On the strongly c-harmoniousness cycle with P_2- or P_3-chord *

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Abstract. The graph $C_n(d; i, j; P_k)$ denote a cycle C_n with path P_k joining two nonconsecutive vertices x_i and x_j of the cycle, where d is the distance between x_i and x_j on C_n. In this paper, we obtain that the graph $C_n(d; i, j; P_k)$ is strongly c-harmonious when k=2, 3 and integer $n \geq 6$.

Key words: harmonious graph; strongly c-harmonious graph; labeling; graph $C_n(d; i, j; P_k)$
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1 Introduction

Only graphs without loops, isolated and multiple edges will be considered in this paper. The symbol Z_n denotes a group of integers modulo n. A graph G of vertex set $V(G)$ and edge set $E(G)$ is said to be a (p, q) graph if it has p vertices and q edges. If there exists an injection f: $V(G) \to Z_q$, such that the induced mapping $f^*(uv) \equiv f(u) + f(v) \pmod q$ is a bijection from $E(G)$ onto Z_q, then f is said to be a harmonious labelling of G. A graph which admits such a labelling is called a harmonious graph. This concept was introduced by Graham and Sloane [1].

Chang, Hsu, and Rogers (see [3]) define an injective labeling f of a graph G with q edges to be strongly c-harmonious labeling if the vertex labels are

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from $\{0,1,\ldots,q-1\}$ and the edge labels induced by $f^*(xy)=f(x)+f(y)$ for each edge xy are $c,c+1,\ldots,c+q-1$. Grace (see [4], [5]) called such a labeling sequential labeling. By taking the edge labels of a sequentially labeled graph with q edges modulo q, we obviously obtain a harmoniously labeled graph. S. Xu in [6] proved that the all cycles with a chord are harmonious except that C_6 and the distance in C_6 between the endpoints of the chord is 2. Graham and Sloane [1] determined the harmonious graphs of order ≤ 5. Seoud and Youssef in [2] determined all harmonious graphs of order 6. Gallian in [7] and [8] surveyed the results on harmonious labeling and strongly c-harmonious labeling of graphs and open the problem whether the cycles with P_k-chord are harmonious or not.

A path of length k and a cycle of length n are denoted by P_k and C_n respectively. Vertices x_1, x_2, ..., x_n on the C_n are connected in a cyclic manner. The internal vertices on the P_k are y_1, y_2, ..., y_{k-1} successively. The graph $C_n(d;i,j;P_k)$ with vertex set $\{x_1,...,x_n,y_1,...,y_{k-1}\}$ and edge set $\{x_1x_2,x_2x_3,...,x_{n-1}x_n,x_nx_1,x_iy_1,y_1y_2,...,y_{k-2}y_{k-1},y_{k-1}x_j\}$ denote a cycle C_n with path P_k joining two nonconsecutive vertices x_i and x_j of the cycle and the distance between x_i and x_j on C_n is d. In this paper, we will discuss strongly c-harmonious problem of the graph $C_n(d;i,j;P_k)$ for k=2 or 3.

Example 1. Figure 1 denote a $C_8(3;1,6;P_3)$. Figure 2 is a labeling of $C_{10}(2;1,3;P_2)$.

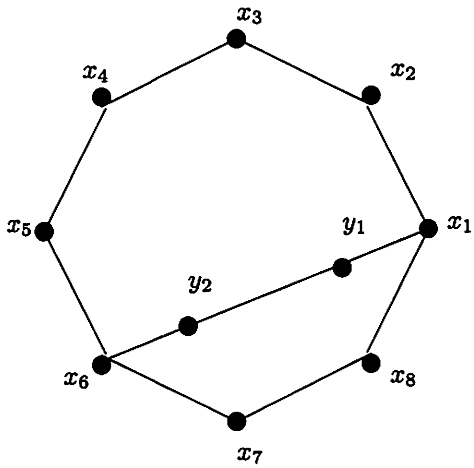

Figure 1

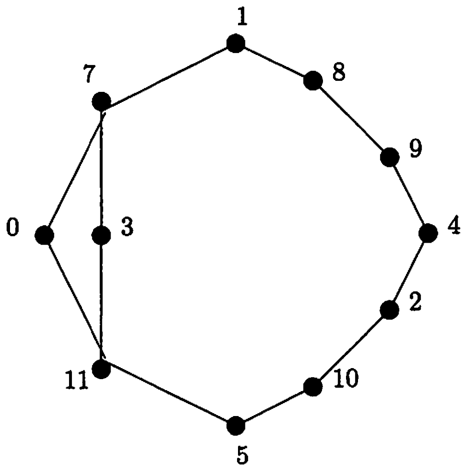

Figure 2

Let Z be the set of all integers and $a \in Z$. The symbols
$[a,b]$ is defined by the set $\{x|\ x \in Z, a \leq x \leq b\}$,
$[a,b]_k$ is defined by the set $\{x|\ x \in Z, a \leq x \leq b, x \equiv a(mod\ k)\}$,
and $\lfloor x \rfloor$ denotes the greatest integer y such that $y \leq x$.
The symbol $f(S)$ denotes the set $\{f(x)|\ x \in S\}$.

Example 2. The symbols $[3,8]=\{3,4,5,6,7,8\}$, $[2,8]_2=\{2,4,6,8\}$, $[3,7]_2=\{3,5,7\}$, $[3,3]=[3,3]_2=\{3\}$ and $[3,8]_2=\{3,5,7\}$. $[a,b]=[a,b]_k=\phi$ if $a > b$.

2 Cycle with P_2-chord

The graph $G=C_n(d;i,j;P_2)$ has $n+1$ vertices and $n+2$ edges. Let $E(G)$ be the edge-set of G and $V(G)$ be the vertex-set of G. In the following, the set behind function f is the label set of corresponding vertices. For instance, "$f(x_{2i-1})=2i-1$ if $i \in [1,m]$, $[1,2m-1]_2$" implies "$f(x_1)=1$, $f(x_3)=3$,..., $f(x_{2m-1})=2m-1$, and the set $[1,2m-1]_2=\{f(x_{2i-1}) \mid i \in [1,m]\}$".

Theorem 2.1. When $n \equiv 0 \pmod 4$ and $n \geq 8$, the graph $C_n(4s-2;2,4s;P_2)$ for $1 \leq s \leq n/8$, the graph $C_n(4s;1,n+1-4s;P_2)$ for $1 \leq s \leq$

$n/8$, and the graph $C_n(4s-1; n/2-2s, (n-2)/2+2s; P_2)$ for $1 \le s \le (n-4)/4$ are strongly $n/2$-harmonious.

Proof. For the graph $G=C_n(4s-2; 2, 4s; P_2)$, we define the function f: $V(G) \to [0, n+1]$ as follows: $f(y_1)=(n-2)/2$,
$f(x_{2i-1})=i-1$ if $i \in [1, (n-4)/4]$, $[0, (n-8)/4]$,
$f(x_{2i-1})=i-2$ if $i \in [(n+4)/4, n/2]$, $[(n-4)/4, (n-4)/2]$,
$f(x_{2i})=n/2+i$ if $i \in [1, n/4]$, $[(n+2)/2, 3n/4]$,
$f(x_{2i})=(n+2)/2+i$ if $i \in [(n+4)/4, (n-4)/4+s]$, $[(3n+8)/4, 3n/4+s]$,
$f(x_{2i})=(n+4)/2+i$ if $i \in [n/4+s, (n-2)/2]$, $[(3n+8)/4+s, n+1]$,
$f(x_n)=n/2$, $f(x_{(n-2)/2})=(3n+4)/4$.

We obtain $f(V(G))=[0, (n-8)/4] \cup [(n-4)/4, (n-4)/2] \cup [(n+2)/2, 3n/4] \cup [(3n+8)/4, 3n/4+s] \cup [(3n+8)/4+s, n+1] \cup \{(n-2)/2, n/2, (3n+4)/4\} = [0, n+1] \setminus \{(3n+4)/4+s\} \subset [0, n+1]$.
Since $|f(V(G))|=n+1$, the f is an injection from $V(G)$ to $[0, n+1]$. By the above definition, we have
$A=\{f^*(x_{2i}x_{2i-1}) | \ i \in [1, (n-4)/4]\}=\{(n-2)/2+2i | \ i \in [1, (n-4)/4]\}$
$=[(n+2)/2, n-3]_2$,
$B=\{f^*(x_{2i}x_{2i+1}) | \ i \in [1, (n-8)/4]\}=\{n/2+2i | \ i \in [1, (n-8)/4]\}$
$=[(n+4)/2, n-4]_2$,
$C=\{f^*(x_{2i}x_{2i-1}), f^*(x_{2i}x_{2i+1}) | \ i \in [(n+4)/4, (n-4)/4+s]\}$
$=\{(n-2)/2+2i, n/2+2i | \ i \in [(n+4)/4, (n-4)/4+s]\}=[n+1, n+2s-2]$,
$D=\{f^*(x_{2i}x_{2i-1}), f^*(x_{2i}x_{2i+1}) | \ i \in [n/4+s, (n-2)/2]\}$
$=\{n/2+2i, (n+2)/2+2i | \ i \in [n/4+s, (n-2)/2]\}=[n+2s, (3n-2)/2]$,
$E=\{f^*(x_2y_1), f^*(x_{4s}y_1), f^*(x_nx_1), f^*(x_{(n-4)/2}x_{(n-2)/2}), f^*(x_{(n-2)/2}x_{n/2}),$
$f^*(x_{n/2}x_{(n+2)/2}), f^*(x_{n-1}x_n)\}$
$=\{n, n+2s-1, n/2, 3n/2, (3n+2)/2, n-1, n-2\}$.
Therefore, $f^*(E(G))=A \cup B \cup C \cup D \cup E=[n/2, (3n+2)/2]$. This implies that the f is a strongly $n/2$-harmonious labeling of G.

For the graph $G=C_n(4s; 1, n+1-4s; P_2)$, we define the function f:

$V(G) \to [0, n+1]$ as follows: $f(y_1)=3n/4$,

$f(x_{2i-1})=3n/4+i$ if $i \in [1, n/4-s]$, $[(3n+4)/4, n-s]$,

$f(x_{2i-1})=(3n+4)/4+i$ if $i \in [(n+4)/4-s, n/4]$, $[n+2-s, n+1]$,

$f(x_{2i-1})=(n-4)/4+i$ if $i \in [(n+4)/4, n/2]$, $[n/2, (3n-4)/4]$,

$f(x_{2i})=(n-4)/4+i$ if $i \in [1, n/4]$, $[n/4, (n-2)/2]$,

$f(x_{2i})=i-(n+4)/4$ if $i \in [(n+4)/4, n/2]$, $[0, (n-4)/4]$.

