SOME IDENTITIES ON THE GENERALIZED HIGHER-ORDER EULER AND BERNOULLI NUMBERS

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ABSTRACT. By the classical method for obtaining the values of the Riemann zeta-function at even positive integral arguments, we shall give some functional equational proof of some interesting identities and recurrence relations related to the generalized higher-order Euler and Bernoulli numbers attached to a Dirichlet character χ with odd conductor d, and shall show an identity between generalized Euler numbers and generalized Bernoulli numbers. Finally, we remark that any weighted short-interval character sums can be expressed as a linear combination of Dirichlet L-function values at positive integral arguments, via generalized Bernoulli (or Euler) numbers.
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1. Introduction

Let d be a fixed positive odd integer and let χ be the Dirichlet's character with conductor d. For a real or complex parameter α, the generalized higher-order Euler numbers $E_{n,\chi}^{(\alpha)}$ and polynomials $E_{n,\chi}^{(\alpha)}(x)$ attached to χ

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are defined by

$$\left(\frac{2\sum_{a=0}^{d-1}(-1)^a\chi(a)e^{at}}{e^{dt}+1}\right)^{\alpha}=\sum_{n=0}^{\infty}E_{n,\chi}^{(\alpha)}\frac{t^n}{n!}, \tag{1.1}$$

$$\left(\frac{2\sum_{a=0}^{d-1}(-1)^a\chi(a)e^{at}}{e^{dt}+1}\right)^{\alpha}e^{xt}=\sum_{n=0}^{\infty}E_{n,\chi}^{(\alpha)}(x)\frac{t^n}{n!},$$

where $|t|<\frac{\pi}{d}$, and $E_{n,\chi}^{(1)}=E_{n,\chi}, E_{n,\chi}^{(1)}(x)=E_{n,\chi}(x)$ signify the generalized Euler numbers and polynomials (cf. [1]). Similarly, $B_{n,\chi}^{(\alpha)}$ and $B_{n,\chi}^{(\alpha)}(x)$ denote the generalized higher-order Bernoulli numbers and polynomials attached to χ, by

$$\left(\frac{\sum_{a=1}^{d}\chi(a)te^{at}}{e^{dt}-1}\right)^{\alpha}=\sum_{n=0}^{\infty}B_{n,\chi}^{(\alpha)}\frac{t^n}{n!},\ |t|<\frac{2\pi}{d}, \tag{1.2}$$

$$\left(\frac{\sum_{a=1}^{d}\chi(a)te^{at}}{e^{dt}-1}\right)^{\alpha}e^{xt}=\sum_{n=0}^{\infty}B_{n,\chi}^{(\alpha)}(x)\frac{t^n}{n!},\ |t|<\frac{2\pi}{d}.$$

Then $B_{n,\chi}^{(1)}=B_{n,\chi}, B_{n,\chi}^{(1)}(x)=B_{n,\chi}(x)$ be the generalized Bernoulli numbers and polynomials (cf.[2], [14]). In particular, if $\chi=\chi_0$ $(d=1)$ is the trivial character, then

$$E_{n,\chi_0}^{(\alpha)}(\frac{\alpha}{2})=2^{-n}E_n^{(\alpha)},\ E_{n,\chi_0}^{(\alpha)}(x)=E_n^{(\alpha)}(x),$$
$$B_{n,\chi_0}^{(\alpha)}=(-1)^nB_n^{(\alpha)},\ B_{n,\chi_0}^{(\alpha)}(x)=(-1)^nB_n^{(\alpha)}(-x),$$

where $E_n^{(\alpha)}, E_n^{(\alpha)}(x), B_n^{(\alpha)}, B_n^{(\alpha)}(x), (n\geq 0)$ be the higher-order Euler numbers and polynomials, higher-order Bernoulli number and polynomials, respectively. Clearly, by (1.1) and (1.2) and the classical method for comparing the coefficients of their generating functions, we have

$$E_{n,\chi}^{(\alpha)}(x)=\sum_{m=0}^{n}\binom{n}{m}E_{m,\chi}^{(\alpha)}x^{n-m},$$
$$B_{n,\chi}^{(\alpha)}(x)=\sum_{m=0}^{n}\binom{n}{m}B_{m,\chi}^{(\alpha)}x^{n-m}$$

