,7) < d(5,7) < d(4,7) < d(3,7) = d_1$, $d_3 = d(5,7) < d(5,8) < d(1,5) = d_1$, $d_4 = d(6,7) < d(6,8) < d(5,8) = d_2$. Now we can see that points 1, 2, 5, 7 lie on a circle, points 1, 3, 5, 8 lie on a circle, points 1, 3, 4, 7 lie on a circle, points 2, 4, 7, 8 lie on a circle. If $d(3,4) = d_4$, $d_4 = d(3,4) < d(3,5) < d(3,6) < d(3,7) = d_1$ and points 3, 4, 6, 7 lie on a circle, points 3, 5, 6, 8 lie on a circle, finally all points of X_D lie on the circle, a contradiction. So $d(3,4) = d_5$. Clearly

$d_4 = d(3,5) < d(4,6) = d_3$, since $\angle 345 < \angle 456$ and $d(3,4) = d(4,5) = d(5,6) = d_5$. Now by Lemma 5, $d_3 = d(4,6) < d(3,6) = d_2$, points 2, 3, 6, 8 lie on a circle. When $d(2,3) = d_4$, points 2, 3, 6, 7 lie on a circle, finally all points of X_D lie on the circle, a contradiction. So $d(2,3) = d_5$. By the same reason, $d(7,8) = d_4$. Denote $\angle 415 = \alpha$, $\angle 627 = \beta$, $\angle 416 = \gamma$, then $\alpha < \beta < \gamma$ and $\gamma = 2\alpha$, $\angle 678 = \pi - \beta - \alpha$, $\angle 567 = \frac{1}{2}(\pi - \beta) + \frac{1}{2}(\pi - \alpha) - \gamma = (\pi - \beta - \alpha) + \frac{1}{2}(\beta - 3\alpha) < \angle 678$, and hence $d_3 = d(5,7) < d(6,8) = d_3$, a contradiction. Therefore $d(6,7) = d_5$.

Suppose $d(3,4) = d_4$. By Lemma 5, $d_4 = d(3,4) < d(3,5) < d(3,6) < d(3,7) = d_1$, $d_4 = d(3,4) < d(2,4) < d(2,5) < d(1,5) = d_1$, and we can see that points 1, 2, 3, 5 lie on a circle, points 2, 3, 6, 8 lie on a circle. Since $d(4,5) = d(5,6) = d(6,7) = d_5$ and $\angle 456 > \angle 765$, we can see that $d_4 = d(5,7) < d(4,6) = d_3 < d(4,7) = d_2$, and hence $d(2,5) = d(1,7) = d_2$. Then until now we can find that points 1, 3, 6, 7 lie on a circle, points 1, 3, 4, 7 lie on a circle, points 1, 2, 5, 7 lie on a circle, and conclude that all points of X_D lie on the circle, a contradiction. Therefore $d(3,4) = d_5$.

Suppose $d(2,3) = d_4$. By Lemma 5, $d_4 = d(2,3) < d(2,4) < d(2,5) < d(2,6) = d_1$. Since $d(3,4) = d(4,5) = d(5,6) = d(6,7) = d_5$, we can see that $d_4 = d(3,5) = d(5,7) < d(4,6) = d_3 < d(4,7) = d(3,6) = d_2$, and hence $d(1,3) = d(5,8) = d(2,5) = d(1,7) = d(3,6) = d_2$. Now we conclude that points 1, 2, 4, 6 lie on a circle, points 2, 3, 6, 8 lie on a circle, points 1, 2, 5, 7 lie on a circle. If $d(7,8) = d_5$, then clearly points 3, 4, 5, 6, 7, 8 lie on a circle, finally we can conclude that all points of X_D lie on the circle, a contradiction. If $d(7,8) = d_4$, then $d(7,8) < d(6,8) < d(5,8) = d_2$ and clearly points 2, 3, 7, 8 lie on a circle, points 2, 4, 6, 8 lie on a circle, finally we can conclude that all points of X_D lie on the circle, a contradiction.