It is easy to verify that the f is an injection from $V(G)$ to $[0, n+1]$. By the above definition, we have

$A=\{f^*(x_{2i}x_{2i-1})|\ i \in [1, n/4-s]\}=\{n-1+2i|\ i \in [1, n/4-s]\}$
$=[n+1, (3n-2)/2-2s]_2$,

$B=\{f^*(x_{2i}x_{2i+1})|\ i \in [1, (n-4)/4-s]\}=\{n+2i|\ i \in [1, (n-4)/4-s]\}$
$=[n+2, (3n-4)/2-2s]_2$,

$C=\{f^*(x_{2i}x_{2i-1})|\ i \in [(n+4)/4-s, n/4]\}=\{n+2i|\ i \in [(n+4)/4-s, n/4]\}$
$=[(3n+4)/2-2s, 3n/2]_2$,

$D=\{f^*(x_{2i}x_{2i+1})|\ i \in [n/4-s, (n-4)/4]\}$
$=\{n+1+2i|\ i \in [n/4-s, (n-4)/4]\}=[(3n+2)/2-2s, (3n-2)/2]_2$,

$E=\{f^*(x_{2i}x_{2i-1})|\ i \in [(n+4)/4, n/2]\}=\{2i-2|\ i \in [(n+4)/4, n/2]\}$
$=[n/2, n-2]_2$,

$F=\{f^*(x_{2i}x_{2i+1})|\ i \in [(n+4)/4, (n-2)/2]\}$
$=\{2i-1|\ i \in [(n+4)/4, (n-2)/2]\}$ $=[(n+2)/2, n-3]_2$,

$G=\{f^*(x_1y_1), f^*(x_{n+1-4s}y_1), f^*(x_nx_1), f^*(x_{n/2}x_{(n+2)/2})\}$
$=\{(3n+2)/2, 3n/2-2s, n, n-1\}$.

Therefore, $f^*(E(G))=A \cup B \cup C \cup D \cup E \cup F \cup G$ $=[n/2, (3n+2)/2]$.

For the graph $G=C_n(4s-1; n/2-2s, (n-2)/2+2s; P_2)$, we define function f: $V(G) \to [0, n+1]$ as follows: $f(y_1)=(3n+4)/4+s$,

$f(x_{2i-1})=i-1$ if $i \in [1, n/2]$, $[0, (n-2)/2]$,

$f(x_{2i})=n/2+i$ if $i \in [1, (n-4)/4]$, $[(n+2)/2, (3n-4)/4]$,

$f(x_{2i})=(n+2)/2+i$ if $i \in [n/4, (n-4)/4+s]$, $[(3n+4)/4, 3n/4+s]$,

$f(x_{2i})=(n+4)/2+i$ if $i \in [n/4+s, (n-2)/2]$, $[(3n+8)/4+s, n+1]$,

$f(x_n)=n/2$.

It is easy to verify that the f is an injection from $V(G)$ to $[0, n+1]$. By the above definition, we have
$A=\{f^*(x_{2i}x_{2i-1}), f^*(x_{2i}x_{2i+1})| \ i \in [1, (n-4)/4]\}$
$=\{(n-2)/2+2i, n/2+2i| \ i \in [1, (n-4)/4]\}=[(n+2)/2, n-2]$,
$B=\{f^*(x_{2i}x_{2i-1}), f^*(x_{2i}x_{2i+1})| \ i \in [n/4, (n-4)/4+s]\}$
$=\{n/2+2i, (n+2)/2+2i| \ i \in [n/4, (n-4)/4+s]\}=[n, n+2s-1]$,
$C=\{f^*(x_{2i}x_{2i-1}), f^*(x_{2i}x_{2i+1})| \ i \in [n/4+s, (n-2)/2]\}$
$=\{(n+2)/2+2i, (n+4)/2+2i| \ i \in [n/4+s, (n-2)/2]\}=[n+1+2s, 3n/2]$,
$D=\{f^*(x_{n/2-2s}y_1), f^*(x_{(n-2)/2+2s}y_1), f^*(x_n x_1), f^*(x_n x_{n-1})\}$
$=\{(3n+2)/2, n+2s, n/2, n-1\}$.
Therefore, $f^*(E(G))=A \cup B \cup C \cup D=[n/2, (3n+2)/2]$. □

Theorem 2.2. When $n \equiv 4 \pmod 8$ and $n \geq 12$, the graph $C_n(n/2; 3, (n+6)/2; P_2)$ is strongly $n/2$-harmonious.

Proof. We define the function f: $V(G) \rightarrow [0, n+1]$ as follows:
$f(y_1)=(n+2)/2$,
$f(x_{2i-1})=(n+2)/2+i$ if $i \in [2, (n+4)/8]$, $[(n+6)/2, (5n+12)/8]$,
$f(x_{2i-1})=(n+4)/2+i$ if $i \in [(n+12)/8, (n+4)/4]$, $[(5n+28)/8, (3n+12)/4]$,
$f(x_{2i-1})=(n+2)/2+i$ if $i \in [(n+12)/4, n/2]$, $[(3n+16)/4, n+1]$,
$f(x_{2i})=i-1$ if $i \in [2, (n+4)/4]$, $[1, n/4]$,
$f(x_{2i})=i$ if $i \in [(n+8)/4, n/2]$, $[(n+8)/4, n/2]$,
$f(x_1)=0$, $f(x_2)=(n+4)/2$, $f(x_{(n+6)/2})=(n+4)/4$.

It is easy to verify that the f is an injection from $V(G)$ to $[0, n+1]$. By the above definition, we have
$A=\{f^*(x_{2i}x_{2i-1})| \ i \in [2, (n+4)/8]\}=\{n/2+2i| \ i \in [2, (n+4)/8]\}$
$=[(n+8)/2, (3n+4)/4]_2$,
$B=\{f^*(x_{2i}x_{2i+1})| \ i \in [2, (n-4)/8]\}=\{(n+2)/2+2i| \ i \in [2, (n-4)/8]\}$
$=[(n+10)/2, 3n/4]_2$,
$C=\{f^*(x_{2i}x_{2i-1})| \ i \in [(n+12)/8, (n+4)/4]\}$

$=\{(n+2)/2+2i|\ i\in[(n+12)/8,(n+4)/4]\}=[(3n+16)/4,n+3]_2$,

$D=\{f^*(x_{2i}x_{2i+1})|\ i\in[(n+4)/8,n/4]\}=\{(n+4)/2+2i|\ i\in[(n+4)/8,n/4]\}$
$=[(3n+12)/4,n+2]_2$,

$E=\{f^*(x_{2i}x_{2i-1})|\ i\in[(n+12)/4,n/2]\}=\{(n+2)/2+2i|\ i\in[(n+12)/4,n/2]\}$
$=[n+7,(3n+2)/2]_2$,

$F=\{f^*(x_{2i}x_{2i+1})|\ i\in[(n+8)/4,(n-2)/2]\}$
$=\{(n+4)/2+2i|\ i\in[(n+8)/4,(n-2)/2]\}=[n+6,3n/2]_2$,

$G=\{f^*(x_3y_1),f^*(x_{(n+6)/2}y_1),f^*(x_nx_1),f^*(x_1x_2),f^*(x_2x_3),$
$f^*(x_{(n+4)/2}x_{(n+6)/2}),f^*(x_{(n+6)/2}x_{(n+8)/2})\}$
$=\{n+4,(3n+8)/4,n/2,(n+4)/2,n+5,(n+2)/2,(n+6)/2\}$.

Therefore, $f^*(E(G))=A\cup B\cup C\cup D\cup E\cup F\cup G=[n/2,(3n+2)/2]$. □

Theorem 2.3. When $n\equiv 1 \pmod 4$ and $n\geq 13$, the graph $C_n(4s+4;(n-5)/2-2s,(n+3)/2+2s;P_2)$ for $1\leq s\leq(n-9)/4$ is strongly $(n+1)/2$-harmonious.

Proof. We define the function f: $V(G)\to[0,n+1]$ as follows:

$f(y_1)=(3n+5)/4+s$,

$f(x_{2i-1})=i-2$ if $i\in[2,(n+7)/4]$, $[0,(n-1)/4]$,

$f(x_{2i-1})=(n-1)/2+i$ if $i\in[(n+11)/4,(n+3)/4+s]$, $[(3n+9)/4,(3n+1)/4+s]$,

$f(x_{2i-1})=(n+1)/2+i$ if $i\in[(n+7)/4+s,(n+1)/2]$, $[(3n+9)/4+s,n+1]$,

$f(x_{2i})=(n+1)/2+i$ if $i\in[1,(n+3)/4]$, $[(n+3)/2,(3n+5)/4]$,

$f(x_{2i})=i-1$ if $i\in[(n+7)/4,(n-1)/2]$, $[(n+3)/4,(n-3)/2]$,

$f(x_1)=(n+1)/2$.

It is easy to verify that the f is an injection from $V(G)$ to $[0,n+1]$. By the above definition, we have

$A=\{f^*(x_{2i}x_{2i-1})|\ i\in[2,(n+3)/4]\}=\{(n-3)/2+2i|\ i\in[2,(n+3)/4]\}$
$=[(n+5)/2,n]_2$,

$B=\{f^*(x_{2i}x_{2i+1})|\ i\in[1,(n+3)/4]\}=\{(n-1)/2+2i|\ i\in[1,(n+3)/4]\}$

$=[(n+3)/2, n+1]_2$,

$C=\{f^*(x_{2i}x_{2i-1})|\ i \in [(n+11)/4, (n+3)/4+s]\}$

$=\{(n-3)/2+2i|\ i \in [(n+11)/4, (n+3)/4+s]\} =[n+4, n+2s]_2$,

$D=\{f^*(x_{2i}x_{2i+1})|\ i \in [(n+7)/4, (n-1)/4+s]\}$

$=\{(n-1)/2+2i|\ i \in [(n+7)/4, (n-1)/4+s]\} =[n+3, n+2s-1]_2$,

$E=\{f^*(x_{2i}x_{2i-1})|\ i \in [(n+7)/4+s, (n-1)/2]\}$

$=\{(n-1)/2+2i|\ i \in [(n+7)/4+s, (n-1)/2]\} =[n+2s+3, (3n-3)/2]_2$,

$F=\{f^*(x_{2i}x_{2i+1})|\ i \in [(n+3)/4+s, (n-1)/2]\}$

$=\{(n+1)/2+2i|\ i \in [(n+3)/4+s, (n-1)/2]\} =[n+2s+2, (3n-1)/2]_2$,

$G=\{f^*(x_{(n-5)/2-2s}y_1), f^*(x_{(n+3)/2+2s}y_1), f^*(x_nx_1), f^*(x_1x_2)$,

$f^*(x_{(n+5)/2}x_{(n+7)/2})\} =\{(3n+1)/2, n+1+2s, (3n+3)/2, n+2, (n+1)/2\}$.

Therefore, $f^*(E(G))=A\cup B\cup C\cup D\cup E\cup F\cup G=[(n+1)/2, (3n+3)/2]$.□

Theorem 2.4. (1) When $n \equiv 1 \pmod 4$ and $n \geq 5$, the graph $C_n(2; 2, n; P_2)$ is strongly $(n-1)/2$-harmonious.

(2) When $n \equiv 1 \pmod 4$ and $n \geq 9$, the graph $C_n(4s-1; (n+5)/2-2s, (n+3)/2+2s; P_2)$ for $1 \leq s \leq (n-5)/4$ is strongly $(n+3)/2$-harmonious.

Proof. For the graph $G=C_n(2; 2, n; P_2)$, we define the function f: $V(G) \to [0, n+1]$ as follows: $f(y_1)=n$,

$f(x_{2i-1})=i-1$ if $i \in [1, (n+1)/2]$, $[0, (n-1)/2]$,

$f(x_{2i})=(n-1)/2+i$ if $i \in [1, (n-1)/2]$, $[(n+1)/2, n-1]$.

