BANANA TREES AND UNION OF STARS ARE INTEGRAL SUM GRAPHS

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ABSTRACT

The concept of integral sum graphs is introduced by Harary [6]. A graph G is an **integral sum graph** or **$\int\Sigma$-graph** if the vertices of G can be labelled with distinct integers so that e = uv is an edge of G if and only if the sum of the labels on vertices u and v is also a label in G. Xu [12] has shown that the union of any three stars and the union of any number of integral sum trees are integral sum graphs. Xu poses the question as to whether all disconnected forests are integral sum graphs. In this paper, we prove that all banana trees and union of any number of stars are integral sum graphs.

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Key words: sum graph or Σ-graph, integral sum graph or $\int\Sigma$-graph, graph $G^+(S)$, banana tree, generalized star, union of stars.
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1. INTRODUCTION

In 1990, Harary [5] introduced the notion of a sum graph. A graph G(V, E) is called a **sum graph** or **Σ-graph** if there is a bijective vertex labeling f from V onto a set of positive integers S such that $xy \in E(G)$ if and only if $f(x) + f(y) \in S$. In 1994, Harary generalized sum graphs by permitting S to be any set of integers and called these graphs **integral sum graphs** or **$\int\Sigma$-graphs [6]**.

Integral sum graph G with a labeling set S is denoted as $G^+(S)$.

Chen [2] obtained several properties of the integral sum labeling of a graph G with $\Delta(G) < |V(G)|-1$. Vilfred and Nicholas [8,10] studied different properties of integral sum graphs G with $\Delta(G) = |V(G)|-1$. Chen [2] proved that trees obtained from a star by extending each edge to a path and trees all of whose vertices of degree not 2 are at least distance 4 apart are integral sum graphs. He conjectures that all trees are integral sum graphs. Wu, Mao, and Le [11] proved that $m.P_n$ are integral sum graphs. Xu [12] has shown that the following classes of graphs are integral sum graphs: the union of any three stars; $T \cup K_{1,n}$ for all trees T; $m.K_3$ for all m; and the union of any number of integral sum trees. Xu also proved that if 2G and 3G are integral sum graphs, then so is mG for all m > 1. Liaw, Kuo and Chang [7] proved that all caterpillars are integral sum graphs. Chen's conjecture [2] on trees still remains open. This motivated us to find new families of trees, which are integral sum graphs. In this paper, we prove that all banana trees and union of any number of stars are integral sum graphs.

All graphs in this paper have neither loops nor multiple edges. For all basic ideas in graph theory, we follow [4] and for further readings on graph labeling problems refer [3].

2. MAIN RESULTS

Chen [2] defined a **generalized star** as a tree obtained from a star by extending each edge to a path and proved the following result.

Theorem 2.1 [2] Every generalized star is an integral sum graph. ■

Definition 2.2 [1] Let G(V,E) be any graph and f be a vertex labeling on G. Then an edge uv of G is said to be **f-proper** if f(u)+f(v) = f(w) for some $w \in V(G)$.

It is easy to prove that the labeling f is an integral sum labeling of the graph G if and only if all edges of G are f-proper and all edges of G^c are not f-proper.

Definition 2.3 [9] A **banana tree** is a family of stars with a new vertex adjoined to one end vertex of each star.

Let T be a banana tree corresponding to the family of stars $\{ K_{1,n_1}, K_{1,n_2}, \ldots, K_{1,n_t} \}$, $t \geq 1$. Let v_i denote the central vertex and $u_{i,j}$, $j = 1, 2, \ldots, n_i$ denote the end vertices of the i^{th} star K_{1,n_i} where $i = 1, 2, \ldots, t$. Let w denote the new vertex joining one vertex $u_{i,1}$ of each star, $i = 1, 2, \ldots, t$. Figure 1 shows banana tree corresponding to $\{ K_{1,1}, 2K_{1,2}, K_{1,3}, K_{1,4} \}$.

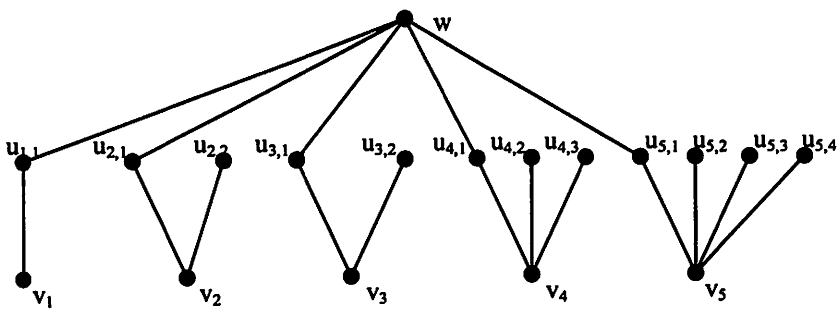

Fig.1

Theorem 2.4 Every banana tree T is an integral sum graph.

