Identities of summations involving powers and inverse of binomial coefficients

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Abstract

In this paper, we give several identities of finite sums and some infinite series involving powers and inverse of binomial coefficients.

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1. Introduction

In [1], the author first used the identity

$$\beta(p,q)=\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

to observe that

$$\binom{n}{k}^{-1}=(n+1)\int_0^1 t^k(1-t)^{n-k}dt.$$

Starting with this observation, it was proved in [1] that

$$\sum_{k=0}^{n}\binom{n}{k}^{-1}=\frac{n+1}{2^n}\sum_{k=0}^{n}\frac{2^k}{k+1}=\frac{n+1}{2^n}\sum_{j\ odd}\binom{n+1}{j}\frac{1}{j}.$$

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In [2], Toutik Mansour presented a method for obtaining a wide class of combinatorial identities, including some ones involving the inverse binomial coefficients.

Sury, Wang and Zhao in [3] showed, among other results, that

$$\sum_{k=m}^{\infty}\binom{n+k}{k}^{-1}=\frac{n}{(n-1)}\binom{m+n-1}{n-1}^{-1},$$

$$\sum_{k=0}^{\infty}\binom{n+k}{k}^{-1}(-1)^k=2^{n-1}(\ln 2-\sum_{k=1}^{n-1}\frac{1}{k})-n\sum_{k=1}^{n-1}(-1)^k\binom{n-1}{k}\frac{2^{n-1-k}}{k}.$$

In this paper, we establish several finite sums and some infinite series which involving powers and inverse of binomial coefficients. It can be found that some of our results are related to the Stirling numbers of the second kind. The identities of this type might not have been presented before.

2. Preliminaries

Lemma 1 [2]. Let $s,n\geq k$ be any nonnegative integer number, let $f(n,k)$ be given by

$$f(n,k)=\frac{(n+s)!}{n!}\int_{u_1}^{u_2}p^k(t)q^{n-k}(t)dt,$$

where $p(t)$ and $q(t)$ are two functions defined on $[u_1,u_2]$. Let $\{a_n\}_{n\geq 0}$ and $\{b_n\}_{n\geq 0}$ be any two sequences, and let $A(x)$, $B(x)$ be the corresponding ordinary generating function. Then

$$\sum_{n\geq 0}\left[\sum_{k=0}^{n}f(n,k)a_kb_{n-k}\right]x^n=\frac{d^s}{dx^s}\left[x^s\int_{u_1}^{u_2}A(xp(t))B(xq(t))dt\right].$$

Lemma 2 [4]. For each $r\geq 0$, the power series with coefficients 'r-th powers' equals:

$$\sum_{k\geq 0}k^rx^k=\sum_{j=0}^{r}j!S(r,j)\frac{x^j}{(1-x)^{j+1}}=\frac{A_r(x)}{(1-x)^{r+1}},\quad |x|<1,$$

where $S(r,j)$ and $A_n(x)$ are Stirling numbers of the second kind and Eulerian polynomials, respectively.

Lemma 3 [5].

$$\sum_{k=0}^{n} k^r x^k = \frac{A_r(x)}{(1-x)^{r+1}} - x^{n+1} \sum_{k=0}^{r} \binom{r}{k} \frac{A_k(x)}{(1-x)^{k+1}} (n+1)^{r-k},$$

where $A_k(x)$ are Eulerian polynomials.

Lemma 4 [4]. Let $B_k(y)$ and B_k are Bernoulli polynomials and Bernoulli numbers, respectively. Then

$$B_k(y) = \sum_{k=0}^{n} \binom{n}{k} B_{n-k} y^k.$$

Lemma 5 [4].

$$\sum_{k=0}^{n} k^r = \frac{1}{r+1}(B_{r+1}(n+1) - B_{r+1}).$$

Lemma 6 [6]. Let r is any nonnegative integer, then

$$\sum_{k=0}^{n} k^r = \sum_{h=0}^{r} \binom{n+1}{h+1} h! S(r,h).$$

where $S(r,j)$ are Stirling numbers of the second kind.