It is easy to verify that the f is an injection from $V(G)$ to $[0, n+1]$. By the above definition, we have

$A=\{f^*(x_{2i}x_{2i-1})|\ i \in [1, (n-1)/2]\}=\{(n-3)/2+2i|\ i \in [1, (n-1)/2]\}$

$=[(n+1)/2, (3n-5)/2]_2$,

$B=\{f^*(x_{2i}x_{2i+1})|\ i \in [1, (n-1)/2]\}=\{(n-1)/2+2i|\ i \in [1, (n-1)/2]\}$

$=[(n+3)/2, (3n-3)/2]_2$,

$C=\{f^*(x_2y_1), f^*(x_ny_1), f^*(x_nx_1)\}=\{(3n+1)/2, (3n-1)/2, (n-1)/2\}$.

Therefore, $f^*(E(G))=A\cup B\cup C=[(n-1)/2, (3n+1)/2]$.

For the graph $G{=}C_n(4s-1;(n+5)/2-2s,(n+3)/2+2s;P_2)$, we define the function f: $V(G)\to[0,n+1]$ as follows: $f(y_1){=}(n+7)/4+s$, $f(x_1){=}0$, $f(x_{2i-1}){=}i$ if $i\in[2,(n+3)/4+s]$, $[2,(n+3)/4+s]$, $f(x_{2i-1}){=}i+1$ if $i\in[(n+7)/4+s,(n+1)/2]$, $[(n+11)/4+s,(n+3)/2]$, $f(x_{2i}){=}(n+3)/2+i$ if $i\in[1,(n-1)/2]$, $[(n+5)/2,n+1]$.

It is easy to verify that the f is an injection from $V(G)$ to $[0,n+1]$. By the above definition, we have

$A{=}\{f^*(x_{2i}x_{2i-1})|\ i\in[2,(n+3)/4+s]\}{=}\{(n+3)/2+2i|\ i\in[2,(n+3)/4+s]\}$
$\quad{=}[(n+11)/2,n+3+2s]_2$,

$B{=}\{f^*(x_{2i}x_{2i+1})|\ i\in[1,(n-1)/4+s]\}{=}\{(n+5)/2+2i|\ i\in[1,(n-1)/4+s]\}$
$\quad{=}[(n+9)/2,n+2+2s]_2$,

$C{=}\{f^*(x_{2i}x_{2i-1})|\ i\in[(n+7)/4+s,(n-1)/2]\}$
$\quad{=}\{(n+5)/2+2i|\ i\in[(n+7)/4+s,(n-1)/2]\}{=}[n+6+2s,(3n+3)/2]_2$,

$D{=}\{f^*(x_{2i}x_{2i+1})|\ i\in[(n+3)/4+s,(n-1)/2]\}$
$\quad{=}\{(n+7)/2+2i|\ i\in[(n+3)/4+s,(n-1)/2]\}{=}[n+5+2s,(3n+5)/2]_2$,

$E{=}\{f^*(x_{(n+5)/2-2s}y_1),f^*(x_{(n+3)/2+2s}y_1),f^*(x_2x_1),f^*(x_nx_1)\}$
$\quad{=}\{(n+7)/2,n+4+2s,(n+5)/2,(n+3)/2\}$.

Therefore, $f^*(E(G)){=}A\cup B\cup C\cup D\cup E{=}[(n+3)/2,(3n+5)/2]$. □

Theorem 2.5. When $n\equiv 1\pmod 4$, the graph $C_9(4;1,5;P_2)$ and the graph $C_n(4;(n-3)/2,(n+5)/2;P_2)$ for $n\geq 13$ are strongly $(n+1)/2$-harmonious.

Proof. For the graph $G{=}C_9(4;1,5;P_2)$, we define the function f: $V(G)\to[0,n+1]$ as follows: $f(y_1){=}1$, $f(x_1){=}10$, $f(x_2){=}4$, $f(x_3){=}3$, $f(x_4){=}2$, $f(x_5){=}8$, $f(x_6){=}0$, $f(x_7){=}6$, $f(x_8){=}7$, $f(x_9){=}5$.

It is easy to verify that the f is an injection from $V(G)$ to $[0,n+1]$. By the above definition, we have

$A{=}\{f^*(x_ix_{i+1})|\ i\in[1,8]\}{=}\{14,7,5,10,8,6,13,12\}$,

$B{=}\{f^*(x_1y_1),f^*(x_5y_1),f^*(x_9x_1)\}{=}\{11,9,15\}$.

Therefore, $f^*(E(G))=A\cup B=\{14,7,5,10,8,6,13,12\}\cup\{11,9,15\}=[5,15]$.

For the graph $G=C_n(4;(n-3)/2,(n+5)/2;P_2)$, we define the function f: $V(G)\to[0,n+1]$ as follows: $f(y_1)=(n-1)/4$, $f(x_{2i-1})=n-i+3$ if $i\in[3,(n+3)/4]$, $[(3n+9)/4,n]$,
$f(x_{2i-1})=n-i+1$ if $i\in[(n+7)/4,(n+1)/2]$, $[(n+1)/2,(3n-3)/4]$,
$f(x_{2i})=(n-1)/2-i$ if $i\in[2,(n-5)/4]$, $[(n+3)/4,(n-5)/2]$,
$f(x_{2i})=(n-3)/2-i$ if $i\in[(n-1)/4,(n+3)/4]$, $[(n-9)/4,(n-5)/4]$,
$f(x_{2i})=(n-1)/2-i$ if $i\in[(n+11)/4,(n-1)/2]$, $[0,(n-13)/4]$,
$f(x_1)=n+1$, $f(x_2)=(n-1)/2$, $f(x_3)=(n-3)/2$, $f(x_{(n+7)/2})=(3n+1)/4$.

It is easy to verify that the f is an injection from $V(G)$ to $[0,n+1]$. By the above definition, we have

$A=\{f^*(x_{2i}x_{2i-1})|\ i\in[3,(n-5)/4]\}=\{(3n+5)/2-2i|\ i\in[3,(n-5)/4]\}$
$\quad=[n+5,(3n-7)/2]_2$,
$B=\{f^*(x_{2i}x_{2i+1})|\ i\in[2,(n-5)/4]\}=\{(3n+3)/2-2i|\ i\in[2,(n-5)/4]\}$
$\quad=[n+4,(3n-5)/2]_2$,
$C=\{f^*(x_{2i}x_{2i-1})|\ i\in[(n-1)/4,(n+3)/4]\}$
$\quad=\{(3n+3)/2-2i|\ i\in[(n-1)/4,(n+3)/4]\}=[n,n+2]_2$,
$D=\{f^*(x_{2i}x_{2i+1})|\ i\in[(n-1)/4,(n-1)/4]\}=\{n+1\}$,
$E=\{f^*(x_{2i}x_{2i-1})|\ i\in[(n+11)/4,(n-1)/2]\}$
$\quad=\{(3n+1)/2-2i|\ i\in[(n+11)/4,(n-1)/2]\}=[(n+3)/2,n-5]_2$,
$F=\{f^*(x_{2i}x_{2i+1})|\ i\in[(n+11)/4,(n-1)/2]\}$
$\quad=\{(3n-1)/2-2i|\ i\in[(n+11)/4,(n-1)/2]\}=[(n+1)/2,n-6]_2$,
$G=\{f^*(x_{(n-3)/2}y_1),f^*(x_{(n+5)/2}y_1),f^*(x_nx_1),f^*(x_1x_2),f^*(x_2x_3),f^*(x_3x_4),$
$\quad f^*(x_{(n+3)/2}x_{(n+5)/2}),f^*(x_{(n+5)/2}x_{(n+7)/2}),f^*(x_{(n+7)/2}x_{(n+9)/2})\}$
$=\{n+3,n-1,(3n+3)/2,(3n+1)/2,n-2,n-4,n-3,(3n-1)/2,(3n-3)/2\}$.

Therefore, $f^*(E(G))=A\cup B\cup C\cup D\cup E\cup F\cup G=[(n+1)/2,(3n+3)/2]$. This implies the graph $C_n(4;(n-3)/2,(n+5)/2;P_2)$ for $n\geq 13$ is strongly $(n+1)/2$-harmonious. □

Theorem 2.6. When $n\equiv 2\pmod 4$ and $n\geq 6$, the graph $C_n(4s-$

$1; (n-2)/2, (n-4)/2+4s; P_2)$ for $1 \le s \le (n+2)/8$ is strongly $n/2$-harmonious.

Proof. We define the function f: $V(G) \to [0, n+1]$ as follows:
$f(y_1)=(n-2)/4$,
$f(x_{2i-1})=n-i+2$ if $i \in [1, (n-2)/4+s]$, $[(3n+10)/4-s, n+1]$,
$f(x_{2i-1})=n-i+1$ if $i \in [(n+2)/4+s, n/2]$, $[(n+2)/2, (3n+2)/4-s]$,
$f(x_{2i})=n/2-i$ if $i \in [1, (n-2)/4]$, $[(n+2)/4, (n-2)/2]$,
$f(x_{2i})=(n-2)/2-i$ if $i \in [(n+2)/4, (n-2)/2]$, $[0, (n-6)/4]$,
$f(x_n)=n/2$.

It is easy to verify that the f is an injection from $V(G)$ to $[0, n+1]$. By the above definition, we have

$A=\{f^*(x_{2i}x_{2i-1}), f^*(x_{2i}x_{2i+1})| \ i \in [1, (n-2)/4]\}$
$=\{(3n+4)/2-2i, (3n+2)/2-2i| \ i \in [1, (n-2)/4]\}=[n+2, 3n/2]$,
$B=\{f^*(x_{2i}x_{2i-1})| \ i \in [(n+2)/4, (n-2)/4+s]\}$
$=\{(3n+2)/2-2i| \ i \in [(n+2)/4, (n-2)/4+s]\}=[n+2-2s, n]_2$,
$C=\{f^*(x_{2i}x_{2i+1})| \ i \in [(n+2)/4, (n-6)/4+s]\}$
$=\{3n/2-2i| \ i \in [(n+2)/4, (n-6)/4+s]\}=[n+3-2s, n-1]_2$,
$D=\{f^*(x_{2i}x_{2i-1})| \ i \in [(n+2)/4+s, (n-2)/2]\}$
$=\{3n/2-2i| \ i \in [(n+2)/4+s, (n-2)/2]\}=[(n+4)/2, n-2s-1]_2$,
$E=\{f^*(x_{2i}x_{2i+1})| \ i \in [(n-2)/4+s, (n-2)/2]\}$
$=\{(3n-2)/2-2i| \ i \in [(n-2)/4+s, (n-2)/2]\}=[(n+2)/2, n-2s]_2$,
$F=\{f^*(x_{(n-2)/2}y_1), f^*(x_{(n-4)/2+4s}y_1), f^*(x_nx_1), f^*(x_{n-1}x_n)\}$
$=\{n/2, n+1-2s, (3n+2)/2, n+1\}$.

Therefore, $f^*(E(G))=A \cup B \cup C \cup D \cup E \cup F=[n/2, (3n+2)/2]$. □

Theorem 2.7. When $n \equiv 2 \pmod 4$ and $n \ge 10$, the graph $C_n(4s+1; 2+2s, n+1-2s; P_2)$ for $1 \le s \le (n-2)/8$, and the graph $C_n(4s+2; n/2-2s, (n+4)/2+2s; P_2)$ for $1 \le s \le (n-6)/4$ are strongly $n/2$-harmonious.