and (also cf. [1, (4)] and [10, (2.12)])

$$E_{k,\chi}(nd) + E_{k,\chi} = 2T_{k,\chi}(nd), \tag{1.3}$$

$$B_{k,\chi}(nd) - B_{k,\chi} = kT'_{k-1,\chi}(nd-1), \tag{1.4}$$

where $T_{k,\chi}(n) = \sum_{l=0}^{n-1}(-1)^l\chi(l)l^k$ and $T'_{k,\chi}(n) = \sum_{l=1}^{n}\chi(l)l^k$.

The main interesting of the generalized Bernoulli numbers is that they give the values at non-positive integers of Dirichlet L-functions $L(s,\chi) = \sum_{n=1}^{\infty}\chi(n)n^{-s}$ $(\sigma > 1)$ attached to χ (e.g. cf.[15, (2)]):

$$L(1-n,\chi) = -\frac{B_{n,\chi}}{n}\ (n \geq 1). \tag{1.5}$$

Various identities for the higher-order generalized resp. twisted Euler and Bernoulli numbers and polynomials have been studied by T. Kim ([1]-[13]), Kurt, G. Liu and several authors (see [16]-[20]), a great deal of interesting and valuable results have been developed by analytic method, algebraic method and elementary method et al. For instance, in [1]–[10], T. Kim et al gave some symmetry properties for the generalized higher-order Euler resp. Bernoulli polynomials by the classical analytic method resp. the symmetry properties of the p-adic invariant integral on $\mathbb{Z}_p$. These and many other interesting results on generalized higher-order Euler and Bernoulli numbers and polynomials, such as q-Euler, q-bernoulli polynomials, the higher-order generalized twisted Euler and Bernoulli numbers and polynomials attached to a Dirichlet character χ, readers may refer to T. Kim et al's work [11]–[22].

The main purpose of this paper, is to prove some identities for the generalized higher-order Euler and Bernoulli numbers by the classical method for obtaining the values of the Riemann zeta-function at even positive integral arguments by comparing the Laurent coefficients of the partial fraction expansion for the hyperbolic cotangent function $\coth x$ (or the cotangent function $\cot x$), which is a form of the Lambert series (e.g. cf. [19, Exercise5.4]). It turns out that some interesting recurrence relationship and multiplication theorem for the generalized higher-order Euler and Bernoulli numbers attached to χ, and that an identity between generalized Euler and Bernoulli numbers, i.e. $E_{n,\chi} = -\frac{2^{n+1}\chi(2)-1}{n+1}B_{n+1,\chi}$ for $d > 1$.

2. The identities of the Generalized Higher-order Euler and Bernoulli Numbers

2.1. Identities related to the generalized higher-order Euler Numbers. Let $\alpha = l$ denotes any positive integer, we shall first consider the following functional equation

$$\sum_{n=0}^{\infty} E_{n,\chi}^{(l)} 2^n \frac{t^n}{n!} = \left(\frac{2\sum_{a=0}^{d-1}(-1)^a \chi(a) e^{2at}}{e^{2dt}+1} \right)^l . \tag{2.1}$$

We start from the definition of the higher-order Euler numbers $E_n^{(l)}$:

$$\left(\frac{2}{e^t+e^{-t}}\right)^l = \sum_{n=0}^{\infty} E_n^{(l)} \frac{t^n}{n!} = \sum_{n=0}^{\infty} E_{2n}^{(l)} \frac{t^{2n}}{(2n)!} \tag{2.2}$$

and an identity derived by Liu[20, (1.25)]

$$E_{2n}^{(l)} = \sum_{k=1}^{n} \rho(k,n) l^k, \tag{2.3}$$

where $\rho(k,n)$ defined by (2.9) (below) and $s(m,k), T(n,m)$ denote the Stirling numbers of the first kind, the central factorial numbers which are defined as following

$$t(t-1)(t-2)\cdots(t-m+1) = \sum_{k=1}^{m} s(m,k) t^k, \tag{2.4}$$

where $m > 0$ (e.g. cf.[21]) and

$$(e^t + e^{-t} - 2)^m = (2m)! \sum_{n=m}^{\infty} T(n,m) \frac{t^{2n}}{(2n)!}. \tag{2.5}$$

