# The Number of Nowhere-Zero Tensions on Graphs and Signed Graphs*

Beifang Chen[1] and Shuchao Li[2,†]

[1]Department of Mathematics, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong

[2]Faculty of Mathematics and Statistics, Central China Normal University, Wuhan 430079, P.R. China

**Abstract.** A nowhere-zero $k$-tension on a graph $G$ is a mapping from the edges of $G$ to the set $\{\pm 1, \pm 2, \ldots, \pm(k-1)\} \subset \mathbb{Z}$ such that, in any fixed orientation of $G$, for each circuit $C$ the sum of the labels over the edges of $C$ oriented in one direction equals the sum of values of the edges of $C$ oriented oppositely. We show that the existence of an *integral tension polynomial* that counts nowhere-zero $k$-tension on a graph, due to Kochol, is a consequence of a general theory of inside-out polytopes. The same holds for tensions on signed graphs. We develop these theories, as well as the related counting theory of nowhere-zero tensions on signed graph with values in an abelian group of odd order. Our results are of two kinds: polynomiality or quasipolynomiality of the tension counting functions, and reciprocity laws that interpret the evaluations of the tension polynomials at negative integers in terms of the combinatorics of the graph.

**Keywords.** nowhere-zero tension, bidirected graph, signed graph, integral tension polynomial, modular tension polynomial, lattice-point counting, rational convex polytope, arrangement of hyperplanes, Tutte polynomial.

## 1. Introduction

The aim of this paper is to study chromatic polynomials of graphs and signed graphs from a new point of view. Chromatic polynomials are well-known functions that have been studied in framework of classical combinatorics and graph theory. For surveys see, e.g., Aigner [1], Biggs [4], Brylawski and Oxley [6], Jaeger [12], Read and Tutte [17], Rota [18], Tutte [19, 20, 21], Welsh [23] and those for singed graphs see, Zaslavsky [25, 26, 27]. We show that the existence of an *integral tension polynomial* that counts nowhere-zero $k$-tensions on a graph, due to Kochol, is a consequence of a general theory of inside-out polytopes. The same holds for tensions on signed graphs. We develop these theories, as well as the related counting theory of nowhere-zero tensions on signed graph with values in an abelian group of odd order.

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†Corresponding author: lscmath@mail.ccnu.edu.cn

Graph coloring problems can be expressed in framework of potentials and tensions. A nice introduction to this approach is in Berge [3]. Let $(G, \varepsilon)$ be a digraph, and let $(C, \varepsilon_c)$ be a directed circuit of $G$. We define a coupling $[\varepsilon, \varepsilon_c] : E(G) \to \{-1, 0, 1\}$ as

$$[\varepsilon, \varepsilon_c](x) = \begin{cases} 1, & \text{if } \varepsilon(x) = \varepsilon_c(x), x \in C; \\ -1, & \text{if } \varepsilon(x) \neq \varepsilon_c(x), x \in C; \\ 0, & \text{if } x \notin C. \end{cases} \tag{1.1}$$

Let $\Gamma$ be an abelian group. A *nowhere-zero tension* of $(G, \varepsilon)$ with values in $\Gamma$ is a function $f : E(G) \to \Gamma$ such that for any directed circuit $(C, \varepsilon_c)$,

$$\sum_{x \in C} [\varepsilon, \varepsilon_c](x) f(x) = 0, \tag{1.2}$$

and which never takes the values 0. (In a certain sense, described below, $f$ is independent of the chosen orientation and the structure of abelian group $\Gamma$.) A *nowhere-zero k-tension* is an integral tension (i.e., $\Gamma = \mathbb{Z}$) whose absolute values are in $[k-1] := \{1, 2, \ldots, k-1\}$. Nowhere-zero tensions are nicely studied in [7, 8, 13].

It has been known that the number of nowhere-zero tension with values in a finite abelian group of order $k$ is a polynomial function of $k$. Recently Kochol [13] discovered the number of nowhere-zero $k$-tensions is also a polynomial in $k$, although not the same polynomial. Recently, Chen also use the group arrangement and Ehrhart theory to study the tension polynomial of graphs [7]. Here we show that this fact is a consequence of a general theory of counting lattice points in inside-out polytopes. Furthermore, we extend Kochol's theorem in two ways: by a reciprocity law that combinatorially interprets negative arguments, and to signed graphs (in which each edge is positive or negative), where the polynomial becomes a quasipolynomial of period two: that is, a pair of polynomials, one for odd value of $k$ and the other for even $k$; and we partially extend to signed graphs Tutte's concept of reciprocity in lattice-point counting leading us to a geometric interpretation of the number of acyclic orientations that are compatible with a given $k$-tension.

Nowhere-zero $\Gamma$- and $k$-tensions in a planar graph correspond to nowhere-zero $\Gamma$ and $k$-flows in the dual graph, respectively. Also the (integral) tension polynomials have their dual counterparts, (integral) flow polynomials, this notion was introduced in [14] and in [2] it was presented in framework of a more general setting.

## 2. The method of polytope, Ehrhart theory and matrix matroid

For the whole exposition is self-contained, we use the basic theory from Beck and Zaslavsky's (see [2, Section 2]). The theory of inside-out polytope was motivated by the problem of counting the integral points of a *rational convex polytope* (the convex hull of finitely many rational points in $\mathbb{R}^d$) that do not lie in any of the members of a particular rational hyperplane arrangement. A (homogeneous, real) *hyperplane arrangement* is a finite set of homogeneous hyperplanes in $\mathbb{R}$ (that is, hyperplanes that contain the origin); it is *rational*, if each hyperplane has a rational normal vector. Suppose we are given a rational convex polytope $P$ spanning $\mathbb{R}^d$ and a rational hyperplane arrangement $\mathcal{H}$. Then $(P, \mathcal{H})$ is a *rational inside-out polytope.* More generally, $P$ and $\mathcal{H}$ may lie in a rational (and homogeneous) subspace $Z$ that is spanned by $P$. We consider inside-out theory essential to understanding our results on integral tensions.