Proof. For the graph $G=C_n(4s+1; 2+2s, n+1-2s; P_2)$, we define the function f: $V(G) \to [0, n+1]$ as follows: $f(y_1)=n-s+1$,
$f(x_{2i-1})=i-1$ if $i \in [1, n/2]$, $[0, (n-2)/2]$,
$f(x_{2i})=(n-2)/2+i$ if $i \in [1, (n+2)/4]$, $[n/2, (3n-2)/4]$,
$f(x_{2i})=n/2+i$ if $i \in [(n+6)/4, n/2-s]$, $[(3n+6)/4, n-s]$,
$f(x_{2i})=(n+2)/2+i$ if $i \in [(n+2)/2-s, n/2]$, $[n-s+2, n+1]$.

It is easy to verify that the f is an injection from $V(G)$ to $[0, n+1]$. By the above definition, we have

$A=\{f^*(x_{2i}x_{2i-1}), f^*(x_{2i}x_{2i+1}) | \ i \in [1, (n+2)/4]\}$
$\quad =\{(n-4)/2+2i, (n-2)/2+2i | \ i \in [1, (n+2)/4]\} = [n/2, n]$,
$B=\{f^*(x_{2i}x_{2i-1}), f^*(x_{2i}x_{2i+1}) | \ i \in [(n+6)/4, n/2-s]\}$
$\quad =\{(n-2)/2+2i, n/2+2i | \ i \in [(n+6)/4, n/2-s]\} = [n+2, 3n/2-2s]$,
$C=\{f^*(x_{2i}x_{2i-1}) | \ i \in [(n+2)/2-s, n/2]\}=\{n/2+2i | i \in [(n+2)/2-s, n/2]\}$
$\quad =[(3n+4)/2-2s, 3n/2]_2$,
$D=\{f^*(x_{2i}x_{2i+1}) | \ i \in [(n+2)/2-s, (n-2)/2]\}$
$\quad =\{(n+2)/2+2i | \ i \in [(n+2)/2-s, (n-2)/2]\}$
$\quad =[(3n+6)/2-2s, (3n-2)/2]_2$,
$E=\{f^*(x_{2+2s}y_1), f^*(x_{n-2s+1}y_1), f^*(x_n x_1)\}$
$\quad =\{(3n+2)/2, (3n+2)/2-2s, n+1\}$.

Therefore, $f^*(E(G))=A \cup B \cup C \cup D \cup E = [n/2, (3n+2)/2]$.

For the graph $G=C_n(4s+2; n/2-2s, (n+4)/2+2s; P_2)$, we define the function f: $V(G) \to [0, n+1]$ as follows: $f(y_1)=(3n+6)/4-s$,
$f(x_{2i-1})=i-1$ if $i \in [1, (n-2)/4]$, $[0, (n-6)/4]$,
$f(x_{2i-1})=i$ if $i \in [(n+2)/4, n/2]$, $[(n+2)/4, n/2]$,
$f(x_{2i})=(n+2)/2+i$ if $i \in [1, (n-2)/4-s]$, $[(n+4)/2, (3n+2)/4-s]$,
$f(x_{2i})=(n+4)/2+i$ if $i \in [(n+2)/4-s, (n-2)/4]$, $[(3n+10)/4-s, (3n+6)/4]$,
$f(x_{2i})=(n+2)/2+i$ if $i \in [(n+6)/4, n/2]$, $[(3n+10)/4, n+1]$,
$f(x_{(n+2)/2})=(n-2)/4$.

It is easy to verify that the f is an injection from $V(G)$ to $[0, n+1]$. By the above definition, we have

$A=\{f^*(x_{2i}x_{2i-1}), f^*(x_{2i}x_{2i+1})|\ i \in [1,(n-2)/4-s]\}$
$=\{n/2+2i, (n+2)/2+2i|\ i \in [1,(n-2)/4-s]\}=[(n+4)/2, n-2s]$,
$B=\{f^*(x_{2i}x_{2i-1})|\ i \in [(n+2)/4-s, (n-2)/4]\}$
$=\{(n+2)/2+2i|\ i \in [(n+2)/4-s, (n-2)/4]\}=[n+2-2s, n]_2$,
$C=\{f^*(x_{2i}x_{2i+1})|\ i \in [(n+2)/4-s, (n-6)/4]\}$
$=\{(n+4)/2+2i|\ i \in [(n+2)/4-s, (n-6)/4]\}=[n+3-2s, n-1]_2$,
$D=\{f^*(x_{2i}x_{2i-1})|\ i \in [(n+6)/4, n/2]\}=\{(n+2)/2+2i|\ i \in [(n+6)/4, n/2]\}$
$=[n+4, (3n+2)/2]_2$,
$E=\{f^*(x_{2i}x_{2i+1})|\ i \in [(n+6)/4, (n-2)/2]\}$
$=\{(n+4)/2+2i|\ i \in [(n+6)/4, (n-2)/2]\}=[n+5, 3n/2]_2$,
$F=\{f^*(x_{n/2-2s}y_1), f^*(x_{(n+4)/2+2s}y_1), f^*(x_nx_1), f^*(x_{(n-2)/2}x_{n/2}),$
$f^*(x_{n/2}x_{(n+2)/2}), f^*(x_{(n+2)/2}x_{(n+4)/2})\}$
$=\{n+1-2s, n+3, n+1, n+2, n/2, (n+2)/2\}$.
Therefore, $f^*(E(G))=A \cup B \cup C \cup D \cup E \cup F=[n/2, (3n+2)/2]$. □

Theorem 2.8. When $n \equiv 2 \pmod 4$, the graph $C_{10}(2;2,10;P_2)$ and the graph $C_n(2;(n-6)/2,(n-2)/2;P_2)$ for $n \geq 14$ are strongly $(n+2)/2$-harmonious.

Proof. For the graph $C_{10}(2;2,10;P_2)$ see Figure 2. For the graph $G=C_n(2;(n-6)/2,(n-2)/2;P_2)$, we define the function f: $V(G) \to [0, n+1]$ as follows: $f(y_1)=(3n-6)/4$,
$f(x_{2i-1})=(n-2)/2+i$ if $i \in [2,(n-6)/4]$, $[(n+2)/2, (3n-10)/4]$,
$f(x_{2i-1})=n/2+i$ if $i \in [(n-2)/4, (n+2)/4]$, $[(3n-2)/4, (3n+2)/4]$,
$f(x_{2i-1})=(n+2)/2+i$ if $i \in [(n+6)/4, n/2]$, $[(3n+10)/4, n+1]$,
$f(x_{2i})=i-2$ if $i \in [2,(n-6)/4]$, $[0,(n-14)/4]$,
$f(x_{2i})=i-2$ if $i \in [(n+6)/4, n/2]$, $[(n-2)/4, (n-4)/2]$,
$f(x_1)=(n-2)/2$, $f(x_2)=n/2$, $f(x_{(n-2)/2})=(3n+6)/4$,
$f(x_{(n+2)/2})=(n-10)/4$.

It is easy to verify that the f is an injection from $V(G)$ to $[0, n+1]$. By the above definition, we have

$A=\{f^*(x_{2i}x_{2i-1})|\ i\in[2,(n-6)/4]\}=\{(n-6)/2+2i|\ i\in[2,(n-6)/4]\}$
$\quad=[(n+2)/2,n-6]_2,$
$B=\{f^*(x_{2i}x_{2i+1})|\ i\in[2,(n-10)/4]\}=\{(n-4)/2+2i|\ i\in[2,(n-10)/4]\}$
$\quad=[(n+4)/2,n-7]_2,$
$C=\{f^*(x_{2i}x_{2i-1})|\ i\in[(n+6)/4,n/2]\}=\{(n-2)/2+2i|\ i\in[(n+6)/4,n/2]\}$
$\quad=[n+2,(3n-2)/2]_2,$
$D=\{f^*(x_{2i}x_{2i+1})|\ i\in[(n+6)/4,(n-2)/2]\}$
$\quad=\{n/2+2i|\ i\in[(n+6)/4,(n-2)/2]\}=[n+3,(3n-4)/2]_2,$
$E=\{f^*(x_{(n-6)/2}y_1),f^*(x_{(n-2)/2}y_1),f^*(x_nx_1),f^*(x_1x_2),f^*(x_2x_3),$
$\quad f^*(x_{(n-6)/2}x_{(n-4)/2}),f^*(x_{(n-4)/2}x_{(n-2)/2}),f^*(x_{(n-2)/2}x_{n/2}),$
$\quad f^*(x_{n/2}x_{(n+2)/2}),f^*(x_{(n+2)/2}x_{(n+4)/2})\}$
$\quad=\{n-5,3n/2,n-3,n-1,n+1,n-4,(3n+2)/2,(3n+4)/2,n-2,n\}.$
Therefore, $f^*(E(G))=A\cup B\cup C\cup D\cup E=[(n+2)/2,(3n+4)/2]$. □

Theorem 2.9. When $n\equiv 3 \pmod 4$ and $n\geq 7$, the graph $C_n(4s+2;2+2s,n-2s;P_2)$ for $0\leq s\leq (n-7)/4$ is strongly $(n-1)/2$-harmonious and the graph $C_n(3;1,4;P_2)$ is strongly $(n+1)/2$-harmonious.

Proof. For the graph $G=C_n(4s+2;2+2s,n-2s;P_2)$, we define the function f: $V(G)\to[0,n+1]$ as follows: $f(y_1)=n-s$,
$f(x_{2i-1})=i-1$ if $i\in[1,(n+1)/2]$, $[0,(n-1)/2]$,
$f(x_{2i})=(n-1)/2+i$ if $i\in[1,(n-1)/2-s]$, $[(n+1)/2,n-1-s]$,
$f(x_{2i})=(n+1)/2+i$ if $i\in[(n+1)/2-s,(n-1)/2]$, $[n+1-s,n]$.

It is easy to verify that the f is an injection from $V(G)$ to $[0,n+1]$. By the above definition, we have

$A=\{f^*(x_{2i}x_{2i-1}),f^*(x_{2i}x_{2i+1})|\ i\in[1,(n-1)/2-s]\}$
$\quad=\{(n-3)/2+2i,(n-1)/2+2i|\ i\in[1,(n-1)/2-s]\}$
$\quad=[(n+1)/2,(3n-3)/2-2s],$
$B=\{f^*(x_{2i}x_{2i-1}),f^*(x_{2i}x_{2i+1})|\ i\in[(n+1)/2-s,(n-1)/2]\}$

$=\{(n-1)/2+2i,(n+1)/2+2i|\ i\in[(n+1)/2-s,(n-1)/2]\}$
$=[(3n+1)/2-2s,(3n-1)/2]$,
$C=\{f^*(x_{2+2s}y_1),f^*(x_{n-2s}y_1),f^*(x_nx_1)\}$
$=\{(3n+1)/2,(3n-1)/2-2s,(n-1)/2\}$.
Therefore, $f^*(E(G))=A\cup B\cup C=[(n-1)/2,(3n+1)/2]$.