Proof Let T be a banana tree corresponding to the family of stars $\{K_{1,n_1}, K_{1,n_2}, \ldots, K_{1,n_t}\}$, $t \geq 1$. Let v_i denote the central vertex and for $j = 1,2,\ldots,n_i$ let $u_{i,j}$ denote the end vertices of the i^{th} star K_{1,n_i} where $i = 1,2,\ldots,t$. Let w denote the new vertex joining one vertex $u_{i,1}$ of each star, $i = 1, 2, \ldots, t$.

If $n_i \leq 2$ for all $i = 1, 2, \ldots, t$, then T is a generalized star and hence is an integral sum graph, using Theorem 2.1. Let $n_i > 2$ for at least one i, $1 \leq i \leq t$. Without loss of generality, let $n_1 \leq n_2 \leq \ldots \leq n_t$. This implies, $n_t \geq 3$. Also a banana tree with t = 1 is actually a general star which is an integral sum graph, using Theorem 2.1.

Let $x > 0$, $y > x$ and $n_0 = 0$. We shall define a vertex labeling f on T with $t \geq 2$ as follows:

$f(w) = x;$

$$f(v_i) = [\prod_{k=1}^{i} (n_{k-1}+1)](x+y) \text{ for } 1 \leq i \leq t-1;$$

$$f(u_{i,j}) = j\,[\prod_{k=1}^{i} (n_{k-1}+1)](x+y) - x \text{ for } 1 \leq i \leq t-1 \text{ and } 1 \leq j \leq n_i;$$

$$f(v_t) = -[\prod_{k=1}^{t} (n_{k-1}+1) - 1](x+y);$$

$$f(u_{t,1}) = [\prod_{k=1}^{t} (n_{k-1}+1)](x+y) - x;$$

$$f(u_{t,j}) = \{(j-1)[\prod_{k=1}^{t} (n_{k-1}+1)] - (j-2)\}(x+y) \text{ for } 2 \leq j \leq n_t.$$

As an example, figure 2 illustrates the vertex labeling f with x = 1 and y = 2 being applied to a banana tree corresponding to $\{K_{1,1}, 2K_{1,2}, K_{1,3}, K_{1,4}\}$, where $t = 5$, $n_0 = 0$, $n_1 = 1$, $n_2 = n_3 = 2$, $n_4 = 3$ and $n_5 = 4$.

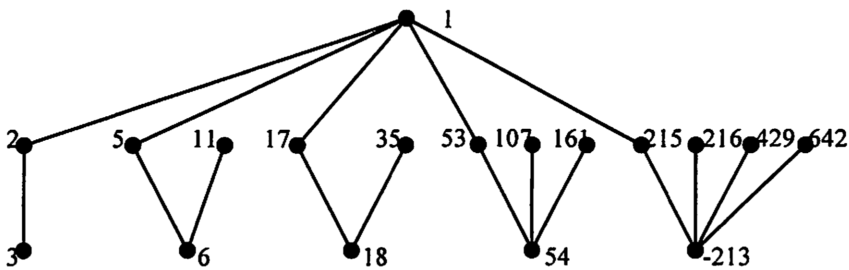

Fig.2

From the definition of the labeling f, we have the following two claims.

Claim 1 The labels of all the vertices in T-w are of the form ax+by where $a \in \{b,b-1\}$.

Claim 2 $f(v_t) < 0 < f(w) < f(u_{1,1}) < f(v_1) < f(u_{1,2}) < \ldots < f(u_{1,n_1}) < f(u_{2,1}) < f(v_2) < f(u_{2,2}) < \ldots < f(u_{t-1,n_{t-1}}) < f(u_{t,1}) < f(u_{t,2}) < \ldots < f(u_{t,n_t})$.

It is easy to check the following:

$f(w) + f(u_{i,1}) = f(v_i)$ for $1 \le i \le t-1$ and $f(w) + f(u_{t,1}) = f(u_{t,2})$;

$f(u_{i,j}) + f(v_i) = f(u_{i,j+1})$ for $1 \le i \le t-1$ and $1 \le j \le n_i - 1$;

$f(u_{i,n_i}) + f(v_i) = f(u_{i+1,1})$ for $1 \le i \le t-1$ and $f(u_{t,1}) + f(v_t) = f(u_{1,1})$;

$f(u_{t,2}) + f(v_t) = f(v_1)$ and $f(u_{t,j}) + f(v_t) = f(u_{t,j-1})$ for $3 \le j \le n_t$.