Lemma 7. Let $m \le n-1$ be nonnegative integer, then

$$\sum_{k=m+1}^{n} \binom{n}{k} \frac{(-1)^k}{k-m} = \sum_{k=0}^{m} \binom{m}{k} (-1)^{m+1-k} \sum_{j=1}^{n} \binom{k+n-j}{k} \frac{1}{j}.$$

Proof. Let $f(z) = \sum_{k=m+1}^{\infty} \frac{(-z)^k}{k-m} = (-1)^{m+1} z^m \ln(1+z)$.

Then

$$\sum_{k=m+1}^{n}\binom{n}{k}\frac{(-1)^k}{k-m}=[z^n]\frac{1}{1-z}f(\frac{z}{1-z})$$

$$=[z^n]\sum_{k=0}^{m}\binom{m}{k}(-1)^{m-k}\frac{1}{(1-z)^{k+1}}\ln(1-z)$$

$$=\sum_{k=0}^{m}\binom{m}{k}(-1)^{m+1-k}\sum_{j=1}^{n}\binom{k+n-j}{k}\frac{1}{j}.$$

The proof is complete. □

By the same way, we can get the following lemma.

Lemma 8. Let $m \le n-1$ be nonnegative integer, then

$$\sum_{k=m+1}^{n}\binom{n}{k}\frac{(-1)^k}{(k-m)2^k}=\frac{1}{2^m}\sum_{k=0}^{m}\binom{m}{k}(-1)^{m-k}\sum_{j=1}^{n}\binom{k+n-j}{k}(\frac{1}{2^j}-\frac{1}{j}).$$

3. Main results

Theorem 1. Let r be any nonnegative integer, then

$$\sum_{k=0}^{n}\binom{n}{k}^{-1}(-1)^k k^r=\frac{n+1}{n+2}\sum_{h=0}^{r}h!S(r,h)(-1)^h\binom{n+h+2}{h}^{-1}+$$

$$(-1)^n(n+1)\sum_{h=0}^{r}h!S(r,h)\sum_{j=0}^{h}\binom{h}{j}\binom{n+j+1}{j}\frac{(-1)^{h-j}}{n+j+2},$$

where $S(r,h)$ are Stirling numbers of the second kind.

Proof. Let

$$S_r(x)=\sum_{n=0}^{\infty}(\sum_{k=0}^{n}(-1)^k k^r\binom{n}{k}^{-1})x^n,$$

and let $a_k=(-1)^k k^r,\ \ b_k=1$ in Lemma 1, so the corresponding generating functions

$$A(x)=\sum_{h=0}^{r}h!S(r,h)\frac{(-x)^h}{(1+x)^{h+1}},\quad B(x)=\frac{1}{1-x}.$$

Then, by Lemma 1

$$\begin{aligned}
S_r(x) &= \frac{d}{dx}(x\sum_{h=0}^{r}h!S(r,h)\int_0^1\frac{(-tx)^h}{(1+tx)^{h+1}}\frac{1}{(1-x+xt)}dt)\\
&= \frac{d}{dx}(x\sum_{h=0}^{r}h!S(r,h)\int_0^1\sum_{j=0}^{h}\binom{h}{j}\frac{(-1)^{h-j}}{(1+tx)^{j+1}(1-x+xt)}dt)\\
&= \sum_{h=0}^{r}h!S(r,h)\sum_{j=0}^{h}\binom{h}{j}(-1)^{h-j}\frac{d}{dx}[-\frac{1}{x^{j+1}}(\ln(1+x)+\ln(1-x)\\
&-\sum_{i=1}^{j}\binom{j}{i}\frac{(-1)^i}{i(1+x)^i}+\sum_{i=1}^{j}\binom{j}{i}\frac{(-1)^i(1-x)^i}{i})]\\
&= \sum_{h=0}^{r}h!S(r,h)\sum_{j=0}^{h}\binom{h}{j}(-1)^{h-j}(\frac{j+1}{x^{j+2}}\ln(1-x^2)+\sum_{i=1}^{j+1}\frac{i-1}{(j+2-i)x^i}\\
&-\frac{1}{x^{j+2}}\sum_{i=1}^{j}\binom{j}{i}(-1)^i\frac{j+1+x(i+j+1)}{i(1+x)^{i+1}}+\frac{2}{x^j(1-x^2)}),
\end{aligned}$$

comparing the coefficients of x^n in the first and the last member of the equalities we complete the proof of theorem. □