For the graph $G=C_n(3;1,4;P_2)$, we define the function f: $V(G)\to[0,n+1]$ as follows: $f(y_1)=(n+5)/2$,
$f(x_{2i-1})=(n+3)/2+i$ if $i\in[2,(n+5)/4]$, $[(n+7)/2,(3n+11)/4]$,
$f(x_{2i-1})=i$ if $i\in[(n+9)/4,(n+1)/2]$, $[(n+9)/4,(n+1)/2]$,
$f(x_{2i})=i$ if $i\in[2,(n+5)/4]$, $[2,(n+5)/4]$,
$f(x_{2i})=(n+3)/2+i$ if $i\in[(n+9)/4,(n-1)/2]$, $[(3n+15)/4,n+1]$,
$f(x_1)=0$, $f(x_2)=(n+3)/2$.

It is easy to verify that the f is an injection from $V(G)$ to $[0,n+1]$. By the above definition, we have
$A=\{f^*(x_{2i}x_{2i-1})|\ i\in[2,(n+5)/4]\}=\{(n+3)/2+2i|\ i\in[2,(n+5)/4]\}$
$=[(n+11)/2,n+4]_2$,
$B=\{f^*(x_{2i}x_{2i+1})|\ i\in[2,(n+1)/4]\}=\{(n+5)/2+2i|\ i\in[2,(n+1)/4]\}$
$=[(n+13)/2,n+3]_2$,
$C=\{f^*(x_{2i}x_{2i-1}),f^*(x_{2i}x_{2i+1})|\ i\in[(n+9)/4,(n-1)/2]\}$
$=\{(n+3)/2+2i,(n+5)/2+2i|\ i\in[(n+9)/4,(n-1)/2]\}$
$=[n+6,(3n+3)/2]$,
$D=\{f^*(x_1y_1),f^*(x_4y_1),f^*(x_nx_1),f^*(x_1x_2),f^*(x_2x_3),f^*(x_{(n+5)/2}x_{(n+7)/2})\}$
$=\{(n+5)/2,(n+9)/2,(n+1)/2,(n+3)/2,n+5,(n+7)/2\}$.
Therefore, $f^*(E(G))=A\cup B\cup C\cup D=[(n+1)/2,(3n+3)/2]$. □

Theorem 2.10. When $n\equiv 3 \pmod 4$ and $n\geq 11$, the graph $C_n(4s;2+2s,n+2-2s;P_2)$ for $1\leq s\leq (n-7)/4$ is strongly $(n-1)/2$-harmonious.

Proof. We define the function f: $V(G)\to[0,n+1]$ as follows:
$f(y_1)=(n+1)/2-s$,

$f(x_{2i-1})=(n+1)/2+i$ if $i\in[2,(n+5)/4]$, $[(n+5)/2,(3n+7)/4]$,
$f(x_{2i-1})=(n-1)/2+i$ if $i\in[(n+13)/4,(n+1)/2]$, $[(3n+11)/4,n]$,
$f(x_{2i})=i-2$ if $i\in[2,(n+9)/4]$, $[0,(n+1)/4]$,
$f(x_{2i})=i-1$ if $i\in[(n+13)/4,(n+1)/2-s]$, $[(n+9)/4,(n-1)/2-s]$,
$f(x_{2i})=i$ if $i\in[(n+3)/2-s,(n-1)/2]$, $[(n+3)/2-s,(n-1)/2]$,
$f(x_1)=(n+1)/2$, $f(x_2)=(n+3)/2$, $f(x_{(n+7)/2})=(n+5)/4$.

It is easy to verify that the f is an injection from $V(G)$ to $[0,n+1]$. By the above definition, we have

$A=\{f^*(x_{2i}x_{2i-1})|\ i\in[2,(n+5)/4]\}=\{(n-3)/2+2i|\ i\in[2,(n+5)/4]\}$
$\quad=[(n+5)/2,n+1]_2$,
$B=\{f^*(x_{2i}x_{2i+1})|\ i\in[2,(n+1)/4]\}=\{(n-1)/2+2i|\ i\in[2,(n+1)/4]\}$
$\quad=[(n+7)/2,n]_2$,
$C=\{f^*(x_{2i}x_{2i-1}),f^*(x_{2i}x_{2i+1})|\ i\in[(n+13)/4,(n+1)/2-s]\}$
$\quad=\{(n-3)/2+2i,(n-1)/2+2i|\ i\in[(n+13)/4,(n+1)/2-s]\}$
$\quad=[n+5,(3n+1)/2-2s]$,
$D=\{f^*(x_{2i}x_{2i-1}),f^*(x_{2i}x_{2i+1})|\ i\in[(n+3)/2-s,(n-1)/2]\}$
$\quad=\{(n-1)/2+2i,(n+1)/2+2i|\ i\in[(n+3)/2-s,(n-1)/2]\}$
$\quad=[(3n+5)/2-2s,(3n-1)/2]$,
$E=\{f^*(x_{2+2s}y_1),f^*(x_{n+2-2s}y_1),f^*(x_nx_1),f^*(x_1x_2),f^*(x_2x_3),$
$\quad f^*(x_{(n+5)/2}x_{(n+7)/2}),f^*(x_{(n+7)/2}x_{(n+9)/2}),\ f^*(x_{(n+9)/2}x_{(n+11)/2}\}$
$\quad=\{(n-1)/2,(3n+3)/2-2s,(3n+1)/2,n+2,n+4,(n+1)/2,$
$\quad(n+3)/2,n+3\}$.

Therefore, $f^*(E(G))=A\cup B\cup C\cup D\cup E=[(n-1)/2,(3n+1)/2]$. □

3 Cycle with P_3-chord

The graph $C_n(d;i,j;P_3)$ has $n+2$ vertices and $n+3$ edges. $f^*(E(G))$ is the set of its edge labels.

Theorem 3.1. When $n\equiv 0 \pmod 4$ and $n\geq 8$, the graph $C_n(4s-1;n/2,(n-2)/2+4s;P_3)$ for $1\leq s\leq n/8$ is strongly $(n+2)/2$-harmonious and the graph $C_n(4s;3,n+3-4s;P_3)$ for $1 \le s \le n/8$ is strongly $(n+4)/2$-harmonious.

Proof. For the graph $G=C_n(4s-1;n/2,(n-2)/2+4s;P_3)$, we define the function f: $V(G) \to [0,n+2]$ as follows:
$f(y_1)=(3n+4)/4$, $f(y_2)=(3n+8)/4$,
$f(x_{2i-1})=n+3-i$ if $i \in [1,n/4]$, $[(3n+12)/4,n+2]$,
$f(x_{2i-1})=n+1-i$ if $i \in [(n+4)/4,n/2]$, $[(n+2)/2,3n/4]$,
$f(x_{2i})=n/2-i$ if $i \in [1,s-1]$, $[(n+2)/2-s,(n-2)/2]$,
$f(x_{2i})=(n-2)/2-i$ if $i \in [s,(n-2)/2]$, $[0,(n-2)/2-s]$,
$f(x_n)=n/2$.

It is easy to verify that the f is an injection from $V(G)$ to $[0,n+2]$. By the above definition, we have
$A=\{f^*(x_{2i}x_{2i-1}),f^*(x_{2i}x_{2i+1})|\ i \in [1,s-1]\}$
$=\{(3n+6)/2-2i,(3n+4)/2-2i|\ i \in [1,s-1]\}$
$=[(3n+8)/2-2s,(3n+2)/2]$,
$B=\{f^*(x_{2i}x_{2i-1})|\ i \in [s,n/4]\}=\{(3n+4)/2-2i|\ i \in [s,n/4]\}$
$=[n+2,(3n+4)/2-2s]_2$,
$C=\{f^*(x_{2i}x_{2i+1})|\ i \in [s,(n-4)/4]\}=\{(3n+2)/2-2i|\ i \in [s,(n-4)/4]\}$
$=[n+3,(3n+2)/2-2s]_2$,
$D=\{f^*(x_{2i}x_{2i-1})|\ i \in [(n+4)/4,(n-2)/2]\}$
$=\{3n/2-2i|\ i \in [(n+4)/4,(n-2)/2]\}=[(n+4)/2,n-2]_2$,
$E=\{f^*(x_{2i}x_{2i+1})|\ i \in [n/4,(n-2)/2]\}$
$=\{(3n-2)/2-2i|\ i \in [n/4,(n-2)/2]\}=[(n+2)/2,n-1]_2$,
$F=\{f^*(x_{n/2}y_1),f^*(y_1y_2),f^*(x_{(n-2)/2+4s}y_2),f^*(x_nx_1),f^*(x_{n-1}x_n)\}$
$=\{n,(3n+6)/2,(3n+6)/2-2s,(3n+4)/2,n+1\}$.
Therefore, $f^*(E(G))=A \cup B \cup C \cup D \cup E \cup F=[(n+2)/2,(3n+6)/2]$.
For the graph $G=C_n(4s;3,n+3-4s;P_3)$, we define the function f: $V(G) \to [0,n+2]$ as follows: $f(y_1)=(n-2)/2$, $f(y_2)=(n+2)/2$,
$f(x_{2i-1})=n+3-i$ if $i \in [1,(n+4)/4]$, $[(3n+8)/4,n+2]$,

$f(x_{2i-1})=n+2-i$ if $i \in [(n+8)/4, n/2]$, $[(n+4)/2, 3n/4]$,
$f(x_{2i})=n/2-i$ if $i \in [2, n/4-s]$, $[n/4+s, (n-4)/2]$,
$f(x_{2i})=(n-2)/2-i$ if $i \in [(n+4)/4-s, (n-4)/4]$, $[n/4, (n-8)/4+s]$,
$f(x_{2i})=n/2-i$ if $i \in [(n+4)/4, n/2]$, $[0, (n-4)/4]$,
$f(x_2)=n/2$, $f(x_{n/2})=(3n+4)/4$.

It is easy to verify that the f is an injection from $V(G)$ to $[0, n+2]$. By the above definition, we have

$A=\{f^*(x_{2i}x_{2i-1}), f^*(x_{2i}x_{2i+1}) |\ i \in [2, n/4-s]\}$
$=\{(3n+6)/2-2i, (3n+4)/2-2i |\ i \in [2, n/4-s]\}=[n+2+2s, (3n-2)/2]$,
$B=\{f^*(x_{2i}x_{2i-1}), f^*(x_{2i}x_{2i+1}) |\ i \in [(n+4)/4-s, (n-4)/4]\}$
$=\{(3n+4)/2-2i, (3n+2)/2-2i |\ i \in [(n+4)/4-s, (n-4)/4]\}$
$=[n+3, n+2s]$,
$C=\{f^*(x_{2i}x_{2i-1}) |\ i \in [(n+8)/4, n/2]\}=\{(3n+4)/2-2i |\ i \in [(n+8)/4, n/2]\}$
$=[(n+4)/2, n-2]_2$,
$D=\{f^*(x_{2i}x_{2i+1}) |\ i \in [(n+4)/4, (n-2)/2]\}$
$=\{(3n+2)/2-2i |\ i \in [(n+4)/4, (n-2)/2]\} =[(n+6)/2, n-1]_2$,
$E=\{f^*(x_3y_1), f^*(y_1y_2), f^*(x_{n+3-4s}y_2), f^*(x_nx_1), f^*(x_1x_2), f^*(x_2x_3),$
$f^*(x_{n/2}x_{(n+2)/2}), f^*(x_{(n+2)/2}x_{(n+4)/2}), f^*(x_{n/2}x_{(n-2)/2})\}$
$=\{3n/2, n, n+2s+1, n+2, (3n+4)/2, (3n+2)/2, (3n+6)/2, n+1, (3n+8)/2\}$.
Therefore, $f^*(E(G))=A \cup B \cup C \cup D \cup E=[(n+4)/2, (3n+8)/2]$. □

Theorem 3.2. When $n \equiv 0 \pmod 4$ and $n \geq 12$, the graph $C_n(4s+1; 2, 3+4s; P_3)$ for $1 \leq s \leq (n-4)/8$ is strongly $n/2$-harmonious, and the graph $C_n(4s+2; (n-2)/2, (n+2)/2+4s; P_3)$ for $1 \leq s \leq (n-4)/8$ is strongly $(n+2)/2$-harmonious.