From now we know it must have $d(2,3) = d(3,4) = d(4,5) = d(5,6) = d(6,7) = d_5$. It is easy to see that $d(3,5) = d(2,4) = d(5,7) = d_4$ and $d(4,6) = d_3$, and hence $d(1,3) = d(3,6) = d(4,7) = d(2,5) = d(5,8) = d(1,7) = d_2$. Now we can conclude that points 1, 3, 4, 7 lie on a circle, points 1, 2, 4, 6 lie on a circle, points 1, 3, 5, 8 lie on a circle, points 1, 3, 6, 7 lie on a circle, points 2, 3, 4, 5 lie on a circle, clearly at last all points of X_D lie on the circle, a contradiction.

Case 2.2: $d(2,7) = D$, $d(3,8) \neq D$ ($d(2,7) \neq D$, $d(3,8) = D$, the proof is similar). Clearly for point 3, $d(3,7) = D$ and $d(3,6) \neq D$. If $d(4,7) = D$, then $DG(\{1,2,4,6,7\}) = C_5$. When $\{1,2,4,6,7\} = R_5$, clearly $d(2,3) = d(3,4)$, $d(4,5) = d(5,6)$, $d(7,8) = d(8,1)$, and hence $R_5 \bigcup \{3,5,8\}$ has at least 6 distances, since $d(3,4) < d(3,5) < d(2,4) < d(3,6) < d(3,8) < d(1,4)$, a contradiction. When $\{1,2,4,6,7\} \neq R_5$, the set $\{1,2,4,6,7\}$ has at least 4 distances, and X_D has at least 6 distances, a contradiction. If $d(4,7) \neq D$, we can obtain a contradiction to 5-distance too.

Case 2.3: $d(2,7) \neq D$, $d(3,8) \neq D$. If $d(3,7) \neq D$, then we know X

contains a subset $Y \in E_8(4)$ by removing three points 1, 4, 6 from X_D with the diameter D been eliminated. So assume $d(3,7) = D$. Then by lemma 5 $d(3,4) \leq d_3$, $d(6,7) \leq d_3$, and $d(x, x+1) \leq d_4$ for the other $x \in X_D$, The proof is similar and more easier than the proof of Case 2.1, we can obtain a contradiction to 5-distance. □

Lemma 8. *Let X be a 11-point 5-distance set and $m = |X_D| = 8$. Then X contains a subset $Y \in \{R_{10} - 2, R_{11} - 3\} \bigcup E_8(4)$.*

Proof. Let $X_D = \{1,2,3,4,5,6,7,8\}$, 8 points are consecutive with counter-clockwise order. If $DG(X_D)$ contains a cycle, by Lemma 3, the cycle is C_3 or C_5 or C_7. Then the remaining points must connect with the points on the cycle, thus there exists a point p such that $d(p) \geq 3$, a contradiction to Lemma 7. So $DG(X_D)$ does not contain a cycle. By Lemma 7, since $DG(X_D)$ has no isolated vertex, X contains a subset $Y \in E_8(4)$ or $DG(X_D)$ may be P_8, $P_6 \bigcup P_2$, $P_5 \bigcup P_3$, $2P_4$, $P_4 \bigcup 2P_2$, $2P_3 \bigcup P_2$ or $4P_2$. Clearly X_D is a 4-distance or 5-distance set since $g(3) = 7$ [1]. If X_D is a 4-distance set, by lemma 1, $X_D = R_8, R_9 - 1 \in E_8(4)$. So assume X_D is a 5-distance set with intervertex distances $D = d_1 > d_2 > d_3 > d_4 > d_5$.