Theorem 2. Let r be any nonnegative integer, then

$$\sum_{k=0}^{n}\binom{n}{k}^{-1}(-1)^k k^r=\frac{n+1}{n+2}\sum_{h=0}^{r}h!S(r,h)(-1)^h\binom{n+h+2}{h}^{-1}+$$
$$(-1)^n\sum_{h=0}^{r}\binom{r}{h}(n+1)^{r+1-h}\sum_{j=0}^{h}j!S(h,j)\frac{(-1)^j}{n+j+2},$$

where $S(r,h)$ are the Stirling numbers of the second kind.

Proof.

$$\sum_{k=0}^{n}\binom{n}{k}^{-1}(-1)^k k^r = (n+1)\sum_{k=0}^{n}(-1)^k k^r \int_0^1 t^k(1-t)^{n-k}dt$$

$$= (n+1)\int_0^1 (1-t)^n \sum_{k=0}^{n} k^r(\frac{-t}{1-t})^k dt$$

$$= (n+1)\sum_{h=0}^{r} h!S(r,h)(-1)^h \int_0^1 t^h(1-t)^{n+1}dt$$

$$+ (-1)^n \sum_{h=0}^{r}\binom{r}{h}(n+1)^{r+1-h}\sum_{j=0}^{h} j!S(h,j)(-1)^j \int_0^1 t^{n+j+1}dt$$

$$= \frac{n+1}{n+2}\sum_{h=0}^{r} h!S(r,h)(-1)^h \binom{n+h+2}{h}^{-1} +$$

$$(-1)^n \sum_{h=0}^{r}\binom{r}{h}(n+1)^{r+1-h}\sum_{j=0}^{h} j!S(h,j)\frac{(-1)^j}{n+j+2}.$$

The proof is complete. □

By Theorems 1 and 2, we get the following corollary.

Corollary 1. Let r is any nonnegative integer, then

$$\sum_{h=0}^{r}\binom{r}{h}(n+1)^{r-h}\sum_{j=0}^{h} j!S(h,j)\frac{(-1)^j}{n+j+2}$$
$$= \sum_{h=0}^{r} h!S(r,h)\sum_{j=0}^{h}\binom{h}{j}\binom{n+j+1}{j}\frac{(-1)^{h-j}}{n+j+2}.$$

From Theorem 1, we can also obtain the following results.

Corollary 2.

$$\sum_{k=0}^{n}\binom{n}{k}^{-1}(-1)^k = \frac{n+1}{n+2}(1+(-1)^n),$$

$$\sum_{k=0}^{n}\binom{n}{k}^{-1}(-1)^k k = \frac{n+1}{(n+2)(n+3)}(-1+(-1)^n((n+1)^2+n)),$$

$$\sum_{k=0}^{n}\binom{n}{k}^{-1}(-1)^k k^2 = \frac{n(n+1)}{(n+2)(n+3)(n+4)}(-1+(-1)^n(7(n+1)^2+n^3)).$$

Theorem 3. Let r be any nonnegative integer, then

$$\sum_{k=0}^{n}\binom{n+k}{k}^{-1}k^r = \frac{1}{n}\sum_{h=0}^{r+1}\binom{n-1}{h}^{-1}h!((n+1)S(r,h)+S(r+1,h))-$$
$$\frac{1}{2n+1}\sum_{k=0}^{r}\binom{r}{k}(n+1)^{r+1-h}(1+\frac{r+1}{r+1-k})\sum_{j=0}^{k}j!S(k,j)\binom{2n}{n+1+j}^{-1}$$
$$-\frac{1}{2n+1}\sum_{h=1}^{r+1}h!S(r+1,h)\binom{2n}{n+1+h}^{-1},$$

where $S(r,h)$ are the Stirling numbers of the second kind.

