A characterization of the lines external to a hyperbolic quadric in PG(3,q).

Dedicated to Professor Franco Eugeni on the occasion of his 70[th] birthday

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Abstract. In this article, the lines not meeting a hyperbolic quadric in PG(3,q) are characterized by their intersection properties with points and planes.

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1. Introduction and Motivations

Let PG(r, q) denote the projective space of dimension r and order q, where q is a prime power. Let ℓ be a line in PG(3, q) and $x=(x_0,x_1,x_2,x_3)$ and $y=(y_0,y_1,y_2,y_3)$ two points on ℓ. The *Plücker coordinates* of the line ℓ are the determinants $l_{ij}=\begin{vmatrix} x_i & x_j \\ y_i & y_j \end{vmatrix}$, with i, $j\in\{0,1,2,3\}$ and $i<j$. They are not all zero and in number $\binom{4}{2}=6$. It is easy to verify that the Plücker coordinates l_{ij} satisfy the equation $l_{01}l_{23}-l_{02}l_{13}+l_{03}l_{12}=0$. Therefore the lines in PG(3,q) are represented by a hyperbolic quadric H_5 in PG(5,q), see [7]. So, a point $L\in H_5$ represents in PG(5,q) a line ℓ of PG(3,q). Moreover a pencil of lines in PG(3,q), i.e. all the

lines through a point contained in the same plane, is represented in PG(5,q) by a line contained in H_5. Thus, two lines in PG(3,q) meeting in a point are represented by two *collinear* points H_5, i.e. two points such that the line through them is completely contained in H_5. In PG(3,q) a maximal set of lines which pairwise meet in a point is either a star of lines i.e. all the lines through a point, or a ruled plane, i.e. a plane considered as the set of its lines. Therefore, the Klein quadric H_5 has two systems of generating planes, called *greek* and *latin* planes for convenience, see [7], which are maximal subspaces on H_5. A latin plane represents a star of lines and a greek plane represents a ruled plane in PG(3,q). One of the most interesting problem in finite geometry is the combinatorial characterization of a remarkable set of lines as point–set of H_5 having suitable incidence properties with respect to the subspaces of H_5. Let K denote a k–set of H_5, i.e. a set of k points of H_5. We recall that the *d–characters* of K, with respect to d–subspaces of H_5, are the numbers $t_i^d=t_i^d(K)$ of d–subspaces of H_5 meeting K in exactly i points, $0\leq i\leq\theta_d$, where $\theta_d=\frac{q^{d+1}-1}{q-1}$ is the number of points in a d–subspace, $d\in\{1,2\}$. A set K is said to be *of class* $[m_1,m_2,...,m_s]_d$ with respect to d–subspaces of H_5, if any d–subspace of H_5 contains either m_1, or m_2, ..., or m_s points of K, where the m_j are non-negative integers with $0\leq m_1<m_2<...<m_s\leq\theta_d$, i.e. if $t_i^d\neq 0\Rightarrow i\in\{m_1,m_2,...,m_s\}$, see [11]. Moreover a set K of class $[m_1,m_2,...,m_s]_d$ is said to be *of type* $(m_1,m_2,...,m_s)_d$ with respect to d–subspaces of H_5, if any d–subspace of H_5 contains either m_1, or m_2, ..., or m_s points of K, and every values occur, i.e. $i\in\{m_1,m_2,...,m_s\}\Leftrightarrow t_i^d\neq 0$, see [11]. A set of type $(m_1,m_2,...,m_s)_d$ is also called *s character set*. Point k–sets on H_5 can be investigated in terms of their numbers of non zero characters, see [9]. Since in H_5 one character k–set with respect to lines is either the empty set or H_5, see [10], in this paper we consider k–set having at least two character with respect to lines different from zero. In order to give a better picture of the current interest in this type of problem the reader is referred to [1], [2], [3], [4], [5] and [6].
The following results enter into this scheme of things.

Result 1 ([3] R. Di Gennaro, N. Durante and D. Olanda, 2004).- If the order q is odd and K is a k–set in H_5 of type $(0,\frac{q-1}{2},\frac{q+1}{2})_1$ and of type $(0,\frac{q^2-q}{2})_2$, then, $k=(q^2+q+1-m)\frac{q^2-q}{2}$ with $m\in\{q+1,q+2,2q+1\}$, necessarily. Moreover if $m=2q+1$ then K represents the family of external lines to a hyperbolic quadric of PG(3,q).

Result 2 ([3] R. Di Gennaro, N. Durante and D. Olanda, 2004).- If the order $q>2$ is even and K is a k–set in H_5 of type $(0,\frac{q}{2})_1$, then, $k=(q^2+q+1-m)\frac{q^2-q}{2}$ with $m\in\{q+1,q+2,2q+1\}$, necessarily. Moreover if $m=2q+1$ then K represents the family of external lines to a hyperbolic quadric of PG(3,q).

In this paper we give a characterization of the set of points of H_5 which represents the set of lines external to a hyperbolic quadric of PG(3,q) as a set of H_5 of class $[0,a,b]_1$ and of type $(m,n)_2$ with respect to subspaces of H_5. In particular we prove the following

Theorem.- *In H_5 a $\frac{(q^2-q)^2}{2}$ –set having exactly $\frac{(q-1)^2q^3(q^3-3q^2+q+3)}{8}$ pairs of non-collinear points and $(q+1)^2(2q^2+1)$ external lines, of class $[0,a,b]_1$ and of type $(m,n)_2$ represents the set of lines external to a hyperbolic quadric in* PG(3,q).