A *region* (more precisely, an *open region*) of $\mathcal{H}$ is a connected component of $\mathbb{R}^d \setminus \bigcup \mathcal{H}$; its closure is a *closed region.* The arrangement *induced* by $\mathcal{H}$ in a subspace $S$ of $\mathbb{R}^d$ is

$$\mathcal{H}^S := \{H \cap S : H \in \mathcal{H}, H \not\supseteq S\}.$$

The *intersection lattice* of $\mathcal{H}$ is

$$\mathcal{L}(\mathcal{H}) := \{\bigcap \mathcal{S} : \mathcal{S} \subseteq \mathcal{H}\},$$

ordered by reverse inclusion [28]; its elements are *flats* of $\mathcal{H}$. $\mathcal{L}$ is a *geometric lattice* with $\hat{0} = \bigcap \emptyset = \mathbb{R}^d$ and $\hat{1} = \bigcap \mathcal{H}$. (For matroids and geometric lattices we refer to [1] or [16] .) The *Möbious function* of $\mathcal{L}$ is the function: $\mu : \mathcal{L} \times \mathcal{L} \to \mathbb{Z}$ defined recursively by

$$\mu(R, S) := \begin{cases} 0, & \text{if } R \not\leq S; \\ 1, & \text{if } R = S; \\ -\sum_{R \leq U < S} \mu(R, U), & \text{if } R < S. \end{cases}$$

We outline the classical Ehrhart theory [10] of lattice-pint enumeration in polytope. We have a rational convex polytope $P$ that spans a subspace $Z$. $P^{\circ}$ denotes the interior of $P$. The *denominator* of $P$ is the least common denominator of all the coordinates of vertices of $P$. We denote by vol$P$ the *volume* of $P$, normalized with respect to $Z \cap \mathbb{Z}^d$; that is, we take the volume of a fundamental domain of the integer lattice in $Z$ to be 1. To explain this last, we note that $Z \cap \mathbb{Z}^d$ is linearly equivalent to $\mathbb{Z}^{\dim Z} \subseteq \mathbb{R}^{\dim Z}$; a *fundamental domain* is a domain in $Z$ that corresponds to the unit hyperplane $[0, 1]^{\dim Z} \subseteq \mathbb{R}^{\dim Z}$, under some invertible linear transformation that carries $Z \cap \mathbb{Z}^d$ to $\mathbb{Z}^{\dim Z}$.

A quasipolynomial is a function $Q(t) = \sum_0^d c_i(t)t^i$ defined on $\mathbb{Z}$ with coefficients $c_i$ that are periodic functions of $t$. Then $Q$ is a polynomial $Q_{\bar{t}}$ on each residue class $\bar{t}$ modulo some integer, called the *period* of $Q$; these polynomials are the *constituents* of $Q$.

The subject of Ehrhart theory is the Ehrhart counting function

$$E_P(t) := \#(tP \cap \mathbb{Z}^d)$$

and the open Ehrhart counting function $E_{P^\circ}(t)$. Ehrhart theorem is that $E_P$ and $E_{P^\circ}$ are quasipolynomials with leading term $(\operatorname{vol} P)t^{\dim P}$ and with periods that divide the denominator of $P$. It follows that one can define both counting functions for negative integers; the Ehrhart-Macdonald Reciprocity Theorem is that $E_{P^\circ}(t) = (-1)^{\dim P} E_P(-t)$.

An *open region* of $(P, \mathcal{H})$ is a nonempty intersection with $P^\circ$ of an open region of $\mathcal{H}$ (thus it is full-dimensional in the span of $P$). A *closed region* of $(P, \mathcal{H})$ is the closure of an open region. A *vertex* of $(P, \mathcal{H})$ is a vertex of any such region. The *denominator* of $(P, \mathcal{H})$ is the least common denominator of all coordinates of all vertices.

The fundamental counting functions associated with $(P, \mathcal{H})$ are two quasipolynomials: the Ehrhart quasipolynomial,

$$E_{P,\mathcal{H}}(t) := \sum_{x \in t^{-1}\mathbb{Z}^d} m_{P,\mathcal{H}}(x),$$

where the *multiplicity* $m_{P,\mathcal{H}}(x)$ of $x \in \mathbb{R}^d$ with respect to $\mathcal{H}$ and $P$ is defined through

$$m_{P,\mathcal{H}}(x) := \begin{cases} \text{the number of closed regions} & \\ \qquad \text{of } (P, \mathcal{H}) \text{ that contains } x, & \text{if } x \in P, \\ 0, & \text{if } x \notin P, \end{cases}$$

and the open *Ehrhart quasipolynomial,*

$$E_{(P,\mathcal{H})^\circ}(t) := \# \left(t^{-1}\mathbb{Z}^d \bigcap \left[P^\circ \setminus \left(\bigcup \mathcal{H}\right)\right]\right).$$

We may think of $(P, \mathcal{H})^\circ$ as the set $P^\circ \setminus (\bigcup \mathcal{H})$, called the *relative interior* of $(P, \mathcal{H})$.

**Theorem 2.1.** ([2]) *If $(P, \mathcal{H})$ is a closed, full dimensional, rational inside-out polytope in $Z \subseteq \mathbb{R}^d$, then $E_{P,\mathcal{H}}(t)$ and $E_{(P,\mathcal{H})^\circ}(t)$ are quasipolynomials in $t$, with period equal to a divisor of the denominator of $(P, \mathcal{H})$, with leading term $(\operatorname{vol} P)t^{\dim P}$, and with constant term $E_{P,\mathcal{H}}(0)$ equals the number of regions of $(P, \mathcal{H})$. Furthermore,*

$$E_{(P,\mathcal{H})^\circ}(t) = (-1)^{\dim P} E_{P,\mathcal{H}}(-t). \tag{2.1}$$

In particular, if $(P, \mathcal{H})$ is integral then $E_{P,\mathcal{H}}$ and $E_{(P,\mathcal{H})^\circ}$ are polynomials. The proof, though more general (and arrived at independently), is similar to Kochol's proof of Theorem 3.1(a) in [14].

Here we need the notion of transversality: $\mathcal{H}$ is called *transverse* to $P$ if every flat $U \in \mathcal{L}(\mathcal{H})$ that intersects $P$ also intersects $P^\circ$, and $P$ does not lie in any of the hyperplanes of $\mathcal{H}$.

**Theorem 2.2** ([2]). *If $P$ and $\mathcal{H}$ are as in Theorem 2.1, then*

$$E_{(P,\mathcal{H})^\circ}(t) = \sum_{U \in \mathcal{L}(\mathcal{H})} \mu(\hat{0}, U) E_{P^\circ \cap U}(t), \tag{2.2}$$

*and if $\mathcal{H}$ is transverse to $P$,*

$$E_{P,\mathcal{H}}(t) = \sum_{U \in \mathcal{L}(\mathcal{H})} |\mu(\hat{0}, U)| E_{P \cap U}(t). \tag{2.3}$$

We shall want a general lemma about matroids of hyperplane arrangements induced by coordinate arrangements. We start with the hyperplane arrangement $\mathcal{H}_m$, consisting of the coordinate hyperplanes in $F^m$ for some field $F$, and we take any subspace $S$. Then $S$ induces an arrangement $\mathcal{H}_m^S$ in $S$.