From the obvious identity

$$\sum_{a=0}^{d-1}(-1)^a \chi(a) e^{at} = \sum_{a=0}^{d-1}(-1)^a \chi(a) \sum_{n=0}^{\infty} \frac{a^n t^n}{n!} = \sum_{n=0}^{\infty} T_{n,\chi}(d) \frac{t^n}{n!}, \quad t < 0 \tag{2.6}$$

and

$$\begin{aligned}\left(\frac{2}{e^t+e^{-t}}\right)^l e^{-lt} &= \sum_{n=0}^{\infty} E_n^{(l)} \frac{t^n}{n!} \sum_{n=0}^{\infty} \frac{(-lt)^n}{n!} = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \binom{n}{k} E_k^{(l)} (-l)^{n-k} \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{n}{2}]} \binom{n}{2k} E_{2k}^{(l)} (-l)^{n-2k} \frac{t^n}{n!},\end{aligned} \tag{2.7}$$

where the last term follows from $E^{(l)}_{2n+1}=0$, we have conclude the RHS of (2.1) is

$$\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}2^m\binom{n}{m}\sum_{\substack{n_1,\cdots,n_l\in\mathrm{N}\\ n_1+\cdots+n_l=m}}\frac{m!}{n_1!\cdots n_l!}T_{n_1,\chi}(d)\cdots T_{n_l,\chi}(d)\right.$$
$$\left.\sum_{k=0}^{[\frac{n-m}{2}]}d^{n-m}\binom{n-m}{2k}(-l)^{n-m-2k}E^{(l)}_{2k}\right)\frac{t^n}{n!},\qquad(2.8)$$

where $\sum\limits_{\substack{n_1,\cdots,n_l\in\mathrm{N}\\ n_1+\cdots+n_l=m}}$ denotes the summation over all non-negative integer $n_1,\cdots,n_l$ such that $n_1+\cdots+n_l=m$.

Substituting (2.3) and comparing the coefficients of $\frac{t^n}{n!}$ on the both sides of (2.1) (which RHS is (2.8)), we obtain the following theorem.

Theorem 2.1. *By the notations above, we have*

$$E^{(l)}_{n,\chi}=\sum_{m=0}^{n}\binom{n}{m}\sum_{\substack{n_1,\cdots,n_l\in\mathrm{N}\\ n_1+\cdots+n_l=m}}\frac{m!}{n_1!\cdots n_l!}T_{n_1,\chi}(d)\cdots T_{n_l,\chi}(d)$$
$$\times\left(\frac{d}{2}\right)^{n-m}\sum_{k=0}^{[\frac{n-m}{2}]}\binom{n-m}{2k}(-l)^{n-m-2k}\sum_{j=1}^{k}\rho(j,k)l^j,$$

where

$$\rho(j,k)=(-1)^j\sum_{m=j}^{k}s(m,j)\frac{2^{-m}(2m)!}{m!}T(k,m)\qquad(2.9)$$

Corollary 2.2. *By the notations above, we have the recurrence relationship*

$$E^{(l)}_{n,\chi}2^n=\sum_{m=0}^{n}\binom{n}{m}E^{(l-1)}_{m,\chi}\sum_{j=0}^{n-m}\binom{n-m}{j}T_{j,\chi}(d)2^{m+j}$$
$$\times(-d)^{n-m-j}\sum_{k=0}^{[\frac{n-m-j}{2}]}\binom{n-m-j}{2k}\sum_{i=1}^{k}\rho(i,k).$$

Proof. The proof follows easily from rewriting (2.1) as

$$\sum_{n=0}^{\infty}E^{(l)}_{n,\chi}2^n\frac{t^n}{n!}=\left(\frac{2\sum_{a=0}^{d-1}(-1)^a\chi(a)e^{2at}}{e^{2dt}+1}\right)^{l-1}\left(\sum_{a=0}^{d-1}(-1)^a\chi(a)e^{2at}\right)\frac{2}{e^{2dt}+1}$$

by (2.6) and (2.7) with $l=1$, comparing the coefficients of $\frac{t^n}{n!}$ on both sides, we complete the proof. □