Thus all the edges of T are f-proper. It suffices to show that every edge $e = uv \in E(T^c)$ is not f-proper. Note that v_t is the only vertex of T with a negative label. If $e = uv_t \in E(T^c)$, then $u \notin V(K_{1,n_t})$, implying that $0 < f(u) < |f(v_t)|$. Thus $f(v_t) < f(u)+f(v_t) < 0$ and then $e = uv_t \in E(T^c)$ is not f-proper for $u \notin V(K_{1,n_t})$.

Let $V(T) - \{v_t\} = A \cup B \cup C$, where $A = \{w\}$, $B = \{u_{t,1}\} \cup \{u_{i,j}: i = 1,2,\ldots,t-1 \text{ and } j = 1,2,\ldots,n_i\}$ and $C = \{v_i : i = 1,2,\ldots,t-1\} \cup \{u_{t,j}: j = 2,3,\ldots,n_t \text{ and } t \ge 2\}$. Clearly the labels of all vertices in B are of the form (b-1)x+by for some positive integer b, while the labels of all vertices in C are of the form bx+by for some positive integer b. Consider any edge $e = uv \in E(T^c)$, where both u and v have positive labels. We shall consider four cases: (1) $u \in A$ and $v \in B \cup C$; (2) $u,v \in B$; (3) $u,v \in C$; (4) $u \in B$ and $v \in C$.

Case 1. $u \in A$ and $v \in B \cup C$. Note that f(u)+f(v) = (a+1)x+ay for some integer a if $u \in A$ and $v \in C$. By Claim 1, $f(u)+f(v) \notin f(V)$. If $v \in B$ and $uv \in E(T^c)$, then $v = u_{r,s}$ for some fixed r and s with $1 \le r \le t-1$ and

$2 \le s \le n_r$, implying $f(v) = s\,[\prod_{k=1}^{r} (n_{k-1}+1)](x+y) - x$. It follows that

$$f(u_{r,s}) < f(u)+f(v) = s\,[\prod_{k=1}^{r} (n_{k-1}+1)](x+y) < (s+1)[\prod_{k=1}^{r} (n_{k-1}+1)](x+y)-x$$

$\le \min \{f(u_{r,s+1}), f(u_{r+1,1})\}$. By Claim 2, $f(u)+f(v) \notin f(V)$.

Case 2. $u,v \in B$. Then f(u)+f(v) = (a-2)x+ay for some integer a. By Claim 1, $f(u)+f(v) \notin f(V)$.

Case 3. $u,v \in C$. If u and v are end vertices of the t^{th} star excluding $u_{t,1}$, by Claim 2 and the fact that $f(u)+f(v) \ne f(u_{t,r})$ for $4 \le r \le n_t$, $f(u)+f(v) \notin f(V)$.

If u and v are central vertices of two different stars, $u = v_r$, $v = v_s$ and $r < s < t$, then by construction $f(v_{r+1}) = (n_r+1)f(v_r)$, $1 \le r \le t-2$. If $f(v_r)+f(v_s) = f(v_k)$ for some k, $r < s < k \le t-1$, then by dividing on both sides by $f(v_s)$, we see that the left side is not an integer whereas the right side is an integer which is not possible and hence $f(u)+f(v) \notin f(V)$. Also, $f(v_r)+f(v_s) \neq f(u_{t,j})$ for any j, $2 \leq j \leq n_t$ and $r < s < t-1$. Thus in this case also $f(u)+f(v) \notin f(V)$.

If $u = v_i$ and $v = u_{t,j}$, $1 \leq i \leq t-1$ and $j = 2,3,\ldots,n_t$, then $f(v_i) < |f(v_t)| < f(u_{t,j})$ for every $i = 1, 2, \ldots, t-1$ and $j = 1, 2, \ldots, n_t$ and hence $f(u)+f(v) \notin f(V)$.

Case 4 $u \in B$ and $v \in C$. In this case both u and v are vertices of different stars or of the same star. Here, the following four sub cases arise:

i) $u = u_{t,1}$ and $v = u_{t,j}$, $j = 2,3,\ldots,n_t$;
ii) $u = u_{t,1}$ and $v = v_i$, $i = 1,2,\ldots,t-1$;
iii) $u = u_{i,j}$ and $v = v_t$, $i = 1,2,\ldots,t-1$ and $j = 1,2,\ldots,n_i$ (already proved for this case) and
iv) $u = u_{i,j}$ and $v = u_{t,k}$, $i = 1,2,\ldots,t-1$, $j = 1,2,\ldots,n_i$ and $k = 1,2,\ldots,n_t$.