Any homogeneous hyperplane arrangement $\mathcal{H}$ has a matroid $M(\mathcal{H})$, whose ground set is the set of hyperplanes and whole rank function is $\operatorname{rk} \mathcal{S} = \operatorname{codim}(\bigcap \mathcal{S})$ for $\mathcal{S} \subseteq \mathcal{H}$. This matroid is simply the linear dependence matroid of the normal vectors of the hyperplanes. The *column matroid* of a matrix $A$, $M(A)$, is the matroid of linear dependence of columns; to keep the notation correct we take the ground set to the set of indices of columns. The *chain-group matroid* of a subspace $S \subseteq F^m$ is the matroid $N(S)$ on $[m]$ whose circuits are the minimal nonempty supports of vectors in $S$. Lat $M$ denotes the lattice of closed sets of a matroid. Thus $\operatorname{Lat} M(\mathcal{H}) \cong \mathcal{L}(\mathcal{H})$.

We refer in the following lemma to orientations of oriented matroids.

**Lemma 2.3** ([2]). *Let $A$ be an $n \times m$ matrix with entries in a field $\mathcal{F}$, let $\mathcal{H}_m = \{H_e : e \in [m]\}$ be the arrangement of coordinate hyperplanes in $F^m$, and let $U = \operatorname{Row} A$, the row space, and $Z = \operatorname{Nul} A$, the null space.*

(i) *The mapping $e \mapsto H_e \cap U$ is a matroid isomorphism from $M(A)$ to $M(\mathcal{H}_k^U)$. Also, $e \mapsto H_e \cap Z$ is a matroid isomorphism of $N(\operatorname{Row} A)$ with $M(\mathcal{H}_m^Z)$.*

(ii) *The mapping*

$$F \mapsto E_U(F) := \{e \in [m] : H_e \supseteq F\}$$

*is the isomorphism of $\mathcal{L}(\mathcal{H}_E^U)$ with* Lat $M(A)$ *induced by the first mapping in* (i). *The mapping*

$$F \mapsto E_Z(F) := \{e \in [m] : H_e \supseteq F\}$$

*is the isomorphism of $\mathcal{L}(\mathcal{H}_E^Z)$ with* Lat $N(\operatorname{Row} A)$ *induced by the second mapping in* (i).

(iii) *If $\mathcal{F}$ is an ordered field, then the regions of $\mathcal{H}_m^U$ corresponding bijectively to the the acyclic orientations of the oriented matroid of columns of $A$, and those of $\mathcal{H}_m^Z$ corresponding bijectively to totally cyclic orientations.*

## 3. Tensions on graphs

A *tension* on a graph $G$ with values in an abelian group $\Gamma$, called a $\Gamma$-tension, is a function $f : E(G) \to \Gamma$ which satisfies (1.2) for every circuit $C$ of $G$ (so it is like a nowhere-zero tension, but the tension value zero is allowed). This definition requires that the edges be oriented in a fixed way. The orientation is arbitrary; it is an artifact of notation, and to overcome this artificiality we define, for an oriented edge $e$, $e^{-1}$ to be the same edge in the opposite orientation and $f(e^{-1}) := -f(e)$. With this law for tension, the validity of Eq.(1.2) is independent of the choice of the orientation of $G$.

The number of nowhere-zero $\Gamma$-tensions on $G$ is a polynomial in $|\Gamma|$, independent of the actual group. We shall write $\tau(G,t)$ for this polynomial and call it the *(strict) modular tension polynomial* of $G$. (Usually $\tau(G,t)$ is called just the "tension polynomial" but we need to distinguish it from the other tension polynomials.) It is well known that $\tau(G,k)$ is the evaluation $(-1)^{r(G)}t(G;1-k,0)$ of the Tutte polynomial (in fact, the classical chromatic polynomial) of $G$. ($r(G)$ is the rank, $|V|-c(G)$, of cycle matroid $M(G)$, where $c(G)$ is the number of connected components.) A nowhere-zero $\mathbb{Z}_k$-tension exists only if there is a nowhere-zero $k$-tension, a *$k$-tension* being an integral-valued tension whose values all satisfy $|f(e)| < k$. These properties in fact follows from the well known properties of the chromatic polynomial. However, the actual number of nowhere-zero $k$-tension for $k > 0$, which we write as $\tau_{\mathbb{Z}}(G,k)$, does not equal the number of nowhere-zero $\mathbb{Z}_k$-tensions and indeed was never known to be a polynomial until the recent work of Kochol [13]. Kochol employed standard Ehrhart theorem combined with a special construction to proved it. We shall show that Kochol's result is a natural consequence of inside-out polytope theory and can be extended to a reciprocity theorem that interprets $\tau_{\mathbb{Z}}(G,k)$ at negative arguments (also see Chen's recent work in [7]).

Recall that a *cut* in a graph $G$ is a partition $\{S,T\}$ of a vertex set $V$ such that the removal of $[S,T]$, the set of all edges between $S$ and $T$, disconnected the graph $G$. For a digraph $(G,\varepsilon)$, a cut $[S,T]$ is said to be *directed* relative to $\varepsilon$ if the edges of $[S,T]$ have the same direction under $\varepsilon$, either all from $S$ to $T$, or all from $T$ to $S$. An orientation of $G$ is *acyclic* if it has no cycles and *totally cyclic* if every edge lies in a cycle. We call an acyclic orientation $\sigma$ and a tension $f$ *compatible* if $f \geq 0$ when it expressed in terms of $\sigma$. Taking the standpoint of the tension $f$, the nonzero edge set $\operatorname{supp} x$ has a preferred orientation, the one in which $f \geq 0$ (we call this $\sigma(f)$; note that it orients only $\operatorname{supp} x$) and the zero edges are free to take up any orientation that makes $G$ acyclic. An *loop* is an edge whose two ends are the same vertex. There is no acyclic orientation if $G$ has a loop.