2.2. Identities related to the generalized higher-order Bernoulli Numbers. Clearly, by the same argument as we stated in §2.1, we also have some identities related to the generalized higher-order Bernoulli Numbers. Recently, it is Liu [20] who applied the classical method for comparing the coefficients of the generating function $\left(\frac{t}{e^t-1}\right)^\alpha$ obtained ;

$$B_n^{(\alpha)} = \sum_{k=1}^{n} \sigma(k,n)\alpha^k, \tag{2.10}$$

where $\sigma(k,n)$ defined by (2.14) and $b(n,k)$ denote the associated Stirling numbers defined by

$$\left(e^t - 1 - t\right)^k = k! \sum_{m=2k}^{\infty} b(m,k)\frac{t^m}{m!}, \tag{2.11}$$

and $s(j,k)$ is the Stirling numbers of the first kind (see Eq.(2.4)).

For an positive integer $\alpha = l$, we may rewrite the functional equation (1.2) as

$$\sum_{n=0}^{\infty} B_{n,\chi}^{(l)} \frac{t^n}{n!} = d^{-l}\left(\sum_{a=1}^{d}\chi(a)e^{at}\right)^l \left(\frac{dt}{e^{dt}-1}\right)^l, \tag{2.12}$$

by the Laurent expansion

$$\sum_{a=1}^{d}\chi(a)e^{at} = \sum_{n=0}^{\infty} T'_{n,\chi}(d)\frac{t^n}{n!} \tag{2.13}$$

and (2.10) we have

Theorem 2.3. *By the notations above, we have*

$$B_{n,\chi}^{(l)} = \sum_{m=0}^{n}\binom{n}{m} \sum_{\substack{n_1,\cdots,n_l\in\mathbb{N}\\ n_1+\cdots+n_l=m}} \frac{m!}{n_1!\cdots n_l!} T'_{n_1,\chi}(d)T'_{n_2,\chi}(d)\cdots T'_{n_l,\chi}(d)$$
$$\times \sum_{k=1}^{n-m} d^{n-m-l}\sigma(k,n-m)l^k;$$
$$B_{n,\chi}^{(l)} = \sum_{m=0}^{n}\binom{n}{m} B_{m,\chi}^{(l-1)} \sum_{k=0}^{n-m}\binom{n-m}{k} d^{k-1}T'_{n-m-k,\chi}(d) \times \sum_{i=1}^{k}\sigma(i,k),$$

where

$$\sigma(k,r) = (-1)^k \sum_{j=k}^{r} s(j,k)\frac{1}{j!\binom{r+j}{j}} b(r+j,j). \tag{2.14}$$

If we appeal to another representation for higher-order Bernoulli numbers

$$B_n^{(\alpha)} = \sum_{k=1}^{n} (-1)^{n-k} \frac{n!}{k!} \sum_{\substack{n_1,\cdots,n_k \in \mathbb{N} \\ n_1+\cdots+n_k=n}} \frac{B_{n_1}\cdots B_{n_k}}{(n_1\cdots n_k) n_1!\cdots n_k!} \alpha^k, \tag{2.15}$$

for $n \geq k$. Similarly, we have the following recurrence relation

Theorem 2.4. *By the notations above, we have*

$$B_{n,\chi}^{(l)} = \sum_{m=0}^{n} \binom{n}{m} B_{m,\chi}^{(l-1)} \sum_{k=0}^{n-m} \binom{n-m}{k} d^{k-1} T'_{n-m-k,\chi}(d) \sum_{i=1}^{k} (-1)^{k-i} \frac{k!}{i!} \times \sum_{\substack{n_1,\cdots,n_i \in \mathbb{N} \\ n_1+\cdots+n_i=k}} \frac{B_{n_1}\cdots B_{n_i}}{(n_1\cdots n_i) n_1!\cdots n_i!}.$$