Proof. For the graph $G=C_n(4s+1; 2, 3+4s; P_3)$, we define the function f: $V(G) \to [0, n+2]$ as follows: $f(y_1)=(n+2)/2$, $f(y_2)=(n+6)/2+2s$,
$f(x_{2i-1})=i-1$ if $i \in [1, (n+4)/4]$, $[0, n/4]$,
$f(x_{2i-1})=(n+2)/2+i$ if $i \in [(n+8)/4, (n+4)/4+s]$, $[(3n+12)/4, (3n+$

$8)/4+s]$,

$f(x_{2i-1})=(n+4)/2+i$ if $i \in [(n+8)/4+s, n/2]$, $[(3n+16)/4+s, n+2]$,

$f(x_{2i})=(n+2)/2+i$ if $i \in [1, 1+2s]$, $[(n+4)/2, (n+4)/2+2s]$,

$f(x_{2i})=(n+4)/2+i$ if $i \in [2+2s, n/4]$, $[(n+8)/2+2s, (3n+8)/4]$,

$f(x_{2i})=i$ if $i \in [(n+4)/4, n/2]$, $[(n+4)/4, n/2]$.

It is easy to verify that the f is an injection from $V(G)$ to $[0, n+2]$. By the above definition, we have

$A=\{f^*(x_{2i}x_{2i-1}), f^*(x_{2i}x_{2i+1}) | \ i \in [1, 1+2s]\}$

$=\{n/2+2i, (n+2)/2+2i | \ i \in [1, 1+2s]\}=[(n+4)/2, (n+6)/2+4s]$,

$B=\{f^*(x_{2i}x_{2i-1}), f^*(x_{2i}x_{2i+1}) | \ i \in [2+2s, n/4]\}$

$=\{(n+2)/2+2i, (n+4)/2+2i | \ i \in [2+2s, n/4]\}=[(n+10)/2+4s, n+2]$,

$C=\{f^*(x_{2i}x_{2i-1}) | \ i \in [(n+8)/4, (n+4)/4+s]\}$

$=\{(n+2)/2+2i | \ i \in [(n+8)/4, (n+4)/4+s]\}=[n+5, n+3+2s]_2$,

$D=\{f^*(x_{2i}x_{2i+1}) | \ i \in [(n+4)/4, n/4+s]\}$

$=\{(n+4)/2+2i | \ i \in [(n+4)/4, n/4+s]\} =[n+4, n+2s+2]_2$,

$E=\{f^*(x_{2i}x_{2i-1}) | \ i \in [(n+8)/4+s, n/2]\}$

$=\{(n+4)/2+2i | \ i \in [(n+8)/4+s, n/2]\}=[n+6+2s, (3n+4)/2]_2$,

$F=\{f^*(x_{2i}x_{2i+1}) | \ i \in [(n+4)/4+s, (n-2)/2]\}$

$=\{(n+6)/2+2i | \ i \in [(n+4)/4+s, (n-2)/2]\}=[n+5+2s, (3n+2)/2]_2$,

$G=\{f^*(x_2y_1), f^*(y_1y_2), f^*(x_{3+4s}y_2), f^*(x_nx_1), f^*(x_{(n+2)/2}x_{(n+4)/2})\}$

$=\{n+3, n+4+2s, (n+8)/2+4s, n/2, (n+2)/2\}$.

Therefore, $f^*(E(G))=A \cup B \cup C \cup D \cup E \cup F \cup G=[n/2, (3n+4)/2]$.

For the graph $G=C_n(4s+2; (n-2)/2, (n+2)/2+4s; P_3)$, we define the function f: $V(G) \rightarrow [0, n+2]$ as follows:

$f(y_1)=(3n+4)/4$, $f(y_2)=(3n+8)/4$,

$f(x_{2i-1})=i-1$ if $i \in [1, n/4+s]$, $[0, (n-4)/4+s]$,

$f(x_{2i-1})=i$ if $i \in [(n+4)/4+s, n/2]$, $[(n+4)/4+s, n/2]$,

$f(x_{2i})=(n+2)/2+i$ if $i \in [1, (n-4)/4]$, $[(n+4)/2, 3n/4]$,

$f(x_{2i})=(n+6)/2+i$ if $i \in [n/4, (n-2)/2]$, $[(3n+12)/4, n+2]$,

$f(x_n)=(n+2)/2$.

It is easy to verify that the f is an injection from $V(G)$ to $[0,n+2]$. By the above definition, we have

$A=\{f^*(x_{2i}x_{2i-1}), f^*(x_{2i}x_{2i+1})|\ i\in[1,(n-4)/4]\}$
$\quad=\{n/2+2i,(n+2)/2+2i|\ i\in[1,(n-4)/4\}=[(n+4)/2,n-1]$,
$B=\{f^*(x_{2i}x_{2i-1})|\ i\in[n/4,n/4+s]\}$
$\quad=\{(n+4)/2+2i|\ i\in[n/4,n/4+s]\}=[n+2,n+2s+2]_2$,
$C=\{f^*(x_{2i}x_{2i+1})|\ i\in[n/4,(n-4)/4+s]\}$
$\quad=\{(n+6)/2+2i|\ i\in[n/4,(n-4)/4+s]\}=[n+3,n+2s+1]_2$,
$D=\{f^*(x_{2i}x_{2i-1})|\ i\in[(n+4)/4+s,(n-2)/2]\}$
$\quad=\{(n+6)/2+2i|\ i\in[(n+4)/4+s,(n-2)/2]\}=[n+2s+5,(3n+2)/2]_2$,
$E=\{f^*(x_{2i}x_{2i+1})|\ i\in[n/4+s,(n-2)/2]\}$
$\quad=\{(n+8)/2+2i|\ i\in[n/4+s,(n-2)/2]\}=[n+4+2s,(3n+4)/2]_2$,
$F=\{f^*(x_{(n-2)/2}y_1), f^*(y_1y_2), f^*(x_{(n+2)/2+4s}y_2), f^*(x_nx_1), f^*(x_{n-1}x_n)\}$
$\quad=\{n,(3n+6)/2,n+3+2s,(n+2)/2,n+1\}$.
Therefore, $f^*(E(G))=A\cup B\cup C\cup D\cup E\cup F=[(n+2)/2,(3n+6)/2]$. □

Theorem 3.3. When $n\equiv 1 \pmod 4$ and $n\geq 9$, the graph $C_n(4s-1;1+2s,n+2-2s;P_3)$ for $1\leq s\leq (n-5)/4$ is strongly $(n+1)/2$-harmonious, and the graph $C_n(4s;(n+1)/2-2s,(n+1)/2+2s;P_3)$ for $1\leq s\leq (n-5)/4$ is strongly $(n-1)/2$-harmonious.

Proof. For the graph $G=C_n(4s-1;1+2s,n+2-2s;P_3)$, we define the function f: $V(G)\to[0,n+2]$ as follows:

$f(y_1)=(n+3)/2+s$, $f(y_2)=(n+3)/2-s$,
$f(x_{2i-1})=i-1$ if $i\in[1,(n+3)/4]$, $[0,(n-1)/4]$,
$f(x_{2i-1})=(n+3)/2+i$ if $i\in[(n+7)/4,(n+1)/2]$, $[(3n+13)/4,n+2]$,
$f(x_{2i})=(n+1)/2+i$ if $i\in[1,s]$, $[(n+3)/2,(n+1)/2+s]$,
$f(x_{2i})=(n+3)/2+i$ if $i\in[1+s,(n-1)/4]$, $[(n+5)/2+s,(3n+5)/4]$,
$f(x_{2i})=i$ if $i\in[(n+3)/4,(n+1)/2-s]$, $[(n+3)/4,(n+1)/2-s]$,
$f(x_{2i})=i+1$ if $i\in[(n+3)/2-s,(n-1)/2]$, $[(n+5)/2-s,(n+1)/2]$.

It is easy to verify that the f is an injection from $V(G)$ to $[0, n+2]$. By the above definition, we have

$A=\{f^*(x_{2i}x_{2i-1}), f^*(x_{2i}x_{2i+1})| \ i \in [1, s]\}$
$=\{(n-1)/2+2i, (n+1)/2+2i| \ i \in [1, s]\}=[(n+3)/2, (n+1)/2+2s]$,
$B=\{f^*(x_{2i}x_{2i-1}), f^*(x_{2i}x_{2i+1})| \ i \in [1+s, (n-1)/4]\}$
$=\{(n+1)/2+2i, (n+3)/2+2i| \ i \in [1+s, (n-1)/4]\}=[(n+5)/2+2s, n+1]$,
$C=\{f^*(x_{2i}x_{2i-1}), f^*(x_{2i}x_{2i+1})| \ i \in [(n+7)/4, (n+1)/2-s]\}$
$=\{(n+3)/2+2i, (n+5)/2+2i| \ i \in [(n+7)/4, (n+1)/2-s]\}$
$=[n+5, (3n+7)/2-2s]$,
$D=\{f^*(x_{2i}x_{2i-1}), f^*(x_{2i}x_{2i+1})| \ i \in [(n+3)/2-s, (n-1)/2]\}$
$=\{(n+5)/2+2i, (n+7)/2+2i| \ i \in [(n+3)/2-s, (n-1)/2]\}$
$=[(3n+11)/2-2s, (3n+5)/2]$,
$E=\{f^*(x_{1+2s}y_1), f^*(y_1y_2), f^*(x_{n+2-2s}y_2), f^*(x_nx_1), f^*(x_{(n+1)/2}x_{(n+3)/2}),$
$f^*(x_{(n+3)/2}x_{(n+5)/2})\}$
$=\{(n+3)/2+2s, n+3, (3n+9)/2-2s, n+2, (n+1)/2, n+4\}$.

Therefore, $f^*(E(G))=A \cup B \cup C \cup D \cup E=[(n+1)/2, (3n+5)/2]$.

For the graph $C_n(4s; (n+1)/2-2s, (n+1)/2+2s; P_3)$, we define the function f: $V(G) \to [0, n+2]$ as follows:

$f(y_1)=(3n+5)/4+s$, $f(y_2)=(3n+1)/4-s$,
$f(x_{2i-1})=(n+1)/2-i$ if $i \in [1, (n+1)/2]$, $[0, (n-1)/2]$,
$f(x_{2i})=n+2-i$ if $i \in [1, (n-1)/4-s]$, $[(3n+9)/4+s, n+1]$,
$f(x_{2i})=n+1-i$ if $i \in [(n+3)/4-s, (n-1)/4+s]$, $[(3n+5)/4-s, (3n+1)/4+s]$,
$f(x_{2i})=n-i$ if $i \in [(n+3)/4+s, (n-1)/2]$, $[(n+1)/2, (3n-3)/4-s]$.