The *real cut space* $Z$ is defined in $\mathbb{R}^E$ by Eq.(1.2). To this space $Z$ we associated the polytope and arrangement

$$P := Z \cap [-1,1]^E, \qquad \mathcal{H} := (\mathcal{H}_E)^Z,$$

where $\mathcal{H}_E$ is the arrangement of coordinate hyperplanes in $\mathbb{R}^E$. A $(k+1)$-tension is then precisely a point $x \in Z \cap \mathbb{Z}^d$ such that $\frac{1}{k}x \in P$ and a nowhere-zero $k$-tension is just a point $x \in Z \cap \mathbb{Z}^d$ such that $\frac{1}{k}x \in P \backslash (\bigcup \mathcal{H})$. Consequently,

$$\bar{\tau}(G, k+1) = E_P(k) \tag{3.1}$$

and

$$\tau_{\mathbb{Z}}(G,k) = E_{(P,\mathcal{H})^{\circ}}(k). \tag{3.2}$$

We call $\tau_{\mathbb{Z}}(G,t)$ the *integral tension polynomial* of $G$ and $\bar{\tau}(G,t)$ the *weak integral tension polynomial.*

**Theorem 3.1.** *Let $G$ be a graph with the real cut space $Z$.*

(i) ([13]) $\tau_{\mathbb{Z}}(G,k)$ *is a polynomial function of $k$ for $k = 1,2,3,\ldots$. It has leading term* $(\operatorname{vol} P)k^{r(G)}$ *if $G$ has no loop; otherwise it is identically zero.*

(ii) *Furthermore, $(-1)^{r(G)}\tau_{\mathbb{Z}}(G,-k)$ for $k \geq 0$ equals the number of non-negative $(k+1)$-tensions counted with multiplicity equal to the number of acyclic orientations of $G$ that are compatible to the tension.*

(iii) *In particular, the absolute of the constant term $|\tau_{\mathbb{Z}}(G,0)|$ counts the number of acyclic orientations of $G$, which equals $(-1)^{r(G)}\tau(G,-1)$.*

(iv) *Finally, the total number of $k$-tensions for $k > 0$, nowhere zero or not, is a polynomial $\bar{\tau}(G,k)$ satisfying $\bar{\tau}(G,k) = (-1)^{r(G)}\bar{\tau}(G,1-k)$, whose leading term is the same as that of $\tau_{\mathbb{Z}}(G,k)$ and whose constant term is $(-1)^{r(G)}$.*

*Proof.* For (i) we apply Theorem 2.1 in $Z$. We call upon the total unimodularity of the matrix of the cycle equations (1.2) to deduce that $P$ is a convex hull of integer lattice points. Thus, $E_{(P,\mathcal{H})^{\circ}}$ is a polynomial, and as we saw in (3.2), it equals $\tau_{\mathbb{Z}}(G,k)$.

Since $E_{P,\mathcal{H}}(k)$ counts the pair of $(x,R)$ where $x \in \mathbb{Z}^d \cap P$ and $R$ is a closed region of $\mathcal{H}$ that contains $x$, parts (ii) follows if we show that the regions of $P$ corresponding to the acyclic orientations of $G$ and a region of $\mathcal{H}$ whose closure contains a chosen point $x \in Z \cap \mathbb{Z}^d$ corresponds to a acyclic orientation that is compatible with $x$ was demonstrated by Green and Zaslavsky in [11], based on the obvious bijection between orthants of $\mathbb{R}^E$ and the orientations of $G$. The second is then easily deduced.

Thus the constant term is the number of acyclic orientations. The fact that this equals $t(G;2,0)$ is a theorem originally due to Las Vergnas [22] and independently proved by Chen in [7, Theorem 1.2].

Part (iv) is standard Ehrhart theory, because a $k$-tension is simply a point $x \in \mathbb{Z}^d$ such that $\frac{1}{k}x \in P^\circ$. That is, $\bar{\tau}(G;k) = E_{P^\circ}(k) = (-1)^{r(G)}E_P(-k)$ by Ehrhart reciprocity. The constant term of $E_P(-k)$ is 1, the Euler characteristic of $P$. It is easy to see that $E_{P^\circ}(k) = E_P(k-1)$ for $k > 0$. Therefore,

$$\bar{\tau}(G;k) = (-1)^{r(G)}E_P(-k) = (-1)^{r(G)}E_{P^\circ}(1-k) = (-1)^{r(G)}\bar{\tau}(G;1-k)$$

if $k$ is a positive integer, whence for all $k$. □

**Problem 3.2.** *Find a formula for, or a combinatorial interpretation of, the leading coefficient* vol $P$ *of the integral tension polynomials.*

**Problem 3.3.** *Is there a combinatorial interpretation of* $\bar{\tau}(G;-k)$ *for* $k \geq 2$?

**Example 1.** *We calculated the integral tension polynomials of some small graphs by counting integral q-tensions on a computer and interpolating to get the polynomial. The graphs were* $C_m$*, the m-cycle, for* $m = 3,4,5,6$ *and* $K_4$*. We state our results along with the modular tension polynomials for comparison; the latter are* $\bar{\tau}(G,t)$ *and* $\tau(G,t)$*. First,* $C_3$*:*

$$\begin{aligned} \bar{\tau}(C_3,q) &= q^2, & \bar{\tau}_{\mathbb{Z}}(C_3,q) &= (q-1)(q-2), \\ \tau(C_3,q) &= 3q^2-3q+1, & \tau_{\mathbb{Z}}(C_3,q) &= 3(q-1)(q-2). \end{aligned}$$

*Next,* $C_4$*:*

$$\begin{aligned} \bar{\tau}(C_4,q) &= q^3, & \bar{\tau}_{\mathbb{Z}}(C_4,q) &= (q-1)(q^2-3q+3), \\ \tau(C_4,q) &= \tfrac{(2q-1)(8q^2-8q+3)}{3}, & \tau_{\mathbb{Z}}(C_4,q) &= \tfrac{2(q-1)(8q^2-22q+21)}{3}. \end{aligned}$$

*Next,* $C_5$*:*

$$\begin{aligned} \bar{\tau}(C_5,q) &= q^4,\ \tau(C_5,q) = (q-1)(q^3-4q^2+6q-4), \\ \bar{\tau}_{\mathbb{Z}}(C_5,q) &= \tfrac{115q^4-230q^3+185q^2-70q+12}{12}, \\ \tau_{\mathbb{Z}}(C_5,q) &= \tfrac{5(q-1)(q-2)(23q^2-41q+36)}{12}. \end{aligned}$$

*Next,* $C_6$*:*

$$\begin{aligned} \bar{\tau}(C_6,q) &= q^5, \tau(C_6,q) = (q-1)(q^4-5q^3+10q^2-10q+5), \\ \bar{\tau}_{\mathbb{Z}}(C_6,q) &= \tfrac{2(2q-1)(44q^4-88q^3+71q^2-27q+5)}{10}, \\ \tau_{\mathbb{Z}}(C_6,q) &= \tfrac{(q-1)(176q^4-839q^3+1571q^2-1404q+620)}{10}. \end{aligned}$$

*Finally, $K_4$:*

$$\bar{\tau}(K_4, q) = q^3, \qquad \bar{\tau}_{\mathbb{Z}}(K_4, q) = (q-1)(q-2)(q-3),$$
$$\bar{\tau}(K_4, q) = (2q-1)(2q^2-2q+1), \qquad \tau_{\mathbb{Z}}(K_4, q) = 4(q-1)(q-2)(q-3).$$

**Problem 3.4.** *Is there any general reason why in some of these examples* ($C_3$ and $K_4$) *both of the integral tension polynomials have integral coefficients and the integral nowhere-zero tension polynomial is a muptiple of the modular polynomial ?*

## 4. Tensions on signed graphs

The best way to understand the equations (1.2) is in terms of the circuit incidence matrix and cocircuit incidence matrix, which we expound in the general context of signed or bidirected graphs.