2.3. **Identities related to the generalized higher-order Euler-Bernoulli Numbers.** Let m, l be fixed positive integers, and K_1, K_2 be odd integers, we shall consider the following functional equation

$$I = \frac{1}{2} \left(\frac{\sum_{a=1}^{d} \chi(a) K_1 t e^{K_1 a t}}{e^{K_1 d t} - 1} \right)^m (e^{dK_1K_2t} + 1) \times \left(\frac{2\sum_{a=0}^{d-1} (-1)^a \chi(a) e^{K_2 a t}}{e^{K_2 d t} + 1} \right)^l . \tag{2.16}$$

Let $M_\chi(t) = \sum_{n=0}^{\infty} (-1)^n \chi(n) e^{nt}$ and $L_\chi(t) = \sum_{n=1}^{\infty} \chi(n) e^{nt}$ attached to a Dirichlet character χ with odd conductor d, then the series convergence absolutely for $\Re(t) < 0$. By the identity

$$\sum_{a=0}^{K_2 d-1} (-1)^a \chi(a) e^{at} = (1 + e^{K_2 d t}) M_\chi(t), \ K_2 d \equiv 1 \ (\mathrm{mod}\ 2) \tag{2.17}$$

we have the Laurent expansion

$$\begin{aligned} (e^{dK_1K_2t} + 1) \frac{\sum_{a=0}^{d-1} (-1)^a \chi(a) e^{K_2 a t}}{e^{K_2 d t} + 1} &= \sum_{j=0}^{K_1 d-1} (-1)^j \chi(j) e^{jK_2 t} \\ &= \sum_{n=0}^{\infty} T_{n,\chi}(K_1 d) \frac{(K_2 t)^n}{n!}, \end{aligned}$$

therefore (2.16) reads

$$I=\left(\sum_{n=0}^{\infty}B_{n,\chi}^{(m)}\frac{K_1^nt^n}{n!}\right)\left(\sum_{n=0}^{\infty}T_{n,\chi}(K_1d)\frac{(K_2t)^n}{n!}\right)\left(\sum_{n=0}^{\infty}E_{n,\chi}^{(l-1)}\frac{K_2^nt^n}{n!}\right)$$

$$=\sum_{n=0}^{\infty}\left(\sum_{i=0}^{n}\binom{n}{i}B_{n-i,\chi}^{(m)}K_2^iK_1^{n-i}\sum_{k=0}^{i}\binom{i}{k}T_{k,\chi}(K_1d)E_{i-k,\chi}^{(l-1)}\right)\frac{t^n}{n!}. \quad (2.18)$$

Similarly, by

$$\sum_{a=1}^{kd}\chi(a)e^{at}=(1-e^{kdt})L_\chi(t),\ k\in\mathbb{N}, \quad (2.19)$$

and (2.13) it's easy to see that

$$\left(\frac{\sum_{a=1}^{d}\chi(a)K_1te^{K_1at}}{e^{K_1dt}-1}\right)(e^{dK_1K_2t}+1)=\sum_{a=1}^{2K_2d}\chi(a)e^{K_1at}\frac{K_1t}{e^{K_1K_2dt}-1}$$

$$=\left(\sum_{n=0}^{\infty}T'_{n,\chi}(2K_2d)\frac{(K_1t)^n}{n!}\right)\frac{1}{K_2d}\sum_{n=0}^{\infty}B_n\frac{(K_1K_2dt)^n}{n!}$$

$$=\sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\binom{n}{j}T'_{j,\chi}(2K_2d)K_1^nB_{n-j}K_2^{n-j-1}d^{n-j-1}\right)\frac{t^n}{n!}. \quad (2.20)$$

Therefore by (2.16) we obtain

$$I=\frac{1}{2}\left(\sum_{n=0}^{\infty}B_{n,\chi}^{(m-1)}\frac{(K_1t)^n}{n!}\right)\left(\sum_{n=0}^{\infty}E_{n,\chi}^{(l)}\frac{(K_2t)^n}{n!}\right)\times$$

$$\left(\sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{n}{j}T'_{j,\chi}(2K_2d)K_1^nB_{n-j}K_2^{n-j-1}d^{n-j-1}\frac{t^n}{n!}\right)$$

$$=\frac{1}{2}\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{k}\binom{k}{i}B_{i,\chi}^{(m-1)}K_1^iE_{k-i,\chi}^{(l)}K_2^{k-i}\times\right. \quad (2.21)$$

$$\left.\sum_{j=0}^{n-k}\binom{n-k}{j}T'_{j,\chi}(2K_2d)K_1^{n-k}B_{n-k-j}K_2^{n-k-j-1}d^{n-k-j-1}\right)\frac{t^n}{n!}$$

Comparing the coefficients of $\frac{t^n}{n}$ of (2.18) and (2.21), we obtain the following Theorem.