If $u = u_{t,1}$ and $v = u_{t,j}$, $2 \leq j \leq n_t$, then $f(u_{t,1})+f(u_{t,j}) > f(u_{i,k})$ for any i and k, $1 \leq i \leq t-1$ and $1 \leq k \leq n_i$ and by Claim 2, $f(u)+f(v) \notin f(V)$.

If $u = u_{t,1}$ and $v = v_i$, $1 \leq i \leq t-1$, then $f(u)+f(v) > f(u_{i,j})$ for any i and j, $1 \leq i \leq t-1$ and $1 \leq j \leq n_i$ and hence $f(u)+f(v) \notin f(V)$.

If $u = u_{i,j}$ and $v = u_{t,1}$, then by Claim 1, $f(u)+f(v) \notin f(V)$, $1 \leq i \leq t-1$ and $1 \leq j \leq n_i$. If $u = u_{i,j}$ and $v = u_{t,k}$, $1 \leq i \leq t-1$, $1 \leq j \leq n_i$ and $2 \leq k \leq n_t$, then $f(u) + f(v) = (b-1)x + by > f(u_{t,k})$ for some positive integer b and for any i and j, $1 \leq i \leq t-1$ and $1 \leq j \leq n_i$. Hence $f(u)+f(v) \notin f(V)$.

Thus in all possible cases, every edge $e = uv \in E(T^c)$ is not f-proper whereas all the edges of T are f-proper. Hence T is an integral sum graph. ■

Xu [12] proved that the union of any three stars is an integral sum graph. Also, Harary [6] proved that every matching is an integral sum graph. Here, we extend the result for the union of stars.

Theorem 2.6 The union of stars is an integral sum graph.

Proof Let $G = K_{1,n_1} \cup K_{1,n_2} \cup \ldots \cup K_{1,n_t}$, $t \geq 1$. Let v_i denote the central vertex and for $j = 1, 2, \ldots, n_i$ let $u_{i,j}$ denote the end vertex of the i^{th} star K_{1,n_i} where $i = 1, 2, \ldots, t$. When $n_i = 1$ for all i, $i = 1,2,\ldots,t$, G is a matching and hence is an integral sum graph [6].

Without loss of generality assume $n_1 \geq 2$ and $t \geq 2$. We shall define a vertex labeling f on G as follows:

$f(u_{1,j}) = 2j - 1$ for $1 \leq j \leq n_1$;

$f(v_1) = 2$;

$$f(u_{i,j}) = [\prod_{k=1}^{i-1} (2.n_k+1)](2j - 1) \text{ for } 2 \leq i \leq t-1 \text{ and } 1 \leq j \leq n_i;$$

$$f(v_i) = 2\,[\prod_{k=1}^{i-1} (2.n_k+1)] \text{ for } 2 \leq i \leq t-1;$$

$$f(v_t) = -\,[\prod_{k=1}^{t} (2.n_k+1)] + 1;$$

$f(u_{t,j}) = j\,[\prod_{k=1}^{t}(2.n_{k-1}+1)] - (j-1)$ for $1 \le j \le n_t$.

As an example, figure 3 illustrates the integral sum labeling f on $G = K_{1,3} \cup K_{1,1} \cup K_{1,2} \cup K_{1,4} \cup K_{1,5}$ as given in this proof. Here $t = 5$, $n_1 = 3$, $n_2 = 1$, $n_3 = 2$, $n_4 = 4$ and $n_5 = 5$.

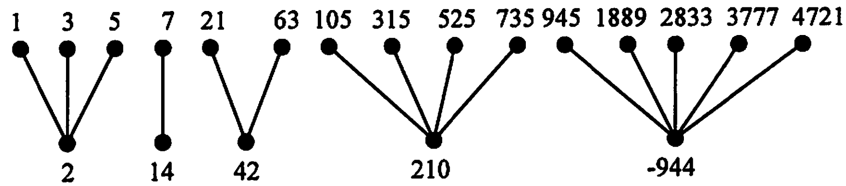

Fig.3

From the definition of the labeling f, we have the following two claims.

Claim 1 The labels of all the end vertices in G are odd positive integers whereas the labels of central vertices are even integers.