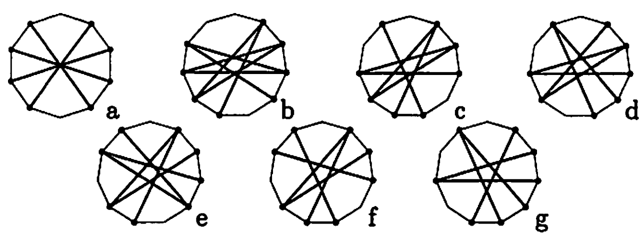

Figure 2:

Assume $DG(X_D) = 4P_2$. Denote by $D = d(4,8) = d(3,7) = d(2,6) = d(1,5) = d_1$. By Lemma 5 $d(x, x+1) \leq d_4$ for every $x \in X_D$. If all the edges of convex 8-point set X_D have the same length, then the 8 points of X_D lie on a circle and X_D must be R_8, but R_8 is a 4-distance set. So it must exist two consecutive edges which have distinct distances, without loss of generality we may assume $d(1,2) = d_5$ and $d(1,8) = d_4$. Then by Lemma 5, $d_4 = d(1,8) < d(2,8) < d(3,8) < d(4,8) = d_1$, and $d_4 = d(1,8) < d(1,7) < d(1,6) < d(1,5) = d_1$, and $d_3 = d(1,7) < d(2,7) = d_2 < d(3,7) = d_1$. Now we prove that $X_D = R_{10} - 2$ (the 2 points absented has D-length).

Suppose $d(2,3) = d_4$. Then by Lemma 5, $d_4 = d(2,3) < d(2,4) < d(2,5) < d(2,6) = d_1$, $d_4 = d(2,3) < d(1,3) < d(3,8) < d(3,7) = d_1$, $d_3 = d(1,3) < d(1,4) < d(1,5) = d_1$. At first assume $d(7,8) = d_4$. Then points 1, 2, 3, 7, 8 lie on a circle, but $\angle 137 \neq \angle 173$ which contradicts the fact $d(1,3) = d(1,7) = d_3$. From this case we can conclude that there exists no four consecutive edges of X_D which have lengths $d_4 - d_4 - d_5 - d_4$. Secondly assume $d(7,8) = d_5$. We know $d(3,4) = d_5$, since otherwise it exists four consecutive edges of X_D which have lengths $d_4 - d_4 - d_5 - d_4$. Now we conclude that points 1, 2, 3, 8 lie on a circle, points 1, 2, 7, 8 lie on a circle, points 3, 4, 7, 8 lie on a circle, and hence points 1, 2, 3, 4, 7, 8 lie on the circle. But $\angle 184 \neq \angle 348$, which contradicts $d(3,8) = d(1,4) = d_2$. From this case we can conclude that there exists no four consecutive edges of X_D which have lengths $d_5 - d_4 - d_5 - d_4$. Therefore $d(2,3) = d_5$.

Suppose $d(7,8) = d_4$. Recall that $d(1,2) = d(2,3) = d_5$ and $d(1,8) = d_4$. Then by Lemma 5, $d_4 = d(7,8) < d(6,8) < d(5,8) < d(4,8) = d_1$, $d_4 = d(7,8) < d(1,7) < d(2,7) = d_2$. At first assume $d(6,7) = d_5$. Then it must have $d(5,6) = d_5$, since otherwise it exists four consecutive edges of X_D which have lengths $d_4 - d_4 - d_5 - d_4$. Now we conclude that points 1, 2, 6, 7 lie on a circle, points 2, 3, 6, 7 lie on a circle, points 1, 2, 5, 6 lie on a circle, and hence points 1, 2, 3, 5, 6, 7 lie on the circle. Since $\triangle 167 \cong \triangle 832$, we can prove that point 8 is also on the circle by the elementary fact (4) in [3]. But $\angle 862 \neq \angle 731$, which contradicts the fact $d(2,8) = d(1,7) = d_3$. Secondly assume $d(6,7) = d_4$. Then by Lemma 5, $d_4 = d(6,7) < d(5,7) < d(4,7) < d(3,7) = d_1$. Then points 1, 6, 7, 8 lie on a circle, points 2, 5, 7, 8 lie on a circle, points 3, 4, 7, 8 lie on a circle. We can see that $\triangle 185 \cong \triangle 874$, $\triangle 187 \cong \triangle 876$, $\angle 581 + \angle 785 = \angle 781 = \angle 876 = \angle 874 + \angle 674$, that is, $\angle 587 = \angle 476$, $\triangle 587 \cong \triangle 476$, $d_3 = d(5,7) = d(4,6)$. By the same reason $d_3 = d(5,7) = d(1,3)$. Then points 1, 3, 4, 6 lie on a circle, points 1, 3, 6, 8 lie on a circle, points 1, 4, 5, 8 lie on a circle, finally all points lie on the circle. But $\angle 862 < \angle 826$, this contradicts $d_3 = d(2,8) = d(6,8)$, which imply $\angle 862 = \angle 826$. From this case we can conclude that there exists no four consecutive edges of X_D which have lengths $d_4 - d_4 - d_5 - d_5$. Hence $d(7,8) = d_5$. Then it must have $d(6,7) = d_5$, since otherwise it exists four consecutive edges of X_D which have lengths $d_4 - d_5 - d_4 - d_5$.