Formally, a *signed graph* $\Sigma = (G, \sigma)$ consists of a graph $G$ and a function $\sigma$ from the set of lingks and loops of $G$ to $\{+, -\}$. (A *link* has two distinct endpoints; a *loop* has two coinciding endpoints. In signed and bidirected graph theory it is convenient to have two more kinds of edges: a *half edge* has one endpoint and a *loose edge* has no endpoints; neither of these has a sign.) If $T \subseteq E$, then $\Sigma|T$ denotes the spanning subgraph whose edge set is $T$. Each circle has a sign, which is the product of the signs of its edges. A subgraph or edge set is called *balanced* if it contains no halfedges and every circle in it has positive sign. (For the general theory of signed graphs see [24].)

The *bias matroid* (or *signed-graphic matroid*) of $\Sigma$ [24], written $M(\Sigma)$, can be defined by its rank function,

$$r(T) = |V| - b(\Sigma|T) \text{ for an edge set } T,$$

where $b(\Sigma|T)$ is the number of components of the subgraph $\Sigma|T$ that are balanced subgraphs, ignoring any loose edges. The circuits of $M(\Sigma)$ are of three kinds: a positive circle, a pair of negative circles that have a single common node, or a pair of node-disjoint negative circles together with a minimal connecting path; here one or both negative circles may be replaced by halfedges. A *coloop* of $M(\Sigma)$ is an edge $e$ whose deletion makes an unbalanced component balanced; or which is an isthmus connecting two components of $\Sigma \setminus e$ of which at least one is balanced. We define the *cyclomatic number* of $\Sigma$ to be $|E| - |V| + b(\Sigma)$. This is the rank of dual $M^{\perp}(\Sigma)$ of the bias matroid.

A graph is *bidirected* when each end of each edge is independently oriented. We express the bidirection by means of an *incidence function* $\eta$ defined on the edge ends: the function is $+1$ if the arrow on that end points into the incident node, and $-1$ otherwise. A bidirection of a graph is really an orientation of a signed graph. A link or loop $e$ with ends $v_1$ and $v_2$ has sign

$$\sigma(e) := -\eta(v_1)\eta(v_2).$$

In plain language, if the two arrows on $e$ in the same direction, then $e$ is positive, but if they are conflict, $e$ is negative. We call $\eta$ an *orientation of the signed graph* $\Sigma$. This notion corresponds to the ordinary notion of graph orientation if we identify an unsigned graph $G$ with all-positive graph $+G$. There exists a subgraph of $G$ corresponding to a circuit of $\Sigma$, and this subgraph can be oriented to be a directed trail. A direction of a circuit $C$ is an orientation $\varepsilon$ on $C$ such that very vertex of $C$ is neither a source nor a sink; we denote the directed circuit as $(C, \varepsilon_c)$. An orientation is *acyclic* if it has no directed circuits and *totally cyclic* if every edge belongs to a directed circuit.

Suppose we have a bidirected graph. *Switching* a node $v$ means changing $\eta$ to $\eta^v$ defined by

$$\eta^v(\varepsilon) = \begin{cases} \eta(\varepsilon), & \text{if } v(\varepsilon) \neq v; \\ -\eta(\varepsilon), & \text{if } v(\varepsilon) = v. \end{cases}$$

The associated switched signed graph is denoted by $\Sigma^v$. It is obtained from $\Sigma$ by negating all links incident with $v$.

Let $(\Sigma, \varepsilon)$ be a bidirected graph, and let $(C, \varepsilon_C)$ be a directed circuit of $\Sigma$. We define a coupling $[\varepsilon, \varepsilon_C] : E(\Sigma) \to \mathbb{Z}$ as

$$[\varepsilon, \varepsilon_C](x) = \begin{cases} 1, & \text{if } x \in E \cap C, \varepsilon(x,v) = \varepsilon_c(x,v) \text{ and } x \text{ is not a circuit bridge;} \\ -1, & \text{if } x \in E \cap C, \varepsilon(x,v) \neq \varepsilon_c(x,v) \text{ and } x \text{ is not circuit bridge;} \\ 2, & \text{if } x \in E \cap C, \varepsilon(x,v) = \varepsilon_c(x,v) \text{ and } x \text{ is a circuit path;} \\ -2, & \text{if } x \in E \cap C, \varepsilon(x,v) \neq \varepsilon_c(x,v) \text{ and } x \text{ is a circuit path;} \\ 0, & \text{otherwise.} \end{cases} \tag{4.3}$$

The coupling $[\varepsilon, \varepsilon_C]$ determines a vector in $\mathbb{R}^E$. We call it a *circuit vector* of $C$ in $(\Sigma, \varepsilon)$. The spanning space of all the circuit vectors of circuits in $(\Sigma, \varepsilon)$ is called *circuit space*, denoted $Z(\Sigma, \varepsilon)$. If fact, it is independent of the orientation $\varepsilon$.

For a *d-bound*$(\mathcal{B}, \varepsilon_b)$ of $(\Sigma, \varepsilon)$. We define a coupling of a bound $[\varepsilon, \varepsilon_b] : E(\Sigma) \to \mathbb{Z}$ as

$$[\varepsilon, \varepsilon_b](x) = \begin{cases} 1, & \text{if } x \in \mathcal{B} \cap [X,Y], \varepsilon^{\nu_x}(x,v) = \varepsilon_b^{\nu_x}(x,v); \\ -1, & \text{if } x \in \mathcal{B} \cap [X,Y], \varepsilon^{\nu_x}(x,v) \neq \varepsilon_b^{\nu_x}(x,v); \\ 2, & \text{if } x \in \mathcal{B} \cap E_X, \varepsilon^{\nu_x}(x,v) = \varepsilon_b^{\nu_x}(x,v); \\ -2, & \text{if } x \in \mathcal{B} \cap E_X, \varepsilon^{\nu_x}(x,v) \neq \varepsilon_b^{\nu_x}(x,v); \\ 0, & \text{otherwise.} \end{cases} \tag{4.4}$$

The coupling $[\varepsilon, \varepsilon_b]$ determines a vector in $\mathbb{R}^E$. We call it a *bond vector* of $\mathcal{B}$ in $(\Sigma, \varepsilon)$. The spanning space of all the bond vectors of bonds in $(\Sigma, \varepsilon)$ is called *bond space*, denoted $U(\Sigma, \varepsilon)$. If fact, it is independent of the orientation $\varepsilon$.