It is easy to verify that the f is an injection from $V(G)$ to $[0, n+2]$. By the above definition, we have

$A=\{f^*(x_{2i}x_{2i-1}), f^*(x_{2i}x_{2i+1})| \ i \in [1, (n-1)/4-s]\}$
$=\{(3n+5)/2-2i, (3n+3)/2-2i| \ i \in [1, (n-1)/4-s]\}$
$=[n+2+2s, (3n+1)/2]$,
$B=\{f^*(x_{2i}x_{2i-1}), f^*(x_{2i}x_{2i+1})| \ i \in [(n+3)/4-s, (n-1)/4+s]\}$

$=\{(3n+3)/2-2i,(3n+1)/2-2i|\ i\in[(n+3)/4-s,(n-1)/4+s]\}$
$=[n+1-2s,n+2s]$,
$C=\{f^*(x_{2i}x_{2i-1}),f^*(x_{2i}x_{2i+1})|\ i\in[(n+3)/4+s,(n-1)/2]\}$
$=\{(3n+1)/2-2i,(3n-1)/2-2i|\ i\in[(n+3)/4+s,(n-1)/2]\}$
$=[(n+1)/2,n-2s-1]$,
$D=\{f^*(x_{(n+1)/2-2s}y_1),f^*(y_1y_2),f^*(x_{(n+1)/2+2s}y_2),f^*(x_nx_1)\}$
$=\{n+1+2s,(3n+3)/2,n-2s,(n-1)/2\}$.
Therefore, $f^*(E(G))=A\cup B\cup C\cup D=[(n-1)/2,(3n+3)/2]$. □

Theorem 3.4. When $n\equiv 2\pmod 4$ and $n\geq 6$, the graph $C_n(4s-1;(n+6)/2-2s,(n+4)/2+2s;P_3)$ for $1\leq s\leq (n+2)/8$ is strongly $(n+4)/2$-harmonious.

Proof. We define the function f: $V(G)\to[0,n+2]$ as follows:
$f(y_1)=(n+10)/4-s$, $f(y_2)=(3n+10)/4+s$,
$f(x_{2i-1})=i$ if $i\in[1,(n+6)/4-s]$, $[1,(n+6)/4-s]$,
$f(x_{2i-1})=i+1$ if $i\in[(n+10)/4-s,(n+2)/4]$, $[(n+14)/4-s,(n+6)/4]$,
$f(x_{2i-1})=i$ if $i\in[(n+10)/4,n/2]$, $[(n+10)/4,n/2]$,
$f(x_{2i})=n/2+i$ if $i\in[1,(n+6)/4]$, $[(n+2)/2,(3n+6)/4]$,
$f(x_{2i})=(n+2)/2+i$ if $i\in[(n+10)/4,(n+2)/4+s]$, $[(3n+14)/4,(3n+6)/4+s]$,
$f(x_{2i})=(n+4)/2+i$ if $i\in[(n+6)/4+s,n/2]$, $[(3n+14)/4+s,n+2]$,
$f(x_{(n+4)/2})=(3n+10)/4$.

It is easy to verify that the f is an injection from $V(G)$ to $[0,n+2]$. By the above definition, we have
$A=\{f^*(x_{2i}x_{2i-1})|\ i\in[1,(n+6)/4-s]\}=\{n/2+2i|\ i\in[1,(n+6)/4-s\}$
$=[(n+4)/2,n+3-2s]_2$,
$B=\{f^*(x_{2i}x_{2i+1})|\ i\in[1,(n+2)/4-s]\}=\{(n+2)/2+2i|\ i\in[1,(n+2)/4-s]\}$
$=[(n+6)/2,n-2s+2]_2$,
$C=\{f^*(x_{2i}x_{2i-1})|\ i\in[(n+10)/4-s,(n+2)/4]\}$

$=\{(n+2)/2+2i|\ i\in[(n+10)/4-s,(n+2)/4]\}=[n+6-2s,n+2]_2$,

$D=\{f^*(x_{2i}x_{2i+1})|\ i\in[(n+6)/4-s,(n-2)/4]\}$

$=\{(n+4)/2+2i|\ i\in[(n+6)/4-s,(n-2)/4]\}=[n+5-2s,n+1]_2$,

$E=\{f^*(x_{2i}x_{2i-1}),f^*(x_{2i}x_{2i+1})|\ i\in[(n+10)/4,(n+2)/4+s]\}$

$=\{(n+2)/2+2i,(n+4)/2+2i|\ i\in[(n+10)/4,(n+2)/4+s]\}$

$=[n+6,n+3+2s]$,

$F=\{f^*(x_{2i}x_{2i-1})|\ i\in[(n+6)/4+s,n/2]\}$

$=\{(n+4)/2+2i|\ i\in[(n+6)/4+s,n/2]\}=[n+5+2s,(3n+4)/2]_2$,

$G=\{f^*(x_{2i}x_{2i+1})|\ i\in[(n+6)/4+s,(n-2)/2]\}$

$=\{(n+6)/2+2i|\ i\in[(n+6)/4+s,(n-2)/2]\}=[n+6+2s,(3n+2)/2]_2$,

$H=\{f^*(x_{(n+6)/2-2s}y_1),f^*(y_1y_2),f^*(x_{(n+4)/2+2s}y_2),f^*(x_nx_1),$

$f^*(x_{(n+4)/2}x_{(n+2)/2}),f^*(x_{(n+4)/2}x_{(n+6)/2}),f^*(x_{(n+6)/2}x_{(n+8)/2})\}$

$=\{n-2s+4,n+5,n+4+2s,n+3,(3n+6)/2,(3n+8)/2,n+4\}$.

Therefore, $f^*(E(G))=A\cup B\cup C\cup D\cup E\cup F\cup G\cup H=[(n+4)/2,(3n+8)/2]$.□

Theorem 3.5. When $n\equiv 2 \pmod 4$ and $n\geq 10$, the graph $C_n(4s+2;n/2-2s,(n+4)/2+2s;P_3)$ for $1\leq s\leq(n-6)/4$, and the graph $C_n(4s+1;1+2s,n-2s;P_3)$ for $1\leq s\leq(n-2)/8$ are strongly $n/2$-harmonious.

Proof. For the graph $G=C_n(4s+2;n/2-2s,(n+4)/2+2s;P_3)$, we define the function f: $V(G)\to[0,n+2]$ as follows:

$f(y_1)=(3n-2)/4-s$, $f(y_2)=(3n+10)/4+s$,

$f(x_{2i-1})=i-1$ if $i\in[1,n/2]$, $[0,(n-2)/2]$,

$f(x_{2i})=(n-2)/2+i$ if $i\in[1,(n-2)/4-s]$, $[n/2,(3n-6)/4-s]$,

$f(x_{2i})=n/2+i$ if $i\in[(n+2)/4-s,(n+2)/4]$, $[(3n+2)/4-s,(3n+2)/4]$,

$f(x_{2i})=(n+2)/2+i$ if $i\in[(n+6)/4,(n+2)/4+s]$, $[(3n+10)/4,(3n+6)/4+s]$,

$f(x_{2i})=(n+4)/2+i$ if $i\in[(n+6)/4+s,n/2]$, $[(3n+14)/4+s,n+2]$.

It is easy to verify that the f is an injection from $V(G)$ to $[0,n+2]$. By the above definition, we have

$A=\{f^*(x_{2i}x_{2i-1}), f^*(x_{2i}x_{2i+1})|\ i \in [1,(n-2)/4-s]\}$
$=\{(n-4)/2+2i, (n-2)/2+2i|\ i \in [1,(n-2)/4-s]\}=[n/2, n-2-2s]$,
$B=\{f^*(x_{2i}x_{2i-1}), f^*(x_{2i}x_{2i+1})|\ i \in [(n+2)/4-s,(n+2)/4]\}$
$=\{(n-2)/2+2i, n/2+2i|\ i \in [(n+2)/4-s,(n+2)/4]\}=[n-2s, n+1]$,
$C=\{f^*(x_{2i}x_{2i-1}), f^*(x_{2i}x_{2i+1})|\ i \in [(n+6)/4,(n+2)/4+s]\}$
$=\{n/2+2i, (n+2)/2+2i|\ i \in [(n+6)/4,(n+2)/4+s]\}=[n+3, n+2+2s]$,
$D=\{f^*(x_{2i}x_{2i-1})|\ i \in [(n+6)/4+s, n/2]\}$
$=\{(n+2)/2+2i|\ i \in [(n+6)/4+s, n/2]\} =[n+4+2s, (3n+2)/2]_2$,
$E=\{f^*(x_{2i}x_{2i+1})|\ i \in [(n+6)/4+s,(n-2)/2]\}$
$=\{(n+4)/2+2i|\ i \in [(n+6)/4+s,(n-2)/2]\}=[n+5+2s, 3n/2]_2$,
$F=\{f^*(x_{n/2-2s}y_1), f^*(y_1y_2), f^*(x_{(n+4)/2+2s}y_2), f^*(x_nx_1)\}$
$=\{n-2s-1, (3n+4)/2, n+3+2s, n+2\}$.
Therefore, $f^*(E(G))=A \cup B \cup C \cup D \cup E \cup F=[n/2,(3n+4)/2]$.

For the graph $G=C_n(4s+1; 1+2s, n-2s; P_3)$, we define the function f: $V(G) \to [0, n+2]$ as follows:
$f(y_1)=(n+4)/2+s$, $f(y_2)=(n+2)/2-s$,
$f(x_{2i-1})=i-1$ if $i \in [1,(n-2)/4]$, $[0,(n-6)/4]$,
$f(x_{2i-1})=i$ if $i \in [(n+2)/4, n/2-s]$, $[(n+2)/4, n/2-s]$,
$f(x_{2i-1})=i+1$ if $i \in [(n+2)/2-s, n/2]$, $[(n+4)/2-s,(n+2)/2]$,
$f(x_{2i})=(n+2)/2+i$ if $i \in [1,s]$, $[(n+4)/2,(n+2)/2+s]$,
$f(x_{2i})=(n+4)/2+i$ if $i \in [s+1,(n-2)/4]$, $[(n+6)/2+s,(3n+6)/4]$,
$f(x_{2i})=(n+2)/2+i$ if $i \in [(n+6)/4, n/2]$, $[(3n+10)/4, n+1]$,
$f(x_{(n+2)/2})=(n-2)/4$.