Proof. We have

$$\sum_{k=0}^{n}\binom{n+k}{k}^{-1}k^r = \sum_{k=0}^{n}k^r(n+k+1)\int_0^1 t^k(1-t)^n dt$$
$$= (n+1)\int_0^1(1-t)^n\sum_{k=0}^{n}k^r t^k dt + \int_0^1(1-t)^n\sum_{k=0}^{n}k^{r+1}t^k dt$$
$$= \frac{1}{n}\sum_{h=0}^{r+1}\binom{n-1}{h}^{-1}h!((n+1)S(r,h)+S(r+1,h))$$
$$-\frac{1}{2n+1}\sum_{k=0}^{r}\binom{r}{k}(n+1)^{r+1-h}(1+\frac{r+1}{r+1-k})\sum_{j=0}^{k}j!S(k,j)\binom{2n}{n+1+j}^{-1}$$
$$-\frac{1}{2n+1}\sum_{j=1}^{r+1}h!S(r+1,j)\binom{2n}{n+1+j}^{-1},$$

which completes the proof. □

From Theorem 3, the next corollary holds.

Corollary 3. We have

$$\sum_{k=0}^{n}\binom{n+k}{k}^{-1} = \frac{n}{n-1}(1-\binom{2n}{n+1}^{-1}), \quad n\geq 2,$$
$$\sum_{k=0}^{n}\binom{n+k}{k}^{-1}k = \frac{n}{(n-1)(n-2)}(1-n^2\binom{2n}{n+1}^{-1}), \quad n\geq 3,$$
$$\sum_{k=0}^{n}\binom{n+k}{k}^{-1}k^2 = \frac{n(n+1-n(n^3-n^2+2)\binom{2n}{n+1}^{-1})}{(n-1)(n-2)(n-3)}, \quad n\geq 4.$$

Analogous to Theorem 3, we can get the theorem below.

Theorem 4. Let r be any positive integer , then

$$\sum_{k=1}^{n}\binom{n}{k}^{-1}\frac{1}{k^r}=\sum_{k=1}^{n}\frac{1}{k^{r-1}}\sum_{j=0}^{k-1}\binom{k-1}{j}\frac{(-1)^{k-j}}{j-n},$$

$$\sum_{k=1}^{n}\binom{n}{k}^{-1}\frac{(-1)^k}{k^r}=\sum_{k=1}^{n}\frac{1}{k^{r-1}}\sum_{j=0}^{k-1}\binom{k-1}{j}\frac{(-1)^{j}}{j-n}.$$

Corollary 4.

$$\sum_{k=1}^{n}\binom{n}{k}^{-1}\frac{1}{k}=\frac{1}{2^n}\sum_{k=1}^{n}\frac{2^k}{k},$$

$$\sum_{k=1}^{n}\binom{n}{k}^{-1}\frac{1}{k^2}=\sum_{k=1}^{n}\frac{1}{k}\sum_{j=0}^{k-1}\binom{k-1}{j}\frac{(-1)^{k-j}}{j-n}.$$

Corollary 5.

$$\sum_{k=1}^{n}\binom{n}{k}^{-1}\frac{(-1)^k}{k}=-\frac{1}{n+1}(1+(-1)^{n-1}),$$

$$\sum_{k=1}^{n}\binom{n}{k}^{-1}\frac{(-1)^k}{k^2}=\sum_{k=1}^{n}\frac{1}{k}\sum_{j=0}^{k-1}\binom{k-1}{j}\frac{(-1)^{j}}{j-n}.$$

Theorem 5. Let r be any nonnegative integer , then

$$\sum_{k=0}^{n}k^r\sum_{j=0}^{k}\binom{n}{j}^{-1}(-1)^j=\frac{n+1}{n+2}(1+(-1)^n)\sum_{h=0}^{r}\binom{n+1}{h+1}h!S(r,h)$$
$$-\sum_{h=0}^{r}h!S(r,h)\sum_{j=0}^{n}\binom{n}{j}^{-1}\binom{j}{h+1}(-1)^j,$$

where $S(r,h)$ are the Stirling numbers of the second kind.