Claim 2 $f(v_t) < 0 < f(u_{1,1}) < f(v_1) < f(u_{1,2}) < \ldots < f(u_{1,n_1}) < f(u_{2,1}) < f(v_2) < f(u_{2,2}) < \ldots < f(u_{2,n_2}) < f(u_{3,1}) < f(v_3) < f(u_{3,2}) < \ldots < f(u_{3,n_3}) < \ldots < f(u_{t-1,1}) < f(v_{t-1}) < f(u_{t-1,2}) < \ldots < f(u_{t-1,n_{t-1}}) < |f(v_t)| < f(u_{t,1}) < f(u_{t,2}) < \ldots < f(u_{t,n_t})$.

It is easy to check the following:

$f(u_{i,j}) + f(v_i) = f(u_{i,j+1})$ for $1 \le i \le t-1$ and $1 \le j \le n_i - 1$;

$f(u_{i,n_i}) + f(v_i) = f(u_{i+1,1})$ for $1 \le i \le t-1$;

$f(u_{t,1}) + f(v_t) = f(u_{1,1})$ and $f(u_{t,j}) + f(v_t) = f(u_{t,j-1})$ for $2 \le j \le n_t$.

Thus all the edges of G are f-proper. It suffices to show that every edge $e = uv \in E(G^c)$ is not f-proper. Note that v_t is the only vertex of G with a negative label and it is even. If $e = uv_t \in E(G^c)$, then $u \notin V(K_{1,n_t})$, implying that $0 < f(u) < |f(v_t)|$. Thus $f(v_t) < f(u)+f(v_t) < 0$ and hence $e = uv_t \in E(G^c)$ is not f-proper for $u \notin V(K_{1,n_t})$.

Consider any edge $e = uv \in E(G^c)$, where both u and v have positive labels. Corresponding to each edge $e = uv$ of G^c, we consider $f(u)+f(v)$, closest integers less than and more than $f(u)+f(v)$ and with same parity of even or odd to show that $f(u)+f(v) \notin f(V)$. $f(u)+f(v)$ is an even integer if both u and v are end vertices or central vertices and is odd otherwise. We shall consider the following four cases.

Case 1. $u = v_i$ and $v = v_j$, $1 \le i,j \le t-1$. Without loss of generality, let $i < j$. Note that from the definition of f, $f(u)+f(v)$ is an even integer and $f(v_j) < f(v_i) + f(v_j) < |f(v_{j+1})|$, $1 \le i < j \le t-1$. By Claim 2, $f(u)+f(v) \notin f(V)$.

Case 2. $u = u_{i,j}$ and $v = u_{i,k}$, $1 \le i \le t$ and $1 \le j < k \le n_i$. Note that $f(v_i) < f(u_{i,j}) + f(u_{i,k}) < |f(v_{i+1})|$ when $1 \le i < t-1$ and $|f(v_t)| < f(u_{i,j}) + f(u_{i,k})$ when $i = t$. Therefore by Claim 2, $f(u)+f(v) \notin f(V)$.

Case 3. $u = u_{i,x}$ and $v = u_{j,k}$, $1 \le i < j \le t$, $1 \le x \le n_i$ and $1 \le k \le n_j$. Note that for $1 \le i < t-1$, $f(v_{j-1}) < f(u_{i,x}) + f(u_{j,k}) < f(v_j)$ when $k = 1$ and $j < t$; $f(v_j) < f(u_{i,x}) + f(u_{j,k}) < |f(v_{j+1})|$ when $k > 1$ and $j < t$, and $|f(v_t)| < f(u_{i,x}) + f(u_{j,k})$ when $j = t$. By Claim 2, $f(u)+f(v) \notin f(V)$.

Case 4. $u = u_{i,j}$ and $v = v_k$, $1 \le k < t$, $i \ne k$, $1 \le i \le t$ and $1 \le j \le n_j$. Note that when $i < k < t$, $f(u_{k,1}) < f(u_{i,j}) + f(v_k) < f(u_{k+1,1})$ if $n_k = 1$ and $f(u_{k,1}) < f(u_{i,j}) + f(v_k) < f(u_{k,2})$ if $n_k > 1$. When $k < i$, $f(u_{i,j}) < f(u_{i,j}) + f(v_k) < f(u_{i+1,1})$ if $j = n_i$ and $f(u_{i,j}) < f(u_{i,j}) + f(v_k) < f(u_{i,j+1})$ if $j < n_i$. By Claim 2, $f(u)+f(v) \notin f(V)$.

Thus in all possible cases, every edge $e = uv \in E(G^c)$ is not f-proper whereas all the edges of G are f-proper. Hence G is an integral sum graph. ■

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