Until now we know that $d(1,2) = d(2,3) = d(7,8) = d(6,7) = d_5$ and $d(1,8) = d_4$, and points 1, 2, 3, 6, 7, 8 lie on a circle. In the following we prove that $d(3,4) = d(5,6) = d_5$ and $d(4,5) = d_4$. When $d(3,4) = d(4,5) = d(5,6) = d_5$, clearly all points of X_D lie on the circle. Assume $\angle 172 = \alpha$, $\angle 148 = \beta$. Since $\angle 136 > \angle 316$, we know that $d_2 = d(1,6) > d(3,6) = d_3$, and $3\alpha = \angle 316 = \angle 137 = \alpha + \beta$, hence $\beta = 2\alpha$. But in this case $\angle 247 = \beta + 2\alpha = 4\alpha = \angle 286$, and then $d_2 = d(2,7) = d(2,6) = d_1$, a contradiction. When at least two of $d(3,4), d(4,5), d(5,6)$ are d_4, it exists four consecutive edges of X_D which have lengths $d_4 - d_4 - d_5 - d_5$ or $d_4 - d_5 - d_4 - d_5$, a contradiction. So in the following we only need to consider the case that it has only one of $d(3,4), d(4,5), d(5,6)$ which is d_4. When $d(3,4) = d_4$ (for $d(5,6) = d_4$, the proof is similar), clearly all points of X_D lie on the circle, and $\angle 418 < \angle 682$, and hence $d_1 = d(4,8) < d(2,6) = d_1$, a contradiction.

At last it remains to consider the case $d(3,4) = d(5,6) = d_5$ and $d(4,5) = d_4$.

By Lemma 5, $d_4 = d(4,5) < d(3,5) < d(2,5) < d(1,5) = d_1$, $d_4 = d(4,5) < d(4,6) < d(4,7) < d(4,8) = d_1$, $d_3 = d(4,6) < d(3,6) < d(2,6) = d_1$. Clearly all points lie on the circle. And we can see that $d(1,3) = d(2,4) = d(5,7) = d(6,8) = d_4$, $d(1,4) = d(5,8) = d_3$, and the line segments $[1,5]$, $[2,6]$, $[3,7]$, $[4,8]$ are four diameters of the circle, since quadrangles 1256 and 3478 are rectangles. Until now very beautifully we obtain the only convex 8-point 5-distance configuration with $DG(X_D) = 4P_2$, that is, $R_{10} - 2$ as shown in Figure 2a.