Proof. By Lemma 6 and Corollary 2,

$$\sum_{k=0}^{n} k^r \sum_{j=0}^{k} \binom{n}{j}^{-1} (-1)^j = \sum_{j=0}^{n} \binom{n}{j}^{-1} (-1)^j \sum_{k=j}^{n} k^r$$
$$= \sum_{j=0}^{n} \binom{n}{j}^{-1} (-1)^j (\sum_{h=0}^{r} \binom{n+1}{h+1} h! S(r,h) - \sum_{h=0}^{r} \binom{j}{h+1} h! S(r,h))$$
$$= \frac{n+1}{n+2}(1+(-1)^n) \sum_{h=0}^{r} \binom{n+1}{h+1} h! S(r,h)$$
$$- \sum_{h=0}^{r} h! S(r,h) \sum_{j=0}^{n} \binom{n}{j}^{-1} \binom{j}{h+1} (-1)^j.$$

which completes the proof. □

Theorem 6. Let r be any nonnegative integer, then

$$\sum_{k=0}^{n} k^r \sum_{j=0}^{k} \binom{n}{j}^{-1} (-1)^j = \frac{n+1}{(n+2)(r+1)}((1+(-1)^n) B_{r+1}(n+1)$$
$$- \sum_{k=0}^{r+1} \sum_{h=0}^{k} \binom{r+1}{k} \binom{n+h-1}{h}^{-1} (-1)^h h! S(k,h) B_{r+1-k})$$
$$+ \frac{(n+1)(-1)^{n+1}}{r+1} \sum_{k=0}^{r+1} \sum_{h=0}^{k} \binom{r+1}{k} h! S(k,h) B_{r+1-k}$$
$$\times \sum_{j=0}^{h} \binom{h}{j} \binom{n+j+1}{j} \frac{(-1)^{h-j}}{n+2+j},$$

where $S(r,h)$ are Stirling numbers of the second kind, and $B_k(y)$ and B_k are Bernoulli polynomials and Bernoulli numbers, respectively.

By Theorem 1 and Lemma 5, we immediately complete the proof of theorem.

Corollary 6.

$$\sum_{k=0}^{n}\sum_{j=0}^{k}\binom{n}{j}^{-1}(-1)^j = \frac{n+1}{n+3}(n+2+(-1)^n),$$
$$\sum_{k=0}^{n}k\sum_{j=0}^{k}\binom{n}{j}^{-1}(-1)^j = \frac{(n+1)(n+3)}{2(n+2)(n+4)}(1+(-1)^n).$$

Theorem 7. Let r be any nonnegative integer , then

$$\sum_{k=0}^{\infty}\binom{n+k}{k}^{-1}k^r = \frac{n}{n-1}\sum_{h=0}^{r}\binom{n-2}{h}^{-1}h!S(r,h),$$

where $S(r,h)$ are the Stirling numbers of the second kind.

Proof. We have

$$\sum_{k=0}^{\infty}\binom{n+k}{k}^{-1}k^r = \int_0^1(1-t)^n\sum_{k=0}^{\infty}k^r(n+k+1)t^k dt$$
$$= (n+1)\sum_{h=0}^{r}h!S(r,h)\int_0^1(1-t)^{n-h-1}t^h dt$$
$$+\sum_{h=0}^{r}h!S(r,h)\int_0^1(1-t)^{n-h-2}t^h(t+h)dt$$
$$=\sum_{h=0}^{r}h!S(r,h)\Big(\frac{n+1}{n}\binom{n-1}{h}^{-1}+\frac{h}{n-1}\binom{n-2}{h}^{-1}+\quad\frac{1}{n}\binom{n-1}{h+1}^{-1}\Big)$$
$$=\frac{n}{n-1}\sum_{h=0}^{r}\binom{n-2}{h}^{-1}h!S(r,h).$$

The proof is complete. □

Corollary 7.

$$\sum_{k=0}^{\infty}\binom{n+k}{k}^{-1}=\frac{n}{n-1},\qquad n\geq 2,$$
$$\sum_{k=0}^{\infty}\binom{n+k}{k}^{-1}k=\frac{n}{(n-1)(n-2)},\qquad n\geq 3,$$
$$\sum_{k=0}^{\infty}\binom{n+k}{k}^{-1}k^2=\frac{n(n+1)}{(n-1)(n-2)(n-3)},\qquad n\geq 4.$$

Analogous to the proof of Theorem 7, we obtain the following identity, by which a infinite series can be reduced to a finite sum.