It is easy to verify that the f is an injection from $V(G)$ to $[0, n+2]$. By the above definition, we have
$A=\{f^*(x_{2i}x_{2i-1}), f^*(x_{2i}x_{2i+1})|\ i \in [1,s]\}$
$=\{n/2+2i, (n+2)/2+2i|\ i \in [1,s]\}=[(n+4)/2,(n+2)/2+2s]$,
$B=\{f^*(x_{2i}x_{2i-1})|\ i \in [s+1,(n-2)/4]\} \cup \{f^*(x_{2i}x_{2i+1})|\ i \in [s+1,(n-6)/4]\}$
$=\{(n+2)/2+2i|\ i \in [s+1,(n-2)/4]\} \cup \{(n+4)/2+2i|\ i \in [s+1,$
$(n-6)/4]\}=[(n+6)/2+2s, n]$,

$C=\{f^*(x_{2i}x_{2i-1})|\ i \in [(n+6)/4, n/2-s]\}=\{(n+2)/2+2i|\ i \in [(n+6)/4, n/2-s]\} =[n+4,(3n+2)/2-2s]_2,$

$D=\{f^*(x_{2i}x_{2i+1})|\ i \in [(n+6)/4,(n-2)/2-s]\} =\{(n+4)/2+2i|\ i \in [(n+6)/4,(n-2)/2-s]\}=[n+5,3n/2-2s]_2,$

$E=\{f^*(x_{2i}x_{2i-1})|\ i \in [(n+2)/2-s, n/2]\}$
$=\{(n+4)/2+2i|\ i \in [(n+2)/2-s, n/2]\}=[(3n+8)/2-2s,(3n+4)/2]_2,$

$F=\{f^*(x_{2i}x_{2i+1})|\ i \in [n/2-s,(n-2)/2]\}$
$=\{(n+6)/2+2i|\ i \in [n/2-s,(n-2)/2]\}=[(3n+6)/2-2s,(3n+2)/2]_2,$

$G=\{f^*(x_{1+2s}y_1), f^*(y_1y_2), f^*(x_{n-2s}y_2), f^*(x_nx_1), f^*(x_{n/2}x_{(n-2)/2}), f^*(x_{n/2}x_{(n+2)/2}), f^*(x_{(n+2)/2}x_{(n+4)/2})\}$
$=\{(n+4)/2+2s, n+3,(3n+4)/2-2s, n+1, n+2, n/2,(n+2)/2\}.$

Therefore, $f^*(E(G))=A\cup B\cup C\cup D\cup E\cup F\cup G=[n/2,(3n+4)/2]$. □

Theorem 3.6. When $n \equiv 3 \pmod 4$ and $n \geq 7$, the graph $C_n(4s;(n-3)/2,(n-3)/2+4s;P_3)$ for $1 \leq s \leq (n+1)/8$ is strongly $(n+1)/2$-harmonious, and the graph $C_n(4s-1;(n+9)/2-4s,(n+7)/2;P_3)$ for $1 \leq s \leq (n+1)/8$ is strongly $(n+3)/2$-harmonious.

Proof. For the graph $G=C_n(4s;(n-3)/2,(n-3)/2+4s;P_3)$, we define the function f: $V(G) \to [0,n+2]$ as follows:

$f(y_1)=(n+1)/4,\ f(y_2)=(3n+11)/4,$

$f(x_{2i-1})=(n+3)/2-i$ if $i \in [1,(n+1)/4], [(n+5)/4,(n+1)/2],$

$f(x_{2i-1})=n+3-i$ if $i \in [(n+5)/4,(n-3)/4+s], [(3n+15)/4-s,(3n+7)/4],$

$f(x_{2i-1})=n+2-i$ if $i \in [(n+1)/4+s,(n+1)/2], [(n+3)/2,(3n+7)/4-s],$

$f(x_{2i})=n+3-i$ if $i \in [1,(n-3)/4], [(3n+15)/4,n+2],$

$f(x_{2i})=(n-1)/2-i$ if $i \in [(n+1)/4,(n-1)/2], [0,(n-3)/4].$

It is easy to verify that the f is an injection from $V(G)$ to $[0,n+2]$. By the above definition, we have

$A=\{f^*(x_{2i}x_{2i-1}), f^*(x_{2i}x_{2i+1})|\ i \in [1,(n-3)/4]\}$
$=\{(3n+9)/2-2i,(3n+7)/2-2i|\ i \in [1,(n-3)/4]\} =[n+5,(3n+5)/2],$

$B=\{f^*(x_{2i}x_{2i-1})|\ i\in[(n+5)/4,(n-3)/4+s]\}$

$\quad=\{(3n+5)/2-2i|\ i\in[(n+5)/4,(n-3)/4+s]\}=[n+4-2s,n]_2,$

$C=\{f^*(x_{2i}x_{2i+1})|\ i\in[(n+1)/4,(n-7)/4+s]\}$

$\quad=\{(3n+3)/2-2i|\ i\in[(n+1)/4,(n-7)/4+s]\}=[n+5-2s,n+1]_2,$

$D=\{f^*(x_{2i}x_{2i-1})|\ i\in[(n+1)/4+s,(n-1)/2]\}$

$\quad=\{(3n+3)/2-2i|\ i\in[(n+1)/4+s,(n-1)/2]\}=[(n+5)/2,n+1-2s]_2,$

$E=\{f^*(x_{2i}x_{2i+1})|\ i\in[(n-3)/4+s,(n-1)/2]\}$

$\quad=\{(3n+1)/2-2i|\ i\in[(n-3)/4+s,(n-1)/2]\}=[(n+3)/2,n+2-2s]_2,$

$F=\{f^*(x_{(n-3)/2}y_1),f^*(y_1y_2),f^*(x_{(n-3)/2+4s}y_2),f^*(x_nx_1),f^*(x_{(n-1)/2}x_{(n+1)/2})\}$

$\quad=\{n+4,n+3,n+3-2s,n+2,(n+1)/2\}.$

Therefore, $f^*(E(G))=A\cup B\cup C\cup D\cup E\cup F=[(n+1)/2,(3n+5)/2]$.

For the graph $G=C_n(4s-1;(n+9)/2-4s,(n+7)/2;P_3)$, we define the function f: $V(G)\to[0,n+2]$ as follows:

$f(y_1)=(3n+7)/4$, $f(y_2)=(3n+3)/4$,

$f(x_{2i-1})=i-1$ if $i\in[1,(n+5)/4]$, $[0,(n+1)/4]$,

$f(x_{2i-1})=(n+1)/2+i$ if $i\in[(n+9)/4,(n+1)/2]$, $[(3n+11)/4,n+1]$,

$f(x_{2i})=(n-1)/2+i$ if $i\in[2,(n+1)/4]$, $[(n+3)/2,(3n-1)/4]$,

$f(x_{2i})=i$ if $i\in[(n+5)/4,(n+1)/2-s]$, $[(n+5)/4,(n+1)/2-s]$,

$f(x_{2i})=i+1$ if $i\in[(n+3)/2-s,(n-1)/2]$, $[(n+5)/2-s,(n+1)/2]$,

$f(x_2)=n+2$.

It is easy to verify that the f is an injection from $V(G)$ to $[0,n+2]$. By the above definition, we have

$A=\{f^*(x_{2i}x_{2i-1}),f^*(x_{2i}x_{2i+1})|\ i\in[2,(n+1)/4]\}$

$\quad=\{(n-3)/2+2i,(n-1)/2+2i|\ i\in[2,(n+1)/4]\}=[(n+5)/2,n],$

$B=\{f^*(x_{2i}x_{2i-1})|\ i\in[(n+9)/4,(n+1)/2-s]\}$

$\quad=\{(n+1)/2+2i|\ i\in[(n+9)/4,(n+1)/2-s]\}=[n+5,(3n+3)/2-2s]_2,$

$C=\{f^*(x_{2i}x_{2i+1})|\ i\in[(n+5)/4,(n+1)/2-s]\}$

$\quad=\{(n+3)/2+2i|\ i\in[(n+5)/4,(n+1)/2-s]\}=[n+4,(3n+5)/2-2s]_2,$

$D=\{f^*(x_{2i}x_{2i-1}),f^*(x_{2i}x_{2i+1})|\ i\in[(n+3)/2-s,(n-1)/2]\}$

$\quad=\{(n+3)/2+2i,(n+5)/2+2i|\ i\in[(n+3)/2-s,(n-1)/2]\}$

$=[(3n+9)/2-2s,(3n+3)/2]$,

$E=\{f^*(x_{(n+9)/2-4s}y_1), f^*(y_1y_2), f^*(x_{(n+7)/2}y_2), f^*(x_nx_1), f^*(x_1x_2),$
$f^*(x_2x_3), f^*(x_{(n+3)/2}x_{(n+5)/2})\}$
$=\{(3n+7)/2-2s,(3n+5)/2,(3n+7)/2,n+1,n+2,n+3,(n+3)/2\}$.

Therefore, $f^*(E(G))=A\cup B\cup C\cup D\cup E=[(n+3)/2,(3n+7)/2]$. □

Theorem 3.7. When $n\equiv 3 \pmod 4$ and $n\geq 11$, the graph $C_n(4s+2;(n-1)/2-2s,(n+3)/2+2s;P_3)$ for $1\leq s\leq (n-7)/4$ is strongly $(n-1)/2$-harmonious.

Proof. We define the function f: $V(G)\to[0,n+2]$ as follows:
$f(y_1)=(3n-1)/4-s$, $f(y_2)=(3n+7)/4+s$,
$f(x_{2i-1})=i-1$ if $i\in[1,(n+1)/2]$, $[0,(n-1)/2]$,
$f(x_{2i})=(n-1)/2+i$ if $i\in[1,(n-3)/4-s]$, $[(n+1)/2,(3n-5)/4-s]$,
$f(x_{2i})=(n+1)/2+i$ if $i\in[(n+1)/4-s,(n+1)/4+s]$, $[(3n+3)/4-s,(3n+3)/4+s]$,
$f(x_{2i})=(n+3)/2+i$ if $i\in[(n+5)/4+s,(n-1)/2]$, $[(3n+11)/4+s,n+1]$.

It is easy to verify that the f is an injection from $V(G)$ to $[0,n+2]$. By the above definition, we have

$A=\{f^*(x_{2i}x_{2i-1}), f^*(x_{2i}x_{2i+1})|\ i\in[1,(n-3)/4-s]\}$
$=\{(n-3)/2+2i,(n-1)/2+2i|\, i\in[1,(n-3)/4-s]\}=[(n+1)/2,n-2-2s]$,

$B=\{f^*(x_{2i}x_{2i-1}), f^*(x_{2i}x_{2i+1})|\ i\in[(n+1)/4-s,(n+1)/4+s]\}$
$=\{(n-1)/2+2i,(n+1)/2+2i|\ i\in[(n+1)/4-s,(n+1)/4+s]\}$
$=[n-2s,n+1+2s]$,

$C=\{f^*(x_{2i}x_{2i-1}), f^*(x_{2i}x_{2i+1})|\ i\in[(n+5)/4+s,(n-1)/2]\}$
$=\{(n+1)/2+2i,(n+3)/2+2i|\ i\in[(n+5)/4+s,(n-1)/2]\}$
$=[n+3+2s,(3n+1)/2]$,

$D=\{f^*(x_{(n-1)/2-2s}y_1), f^*(y_1y_2), f^*(x_{(n+3)/2+2s}y_2), f^*(x_nx_1)\}$
$=\{n-1-2s,(3n+3)/2,n+2+2s,(n-1)/2\}$.

Therefore, $f^*(E(G))=A\cup B\cup C\cup D=[(n-1)/2,(3n+3)/2]$. □

4 Main theorem

From the sections 2 and 3, we obtain the following theorem.

Theorem When integer $n \geq 6$, the graph $C_n(d; i, j; P_k)$ for k=2, 3 is strongly c-harmonious. Therefore, the graph $C_n(d; i, j; P_k)$ for k=2, 3 is harmonious also.

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