Theorem 2.5. *By the notations above, we have*

$$\sum_{i=0}^{n}\binom{n}{i}B_{n-i,\chi}^{(m)}K_2^iK_1^{n-i}\sum_{k=0}^{i}\binom{i}{k}T'_{k,\chi}(K_1d)E_{i-k,\chi}^{(l-1)}$$
$$=\frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{k}\binom{k}{i}B_{i,\chi}^{(m-1)}K_1^{n+i-k}E_{k-i,\chi}^{(l)}\times$$
$$\sum_{j=0}^{n-k}\binom{n-k}{j}T'_{j,\chi}(2K_2d)B_{n-k-j}K_2^{n-i-j-1}d^{n-k-j-1}.$$

For $K_1 = K_2 = 1$, we have

Corollary 2.6. *By the notations above, we have*

$$\sum_{i=0}^{n}\binom{n}{i}B_{n-i,\chi}^{(m)}\sum_{k=0}^{i}\binom{i}{k}T'_{k,\chi}(d)E_{i-k,\chi}^{(l-1)}$$
$$=\frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{k}\binom{k}{i}B_{i,\chi}^{(m-1)}E_{k-i,\chi}^{(l)}\sum_{j=0}^{n-k}\binom{n-k}{j}T'_{j,\chi}(2K_2d)B_{n-k-j}d^{n-k-j-1}.$$

Remark 2.7. *We remark that by the parity argument,*

$$\sum_{\substack{n=0\\2|n}}^{\infty}\chi(n)e^{nt}-\sum_{\substack{n=0\\2\nmid n}}^{\infty}\chi(n)e^{nt}=\sum_{\substack{n=0\\2|n}}^{\infty}\chi(n)e^{nt}-\left(\sum_{n=0}^{\infty}\chi(n)e^{nt}-\sum_{\substack{n=0\\2|n}}^{\infty}\chi(n)e^{nt}\right),$$

we have

$$M_\chi(t)=2\chi(2)L_\chi(2t)-L_\chi(t),\ (d>1), \tag{2.22}$$

where we omit $n = 0$ since $\chi(0) = \chi(d) = 0$ for $d > 1$. By the obvious identity

$$\sum_{a=0}^{d-1}(-1)^a\chi(a)e^{at}=(e^{dt}+1)M_\chi(t)$$

and (2.19), *recalling the generating function* (1.1) *and* (1.2), *we have the Laurent expansion*

$$M_\chi(t)=\sum_{n=0}^{\infty}E_{n,\chi}\frac{t^n}{n!},\quad L_\chi(t)=-\sum_{n=0}^{\infty}B_{n,\chi}\frac{t^{n-1}}{n!}.$$

Comparing the coefficients on both sides of (2.22), *we obtain*

$$E_{n,\chi}=-\frac{2^{n+1}\chi(2)-1}{n+1}B_{n+1,\chi}.$$

3. On the weighted short-interval character sums

For a Dirichlet character χ modulo d, and N be a multiple of d, say $N = ud$, then $L_\chi(t)$ denote the Lambert series associated to χ:

$$L_\chi(-t) = \sum_{n=1}^{\infty} \chi(n)e^{-nt}, \quad \mathrm{Re}\, t > 0, \tag{3.1}$$

which corresponds to the hyperbolic cotangent function $\coth x$, and let r be a positive integer prime to N. The essential case is $u \le r$, which we so assume. Yamamoto [25] defined the weighted short-interval sums associated to χ as

$$S^\kappa_{r,N}(\chi) = \sum_{1\le a\le \frac{N}{r}}{}' \chi(a) f\left(\frac{a}{d}\right), \tag{3.2}$$

and the conjugate character sum $T^\kappa_{r,ud}(\chi)$:

$$T^\kappa_{r,ud}(\chi) = \sum_{a=0}^{d-1} \chi(a) \tilde{f}\left(\frac{a}{d}\right),$$

for a character χ modulo $d > 1$, where $\tilde{f}$ is the conjugate function (cf. [25, p.285]) of f:

$$f(x) = \begin{cases} x^\kappa, & 0 \le x < \alpha \\ 0, & \alpha < x < 1. \end{cases}$$