Theorem 8. Let r be any nonnegative integer, then

$$\begin{aligned}
&\sum_{k=0}^{\infty}\binom{n+k}{k}^{-1}k^r(-1)^k\\
&=2^{n-1}n\ln 2\sum_{h=0}^{r}h!S(r,h)(-\frac{1}{2})^h\sum_{j=0}^{h}\binom{h}{j}\binom{n+j}{j}(2-\frac{h+1}{j+1})\\
&+n2^n\sum_{h=0}^{r}h!S(r,h)\sum_{j=0}^{h}\binom{h}{j}(-1)^{h-j}\sum_{k=0,k\neq j}^{n}\binom{n}{k}\frac{(-1)^k}{k-j}(\frac{1}{2^j}-\frac{1}{2^k})\\
&+2^{n-1}\sum_{h=0}^{r}h!S(r,h)\sum_{j=0}^{h}\binom{h}{j}(-1)^{h-j}\sum_{k=0,k\neq j+1}^{n}\binom{n}{k}\frac{(-1)^k}{k-j-1}(\frac{1}{2^j}-\frac{1}{2^k}),
\end{aligned}$$

where $S(r,h)$ are the Stirling numbers of the second kind.

Corollary 8.

$$\sum_{k=0}^{\infty}\binom{n+k}{k}^{-1}(-1)^k=2^{n-1}n(\ln 2-\sum_{k=1}^{n-1}\frac{1}{k2^k}),$$
$$\sum_{k=0}^{\infty}\binom{n+k}{k}^{-1}(-1)^k k=2^{n-2}n(n+1)(\ln 2+\sum_{k=1}^{n}\frac{1}{k2^k})+\frac{n-1}{4},$$

$$\sum_{k=0}^{\infty}\binom{n+k}{k}^{-1}(-1)^k k^2 = 2^{n-3}n^2(n+3)(\ln 2 - \sum_{k=1}^{n}\frac{1}{k2^k}) + n2^{n-4}(n+16) - \frac{n(n+1)}{8}.$$

Where $H_n := \sum_{k=1}^{n}\frac{1}{k}$.

Remark. The first identity of Corollary 8 is a result in the paper [3]. Here it has been further simplified.

In addition, the next two theorems can be verified.

Theorem 9.

$$\sum_{k=1}^{\infty}\binom{n+k}{k}^{-1}\frac{(-1)^k}{k} = 2^n(\sum_{k=1}^{n}\frac{1}{k2^k} - \ln 2),$$

$$\sum_{k=1}^{\infty}\binom{n+k}{k}^{-1}\frac{(-1)^k}{k^2} = \frac{2^{n+1}}{n+1}(\sum_{k=1}^{n}\frac{1}{k2^k} - \ln 2) + \sum_{k=1}^{n+1}\binom{n+1}{k}\frac{1}{k}\sum_{j=1}^{k}\frac{(-1)^j}{j} - \frac{\pi^2}{12}.$$

Theorem 10.

$$\sum_{k=1}^{\infty}\binom{n+k}{k}^{-1}\frac{1}{k} = \frac{1}{n},$$

$$\sum_{k=1}^{\infty}\binom{n+k}{k}^{-1}\frac{1}{k^2} = \sum_{k=1}^{n+1}\binom{n+1}{k}\frac{(-1)^k}{k}H_k + \frac{1}{(n+1)^2} + \frac{\pi^2}{6}.$$

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References

[1] B. Sury, Sum of the reciprocals of the binomial coefficients. European J. Combin. 14(1993), 351-353.

[2] T. Mansour, Combinatorial identities and inverse binomial coefficients. Adv. in Appl. Math. 28 (2002), 196-202.

[3] B. Sury, T. Wang and F. Z. Zhao, Identities involing reciprocals of binomial coefficients, Journal of Integer Sequences. 7 (2004), Article 04.2.8.

[4] L. Comtet, Advanced Combinatorics, D. Reidel Publishing Co., Dordrecht, 1974.

[5] L. C. Hsu, E.L. Tan, A Refiniement of DeBruyn's Formulas for $\sum a^k k^p$, The Fibonacci Quarterly. 38 (2) (2000), 56-60.

[6] L. C. Hsu, A summation rule using Stirling numbers of the second kind, The Fibonacci Quarterly. 31 (1993), 256-262.