Let $\Sigma^\varepsilon$ be a directed graph on the edge set $\{e_1, e_2, \ldots, e_n\}$ with orientation $\varepsilon$ such that the family of circuits is $\mathcal{C}(\Sigma^\varepsilon) = \{C_1, C_2, \ldots, C_m\}$. The *circuit incidence matrix* $A(\Sigma)$ of $\Sigma$ is the $m \times n$ matrix $a_{ij}$ in which $a_{ij} := [\varepsilon, \varepsilon_C](x)$ defined in (4.3). Similarly, the *bond incidence matrix* $H(\Sigma)$ is the $k \times n$ matrix $(h_{ij})$ in which $h_{ij} := [\varepsilon, \varepsilon_b](x)$ defined in (4.4). Clearly, the $A(\Sigma)$ and $H(\Sigma)$ are not the unimodular matrices. However, by a result of Chen and Wang in [9] the row vector of $A(\Sigma)$ is orthogonal to the collum vector of $H(\Sigma)^T$; that is to say,

$$A(\Sigma)H(\Sigma)^T = 0. \tag{4.5}$$

They also prove the inner product space $\mathbb{R}^E$ of signed graph $(\Sigma, \varepsilon)$ is the orthogonal directed sum of the circuit space $Z(\Sigma)$ and the bond space $U(\Sigma)$, i.e.,

$$\mathbb{R}^E = Z(\Sigma) \oplus U(\Sigma).$$

With these convention we define a *tension* on $\Sigma$ *with values in* $\Gamma$ as any $f \in \Gamma^E$ for which

$$A(\Sigma)f = 0, \tag{4.6}$$

in other words, for which $f \in \operatorname{Nul} A(\Sigma)$. This definition generalizes that of a tension on a graph; Naturally, then, we generalize theorems on tensions. A $k$-tension is an integral tension $f$ for which every $|f(e)| < k$, just as before. The set of all tensions of $(\Sigma, \varepsilon)$ is an abelian group, denoted $\tau(\Sigma, \varepsilon; \Gamma)$. Let $\tau_{\text{nz}}(\Sigma, \varepsilon; \Gamma)$ be the set of all nowhere-zero tensions.

### 4.1. Group-valued tensions.

We begin our treatment of signed graphs with the analog of the modular tension polynomial, since as far as we know it has not been published. The Tutte polynomial $t(\Sigma; x, y)$ of $\Sigma$ is defined to be that of $M(\Sigma)$. (It is not equal to that of the underlying graph unless $\Sigma$ is balanced; see [6] for the Tutte polynomial of a matroid.)

**Theorem 4.1.** *For each signed graph $\Sigma$ there is a polynomial $\tau(\Sigma, k)$ such that the number of nowhere-zero tensions on $\Sigma$ with values in a finite abelian group $\Gamma$ of odd order is $\tau(\Sigma, |\Gamma|)$. In fact,*

$$\tau(\Sigma, x) = (-1)^{r(\Sigma)} t(\Sigma; 1 - x, 0).$$

*Proof.* Let $\Sigma = (G, \sigma)$ be a signed graph. Let $\Gamma$ be an abelian group. A *coloring* of $\Sigma$ with the color set $\Gamma$ is a function $f : V \to \Gamma$. $f$ is called *proper* if it satisfies $f(v) \neq f(u)\sigma(e : vu)$ for all $e \in E(\Sigma)$. Let $K(\Sigma, \Gamma)$ denote the set of all colorings of $\Sigma$ with colors in $\Gamma$ and let $K_{\rm nz}(\Sigma, \Gamma)$ denote the set of all proper colorings. If $|\Gamma| = q$ is odd positive integer, it is well-known that the counting function

$$\chi(\Sigma, q) := |K_{\rm nz}(\Sigma, \Gamma)| \tag{4.7}$$

is a polynomial function of $q$, depending only on the order of $\Gamma$, not on the group structure; $\chi(\Sigma, t)$ is called the *chromatic polynomial* of $\Sigma$.

Let $\varepsilon$ be an orientation of $\Sigma$. We denote by $T(\Sigma; \varepsilon, \Gamma)$ the abelian group of all tensions of the $(\Sigma, \varepsilon)$ with values in $\Gamma$, called the *tension group* of $(\Sigma, \varepsilon)$, and by $T_{\rm nz}(\Sigma, \varepsilon; \Gamma)$ the set of all nowhere-zero tensions. If $\Gamma$ is finite, we shall see that $|T(\Sigma; \varepsilon, \Gamma)|$ and $|T_{\rm nz}(\Sigma, \varepsilon; \Gamma)|$ depend only on the order of $\Gamma$, but not on the abelian group structure. So, for $|\Gamma| = q$, we define the counting function

$$\tau(\Sigma, q) := |T_{\rm nz}(\Sigma, \varepsilon; \Gamma)|. \tag{4.8}$$

We shall see that $\tau(\Sigma, q)$ is a polynomial function of positive odd integers $q = |\Gamma|$, and is independent of the orientation $\varepsilon$ and the abelian group structure of $\Gamma$.

Note that a coloring of $\Sigma$ may be viewed as a potential on $\Sigma$. There is natural difference operator $d : \Gamma^V \to \Gamma^E$ defined by

$$(df)(e) = \varepsilon(u, e)f(u) + \varepsilon(v, e)f(v), \tag{4.9}$$

where $e = uv$ is an edge with the orientation $\varepsilon$; see [9, 27].

In order to obtain our result, we shall use the following lemma, which is obtained by Chen and Wang in [9].

**Lemma 4.2.** (a) $\operatorname{Im}(d) = T(\Sigma, \varepsilon; \Gamma)$.

(b) $d : K(\Sigma, \Gamma) \to T(\Sigma, \varepsilon; \Gamma)$ *is a group homomorphism with* $\operatorname{Ker}(d) \simeq \Gamma^{b(\Sigma)}$, *where* $b(\Sigma)$ *is the number of balanced components of* $\Sigma$.

(c) *The restriction* $d : K_{\rm nz}(\Sigma, \Gamma) \to T_{\rm nz}(\Sigma, \varepsilon; \Gamma)$ *is well defined.*

**Corollary 4.3.** *The chromatic polynomial* $\chi(\Sigma, t)$ *and the modular tension polynomial* $\tau(\Sigma, t)$ *are related by*

$$\chi(\Sigma, t) = t^{b(\Sigma)}\tau(\Sigma, t), \tag{4.10}$$

*where* $b(\Sigma)$ *is the number of balanced components of* $\Sigma$.