If $DG(X_D) = P_8$, $P_6 \bigcup P_2$, $P_5 \bigcup P_3$, $2P_4$, $P_4 \bigcup 2P_2$ or $2P_3 \bigcup P_2$, then similarly we can prove that $X_D = R_{11} - 3$ as shown in Figure 2b–2g. □

Lemma 9. *[6] Let $G = DG(X)$ be the diameter graph of X with $|X| = n$. If $G \neq C_n$, then we have $\alpha(G) \geq \lceil n/2 \rceil$.*

Lemma 10. *Let X be a 11-point 5-distance set. Then X contains a subset $Y \in \{R_7, R_9, R_{10} - 2, R_{11} - 3\} \bigcup E_8(4)$.*

Proof. Let X be the 11-point 5-distance set. If $m = |X_D| \geq 9$, then by Lemma 2 and Lemma 1, the subset $X_D \subset X$ is a convex set, and X contains a subset $Y \in \bigcup_{k \leq 5} M_9(k) = \{R_9, R_{10}-1, R_{11}-2\}$. If $m = 8$, then by Lemma 8, X contains a subset $Y \in \{R_{10} - 2, R_{11} - 3\} \bigcup E_8(4)$. If $m = 7$, and if $DG(X_D) = C_7$, then we can prove that $X_D = R_7$; and if $DG(X_D) \neq C_7$, then by Lemma 9, $\alpha(DG(X_D)) \geq 4$, X contains a subset $Y \in E_8(4)$ by removing three points from X_D with the diameter D been eliminated. If $m \leq 6$, then $11 - \lceil \frac{m}{2} \rceil \geq 8$, by Lemma 2, X contains a subset $Y \in E_8(4)$. □

3 Classification of 11-point 5-distance sets

Lemma 11. *[1] $g(4) = 9$ and every 9-point 4-distance set in the plane is isomorphic to R_9 or one of the three configurations given in Figure 3a-3c.*

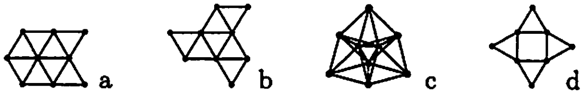

Figure 3: 9-point 4-distance sets, and a special 8-point 4-distance set

Lemma 12. *[6] Every 8-point 4-distance set in the plane is isomorphic to R_8, R_7^+, an 8-point subset of a 9-point 4-distance set or figure 3d.*

Theorem 13. *There are four 11-point 5-distance sets in the plane to within isomorphism, that are R_{11} and the three configurations given in Figure 4.*

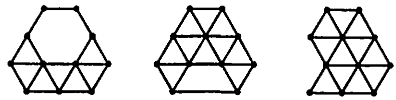

Figure 4: 11-point 5-distance sets

Proof. Let X be the 11-point 5-distance set. By Lemma 10, X contains a subset $Y \in \{R_7, R_9, R_{10}-2, R_{11}-3\} \bigcup E_8(4)$. If X contains a subset $Y \in \{R_7, R_9, R_{10}-2\}$, then it is clear that they can not be extended to a 11-point 5-distance set. If X contains a subset $R_{11}-3$, then it is clear that they can be extended to a 11-point 5-distance set R_{11}. Now assume X contains a subset $Y \in E_8(4)$. From Lemma 12 and Lemma 11 we know all the configurations of $E_8(4)$ as considered in the following. At first assume X contains a subset $Y \in \{R_7^+, R_8, R_9-1\}$, then it is clear that they can not be extended to 11-point 5-distance sets. Secondly assume X contains an 8-point subset of a 9-point 4-distance set of Figure 3c. The proof is similar as in [6], it can not be extended to a 11-point 5-distance set. Thirdly assume X contains an 8-point subset of a 9-point 4-distance set of Figure 3a, or Figure 3b. Clearly $X \subset L_{\triangle} = \{a(1,0) + b(\frac{1}{2}, \frac{\sqrt{3}}{2}) : a, b \in \mathbb{Z}\}$ and X is one of the three 11-point 5-distance sets as shown in Figure 4. At last, assume X contains the special 8-point 4-distance set of Figure 3d. Also we can prove that it can not be extended to a 11-point 5-distance set. The proof is complete. □

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