In the notation of Yamamoto [25, p. 280], say, $S^\kappa_{r,N}(\chi) = S_{\frac{u}{r}} = S_{\frac{N/d}{r}}$. Comparing the coefficients of $\frac{t^{n-1}}{n!}$ on both sides of (2.19) and using (1.2) with $\alpha = 1$, they deduced [15, (6), p. 276]

$$(\kappa+1)(rd)^\kappa S^\kappa_{r,N}(\chi) = -B_{\kappa+1,\chi} r^\kappa + \frac{\overline{\chi}(N)}{\varphi(N)} \sum_\psi \overline{\psi}(-N) B_{\kappa+1,\chi\psi}(N). \tag{3.3}$$

where the sum is over all Dirichlet characters ψ modulo r and φ being Euler φ-function. Therefore by Yamamoto's results[25] or (3.3) or [26], we conclude:

1. As has been completely demonstrated in this note, by applying the functional equations of $T_{k,\chi}(n)$ resp. $T'_{k,\chi}(n)$, any weighted short interval character sum may be expressed as a linear combination of $L(1,\chi)$'s and inevitably in terms of the class number

$$h(d) = \frac{w\sqrt{|d|}}{2\pi} L(1, \chi_{-|d|}), \tag{3.4}$$

via generalized Euler resp. Bernoulli numbers, where $h(d)$, $\chi_{-|d|}(a) = \left(\frac{a}{|d|}\right)$, w denote the class number of the imaginary quadratic field $\mathbb{Q}(\sqrt{d})$

with discriminant $d < 0$, the corresponding Kronecker character, the number of roots of unity in $\mathbb{Q}(\sqrt{d})$, respectively. Berndt [23, pp.413-445] contains a number of useful formulas, but of course is not exhaustive (cf. formulas in [22, Lemma1.3-1.4]). But now that we have Yamamoto's colossal theory[25], we are supposed to use it.

2. We notice that in [22, Lemma1.3-1.4]) or [26] we consider weighted short interval character sums with polynomial weight, and *a fortiori*, of Bernoulli polynomial weight, and the final formulas contain Bernoulli numbers and class numbers of imaginary quadratic fields (cf. [22], for $p \equiv 1 \pmod 4$, $\chi_{-4}\chi_p$ is odd, and for $p \equiv 3 \pmod 4$, χ_{-p} is odd) as in Berndt [23, pp. 413-445]. But Yamamoto also treats the case of Clausen function weight, or what is the same thing, log sin weight. Therefore, it is very intriguing to pursue research on class numbers of real quadratic fields as in Chowla [24].

3. Considering Euler number congruences to the higher prime power modulus is important from p-adic theoretic point of view. As the example of Shiratani-Yokoyama [27], some relations on Bernoulli or Euler numbers can be deduced by p-adic argument.

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REFERENCES

[1] T. Kim, *Symmetry Properties of the Generalized Higher-order Euler Polynomials*, Proceedings of the Jangjeon mathematical Society, Vol 13 (2010), 13-16.

[2] T. Kim, K. Hwang, and Y. Kim, *Symmetry Properties of Higher-Order Bernoulli Polynomials*, Advances in Difference Equations **2009** (2009), Article ID 318639, 6 pages. doi:10.1155/2009/318639

[3] T. Kim, *Symmetry of power sum polynomials and multivariate fermionic p-adic invariant integral on $\mathbb{Z}_p$*, Russian Journal of Mathematical Physics, **16** (2009), 93-96.

[4] T. Kim, *Symmetry p-adic invariant integral on $\mathbb{Z}_p$ for Bernoulli and Euler polynomials*, Journal of Difference Equations and Applications, **14** (2008), 1267-1277.