*Proof.* It follows from (b) and (c) of Lemma 4.2. ☐

By a result of Zaslavsky in [27], we have the relationship between the chromatic polynomial $\chi(\Sigma, t)$ and the characteristic polynomial $p(t)$ as the follows;

$$\chi(\Sigma, t) = \sum_{S \subseteq E(\Sigma)} t^{b(S)}(-1)^{\#S} = \sum_{A \in \mathrm{Lat}\Sigma} \mu(\emptyset, A) t^{b(A)} = t^{b(\Sigma)} p(t). \quad (4.11)$$

By (6.20) of [6], the characteristic polynomial $p(t)$ of $\Sigma$ is related to the Tutte polynomial $t(\Sigma; x, y)$ by

$$p(t) = (-1)^{r(\Sigma)} t(\Sigma; 1-t, 0). \quad (4.12)$$

By Eqs. (4.10)-(4.12) we obtain

$$\tau(\Sigma, x) = (-1)^{r(\Sigma)} t(\Sigma; 1-x, 0).$$

□

**Problem 4.4.** *Is there any significance to $\tau(\Sigma, k)$ evaluated at even natural number $k$?*

Theorem 4.1 means there is a polynomial $\tau(\Sigma, x)$, which we call the *(strict) modular tension polynomial*, such that for any odd positive number $k$, $\tau(\Sigma, k)$ is the number of nowhere-zero tensions on $\Sigma$ with values in any fixed abelian group of order $k$. In [2] the authors guess that there could be something similar with the modular flow polynomial, and whether flows and colorings might be connected through duality of signed graphs, analogously to the duality of colorings and flows on planar graphs. In fact Theorem 4.1 has positively answer their suspection; precisely, it is just the tensions and flows which are connected through duality of signed graphs, analogously to the duality of colorings and flows on planar graphs.

**Corollary 4.5.** *The number of acyclic orientations of $\Sigma$ equals $(-1)^{r(\Sigma)}\tau(\Sigma, -1)$.*

*Proof.* The number of acyclic reorientations of an orientation of a matroid $M$ is $t(M; 2, 0)$ [22]. Since cycles in an orientation of $\Sigma$ are the same as cycles in the corresponding orientation of $M(\Sigma)$ [27], the number of acyclic orientations of $\Sigma$ equals $t(M(\Sigma); 2, 0) = (-1)^{r(\Sigma)}\tau(\Sigma, -1)$. □

### 4.2. Integral $k$-tensions on signed graphs

It is time for integral tensions. A $k$-tension for a signed graph is an integral tension $f$ for which every $|f(e)| < k$, $e \in E(\Sigma)$, just as before. For $k > 0$ let

$$\tau_{\mathbb{Z}}(\Sigma, k) := \text{the number of nowhere-zero } k\text{-tensions on } \Sigma.$$

As with abelian-group tensions, $\tau_{\mathbb{Z}}(\Sigma, k) = 0$ if there is a loop in $M(\Sigma)$. Let

$$\bar{\tau}_{\mathbb{Z}}(\Sigma, k) := \text{the number of all } k\text{-tensions on } \Sigma,$$

for all $k > 0$. We take $U := U(\Sigma)$ to be the real bond space $\operatorname{Nul} A(\Sigma)$, which is a solution space of Eq.(4.6) and just as unsigned graphs,

$$P := U \cap [-1,1]^E, \qquad \mathcal{H} := (\mathcal{H}_E)^U,$$

where $\mathcal{H}_E$ is the arrangement of coordinate hyperplanes in $\mathbb{R}^E$. As with ordinary graphs, a tension $f$ and an orientation $\varepsilon$ are *compatible* if $f \geq 0$ when expressed in terms of $\varepsilon$.

**Theorem 4.6.** (a) *For any signed graph $\Sigma$, $\tau_{\mathbb{Z}}(\Sigma, k)$ is a quasipolynomial function of $k$ for $k = 1, 2, 3, \ldots$. It's period is 1 or 2, and is 1 if $\Sigma$ is balanced. $\tau_{\mathbb{Z}}(\Sigma, k)$ has leading term* (volP)$k^{r(\Sigma)}$ *if $M(\Sigma)$ has no loops; otherwise $\tau_{\mathbb{Z}}(\Sigma, k)$ is identically zero.*

(b) *Furthermore, $(-1)^{r(\Sigma)}\tau_{\mathbb{Z}}(\Sigma, -k)$ equals the number of $(k+1)$-tensions counted with multiplicity equals to the number of compatible acyclic orientations of $\Sigma$.*

(c) *In particular, the constant term $\tau_{\mathbb{Z}}(\Sigma, 0)$ equals the number of acyclic orientations of $\Sigma$, which equals $(-1)^{r(\Sigma)}\tau(\Sigma, -1)$.*

(d) *Finally, $\bar{\tau}_{\mathbb{Z}}(\Sigma, k)$ is a quasipolynomial of period 1 or 2 (period 1 if $\Sigma$ is balanced) whose leading term is the same as that of $\tau_{\mathbb{Z}}(\Sigma, k)$ and whose constant term is $(-1)^{r(\Sigma)}$. Furthermore, $\bar{\tau}_{\mathbb{Z}}(\Sigma, k) = (-1)^{r(\Sigma)}\bar{\tau}_{\mathbb{Z}}(\Sigma, 1-k)$.*

**Lemma 4.7.** *The vertices of $(P, \mathcal{H})$ are half integral.*

*Proof.* For the fundamental circuits according to a base of $M(\Sigma)$ we choose the elements suitably such that the matrix $A(\Sigma)$ contains an identity matrix $I_m$. A vertex is a solution of $A(\Sigma)f = 0$ with $|E| - m$ coordinates of $f$ set equal to fixed values in $\{0, 1, -1\}$. Let $B$ be the edge set whose coordinates in $f$ are left undetermined, let $B^c := E \setminus B$, and write $f = (f_B, f_{B^c})^T$. Then $f$ is the unique solution of $H(\Sigma|B)f_B = -H(\Sigma|B^c)f_{B^c}$.

The null space $\operatorname{Nul} A(\Sigma)$ is 2-regular (see Proposition 9.1 of [15]) and if $Q$ is a nonsingular square matrix for which $\operatorname{Nul}[I|Q]$ is 2-regular, then $Q^{-1}b$ is half integral for every integral vector $b$ (a special case of [18, Proposition 6.1]). These facts applied to $Q = H(\Sigma|B)$ imply that the solution of $H(\Sigma|B)f = b$ is half integral for any $b \in \mathbb{Z}^B$. Apply this to $b = -H(\Sigma|B^c)f_{B^c}$. □

*Proof of Theorem 4.6.* This proof is similar to that of Theorem 3.1. In (a) and (d), instead of total unimodularity we have Lemma 4.7 to tell us that the denominator of $(P, \mathcal{H})$, hence the period of the Ehrhart quasipolynomials, divides 2.

For (b) we need to show that the regions of $\mathcal{H}$ corresponds to the acyclic orientations of $\Sigma$. The latter are the acyclic reorientations of the natural orientation of $M(\Sigma)$, which is the oriented matroid of columns of $A(\Sigma)$; see Theorem 3.3 of [27]. Now we apply Lemma 2.3 (ii).

For (c) we use Corollary 4.5. □

### 4.3. Nowhere-zero tensions reduce to tension with Möbius complications

The final main result express the nowhere-zero integral tension polynomial in terms of the weak integral tension polynomials of subgraphs. We begin with structural lemmas. As before, $U$ is the bond space, $P := [-1,1]^E \cap U$, and $\mathcal{H} := \mathcal{H}_E^U$. For a flat $F \in \mathcal{L}(\mathcal{H})$ we define

$$E(F) := \{e \in E : F \subseteq H_e\} = \{e \in E : f(e) = 0 \text{ for all } f \in F\}$$

This is the $E_U(F)$ of Lemma 2.3 (ii). We see that $E(F)^c$ is the union of the supports of the vectors in $F$.

**Lemma 4.8.** *The lattice of flats of $\mathcal{H}$ is isomorphic to the lattice of closed sets of the biased matroid $M(\Sigma)$. The isomorphism is given by $F \mapsto E(F)$. The corresponding matroid isomorphism $M(\Sigma) \cong M(\mathcal{H})$ is given by $e \in H_e \cap U$.*

*Proof.* There is an application of Lemma 2.3. The matrix is $A(\Sigma)$ and the matroid $M(A(\Sigma))$ is the dual biased matroid $M^{\perp}(\Sigma)$ by Theorem 8A.1 of [27], so $M(\Sigma)$ is the chain-group matroid of Row $A(\Sigma)$. The lemma applies since the real bond space $U = \operatorname{Nul} A(\Sigma)$. □

**Lemma 4.9.** *$\tau_{\mathbb{Z}}(\Sigma, k)$ is identically zero if and only if $M(\Sigma)$ has a loop.*

*Proof.* By a result of Bouchet's on integral chain-group matroids [5]: the chain-group has a nowhere-zero chain if and only if the dual matroid has no loop. In our case the chain-group is the group of the integral tensions, $\operatorname{Nul}A(\Sigma)$ by definition. Its chain-group matroid is dual to that of $\operatorname{Row}A(\Sigma)$, which is dual to the column matroid of $A(\Sigma)$, which is $M(\Sigma)$. □

**Lemma 4.10.** *A flat $F$ of $\mathcal{H}$ can be represented as* $[\operatorname{Nul} A(\Sigma | E(F)^c)] \times \{0\}^{E(F)}$.

*Proof.* The lemma is follows by the definitions of $U$ and $E(F)$. □

**Theorem 4.11.** *Take a signed graph $\Sigma$. Letting $S$ ranges over all subsets of $E$, or merely over all for which $M(\Sigma)|S$ has no loops, then*

$$\tau_{\mathbf{Z}}(\Sigma, -k) = \sum_S |\mu(\hat{0}, S^c)| \bar{\tau}_{\mathbf{Z}}(\Sigma|S, k+1) \tag{4.13}$$

*and*

$$\tau_{\mathbf{Z}}(\Sigma, k) = \sum_S \mu(\hat{0}, S^c) \bar{\tau}_{\mathbf{Z}}(\Sigma|S, k+1), \tag{4.14}$$

*where $\mu$ is the Möbius function of $M(\Sigma)$ and $\hat{0}$ is the set of loops of $M(\Sigma)$.*

*Proof.* The polytope and the arrangement are transversal because $(\bigcap \mathcal{H}) \bigcap P^0 \neq \emptyset$.

By Lemma 4.9 $\tau_{\mathbf{Z}}(\Sigma|S, k) = 0$ if $M(\Sigma)|S$ has a loop, therefore the two ranges of summation are equivalent. For $F \in \mathcal{L}(\mathcal{H})$, by Lemma 4.10 we know that

$$P \cap F = [-1, 1]^E \cap ([\operatorname{Nul} A(\Sigma|E(F)^c)] \times \{0\}^{E(F)}).$$

Take $S = E(F)^c$, then

$$P \cap F = ([-1, 1]^S \cap Z') \times \{0\}^{S^c},$$

where $Z'$ is the real bond space of $\Sigma|S$. Its Ehrhart polynomial equals $\bar{\tau}_{\mathbf{Z}}(\Sigma|S,\ k+1)$.

Now the result follows from Lemma 4.8, Theorem 2.2, Eq.(3.2), and Theorem 3.1(iv). □

It may be helpful to list some characterization of the edge sets that support nowhere-zero integral tensions.

**Theorem 4.12.** *For $S \subseteq E := E(\Sigma)$, the following properties are equivalent.*

(i) *$M(\Sigma)|S$ has no loops;*
(ii) *$\Sigma|S$ has an acyclic orientation;*
(iii) *$\Sigma|S$ has a nowhere-zero integral tension;*
(iv) *$\Sigma|S$ has a nowhere-zero real tension;*
(v) *$S^c$ is closed in the bias matroid $M(\Sigma)$;*
(vi) *$S = E(F)^c$ for some flat $F \in \mathcal{L}((\mathcal{H}_E)^U)$.*

*Proof.* (i) $\Leftrightarrow$ (ii): We could prove it for signed graphs via oriented matroids. We know the number of acyclic reorientations of an orientation of a matroid $M$ is $t(M; 2, 0)$ [22] and that this equals 0 if and only if $M$ has a loop. Apply that to the natural orientation of $M(\Sigma)$.

(i) $\Leftrightarrow$ (iii) by Lemma 4.9.

(iii) $\Leftrightarrow$ (iv) is trivial.

(iv) $\Leftrightarrow$ (i) by the proof of [5, Lemmas 2.4 and 2.5], which amounts to say that any tension on $\Sigma$ with values in an abelian group where $2a = 0$ implying $a = 0$ must be zero on every loop. Here the group is the additive group of $\mathbb{R}$.

(v) $\Leftrightarrow$ (i): By matroid duality the complements of the closed sets in $M^{\perp}(\Sigma)$ are the edge sets that do not contain a loop of $M(\Sigma)$.

(v) $\Leftrightarrow$ (vi): This is Lemma 4.8. □


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