[5] T. Kim, *Symmetry identities for the twisted generalized Euler polynomials*, Adv. Stud. Contemp. Math. **19** (2009), 111-118.

[6] Y. Kim, K. Hwang, *A symmetry of power sum and twisted Bernoulli polynomials*, Adv. Stud. Contemp. Math. **18** (2009), 127-133.

[7] T. Kim, *Symmetry identities for the twisted generalized Euler polynomials*. Adv. Stud. Contemp. Math. (Kyungshang) **19** (2009), no. 2, 151-155.

[8] T. Kim, L. Jang, Y. Kim, K. Hwang, *on the identities of symmetry for the generalized Bernoulli polynomials attatched to χ of higher order*. J. Inequal. Appl. 2009, Art. ID 640152, 7 pages.

[9] T. Kim, *Barnes-type multiple q-zeta functions and q-Euler polynomials*, J. Physics A: Mathematical and Theoretical **43** (2010), Article ID 255201, 11pp, Art. ID 273545, 9pp.
[10] T. Kim, S. Rim B. Lee, *Some identities of symmetry for the generalized Bernoulli numbers and polynomials*, Abstr. Appl. **2009** (2009). Art, ID 848943, 8 pp.
[11] T. kim, *On the symmetries of the q-Bernoulli Polynomials*, Abstr. Appl. Anal, **2008** (2008), Art, ID 914367, 7 pp.
[12] S. Rim, Y. Kim, B. Lee, T. Kim, *Some identities of the generalized twisted Bernoulli numbers and polynomials of higher order*, J. Comput. Anal. Appl. **12**(2010), no. 3, 695-702.
[13] T. Kim, *Note on the Euler q-zeta functions*, J. Number Theory, **129** (2009), no.7, 1798-1804.
[14] K.Chen, *Sums of the Products of Gneralized Bernoulli Polinomials*, Pacific Journal of Mathematics. **208** (2003),no.1, 39-51.
[15] J. Szmidt, J. Urbanowicz and D. Zagier, *Congruences among generalized Bernoulli numbers*, Acta Arith. **3** (1995), 273-278.
[16] V. Kurt, *A Further symmetric relation on the analogue of the Apostol-Bernoulli and the analogue of the Apostol-Genocchi polynimials*. Applied Mathematical Sciences, **3**, (2009), no. 56, 2757-2764.
[17] E. Deeba, D. Rodriguez, *Stirlings series and Bernoulli numbers*, The American Mathematical Monthly, **98** (1991), no.5, 423-426.
[18] Y. Jang, M. Kim, J. Son, *An analogue of Bernoulli numbers and their congruences*, Proceedings of the Jangjeon Mathematical Society **1** (2000), 133-144.
[19] S. Kanemitsu and H. Tsukada, *Vistas of special functions*, World Sci. New Jersey-London-Singapore-··· 2007.
[20] G. Liu *Generating functions and generalized Euler numbers*, Proc. Japan Acad. **84** Ser. A (2008), 29-34.
[21] H.M. Srivastava and J.S. Choi, *Series Associated with the zeta related functions*, Kluwer Acad. publ. Dordrecht. Boston and London, (2001).
[22] N. Wang, J. Li and D.Liu, *Euler numbers congruences and Dirichlet L-functions*, J. Number Theory (6) **129** (2009), 1522-1531.
[23] B. C. Berndt, *Classical theorems on quadratic residues*, Enseign. Math.(2) **22** (1976), 261-304.
[24] P. Chowla, *On the class number of real quadratic field*, J. Reine Angew. Math. **230** (1968), 51-60.
[25] Y. Yamamoto, *Dirichlet series with periodic coefficients*, Proc. Intern. Sympos. "Algebraic Number Theory", Kyoto 1976, 275-289. JSPS, Tokyo 1977.
[26] S. Kanemitsu, H. Li and N. Wang, *On Weighted Short-interval Sums*, to appear.
[27] K. Shiratani and S. Yokoyama, *An aplication of p-adic convolutions*, Mem. Fac. Sci., Kyushu Univ. Ser. Math. **36** (1982), 73-83.