2. The proof of the Theorem

Suppose that K is a k–set of type $(m,n)_2$ in H_5. Let α denote a latin (greek) plane. By counting in double way the total number of latin (greek) planes, of incident point–planes pairs (P,α) with $P\in K\cap\alpha$, and triples (P,Q,α) with $P,Q\in K\cap\alpha$, we have what are referred to as the *standard equations* on the integers $t_m=t_m^2(K)$ and $t_n=t_n^2(K)$, see [9],

$$(2.1)\qquad \begin{cases} t_m+t_n=q^3+q^2+q+1 \\ mt_m+nt_n=k(q+1) \\ m(m-1)t_m+n(n-1)t_n=k(k-1)-2\tau \end{cases},$$

where τ denotes the number of pairs of non collinear points. Thus, a two character set with respect to latin (greek) planes depends by four parameters k, τ, m and n and a complete classification seems to be extremely difficult, see [1], [6], [8], [10] and [12].

For $k=\frac{(q^2-q)^2}{2}$ and $\tau=\frac{(q-1)^2q^3(q^3-3q^2+q+3)}{8}$ the system of equations (2.1) becomes

$$(2.2)\qquad \begin{cases} t_m+t_n=(q+1)(q^2+1) \\ mt_m+nt_n=q^2(q-1)^2(q+1)/2 \\ m(m-1)t_m+n(n-1)t_n=q^2(q-1)^2(q+1)^2(q-2)/4 \end{cases}.$$

From the first two equations of (2.2), we get

(2.3) $$\begin{cases} t_m = [2n(q^2+1) - q^2(q-1)^2](q+1)/(2n-2m) \\ t_n = [q^2(q-1)^2 - 2m(q^2+1)](q+1)/(2n-2m) \end{cases}.$$

Since $t_n>0$, by the second equation of (2.3) we have that

$$0 \le m < q^2(q-1)^2/(2q^2+2) = (q^2-2q)/2 + q/(q^2+1).$$

Since $q/(q^2+1)<1/2$ we get

(2.4) $$\begin{cases} 0 \le m \le (q^2-2q)/2 & \text{if q is even} \\ 0 \le m \le (q^2-2q-1)/2 & \text{if q is odd} \end{cases}.$$

Firstly, we observe that if $q=2$, then, by (2.4), $m=0$.
Let us suppose that $q\ge3$.
From equations (2.2), we get

(2.5) $$2(q^2+1)mn = q^2(q-1)^2[m+n-q(q-1)/2].$$

Since GCD(q^2, q^2+1)=1 we have that $2mn\equiv0 \pmod{q^2}$.
We claim that $m=0$ and, by (2.5), $n=q(q-1)/2$.
Indeed, if $m>0$, we have the following three possible cases:
1) $m\equiv0 \pmod{q^2}$ and $m>0$.
In this case we have that $m\ge q^2$ which leads, taking into account (2.4), a contradiction.
2) $n\equiv0 \pmod{q^2}$.
Since $0<n\le q^2+q+1$ we have that $n=q^2$, necessarily. By (2.5) we obtain that $m=\dfrac{q(q-1)^2}{2(q+1)}=\dfrac{q(q-3)}{2}+2-\dfrac{2}{q+1}$ which is not an integer, a contradiction.
3) $2mn\equiv0 \pmod{q^2}$, $m\not\equiv0 \pmod{q^2}$ and $n\not\equiv0 \pmod{q^2}$.
Firstly let us consider the case $q=p^h$ with p an odd prime.
Thus $mn\equiv0 \pmod{q^2}$.
We have that $m=ap^s$, $n=bp^t$, $a\not\equiv0 \pmod p$, $b\not\equiv0 \pmod p$, $1\le s\le 2h-1$, $1\le t\le 2h-1$.
Let r denote the minimum between s and t.
If $r\le h$, then from (2.5) we obtain

(2.6) $$2(q^2+1)abp^{s+t} = p^{2h+r}(q-1)^2[ap^{s-r}+bp^{t-r}-p^{h-r}(q-1)/2].$$

So $s+t\ge2h+r$. If $r=s$ then $t\ge2h$, a contradiction. If $r=t$ then $s\ge2h$, a contradiction, too.
If $r\ge h+1$, then from (2.5) we obtain

(2.7) $$2(q^2+1)abp^{s+t} = p^{3h}(q-1)^2[ap^{s-h}+bp^{t-h}-(q-1)/2].$$

Since $r-h\ge1$ we have that $s-h\ge1$ and $t-h\ge1$. So $[ap^{s-h}+bp^{t-h}-(q-1)/2]\not\equiv0 \pmod p$. Since $(q^2+1)ab\not\equiv0 \pmod p$, by (2.7) we get $s+t=3h$. So $mn=abq^3$ and (2.7) becomes

(2.8) $$2(q^2+1)ab=(q-1)^2[ap^{s-h}+bp^{t-h}-(q-1)/2].$$

Equation (2.8) implies that $2(q^2+1)ab\equiv 0 \pmod{(q-1)^2}$.

Since $\mathrm{GCD}(p^{2h}+1, p^h-1)=2$ we have that $ab\equiv 0 \pmod{(q-1)^2/4}$. Hence

(2.9) $$mn=abq^3\geq q^3(q-1)^2/4.$$

By $n\leq q^2+q+1$ and (2.4) in the case q odd we obtain

(2.10) $$mn\leq(q^2+q+1)(q^2-2q-1)/2.$$

From (2.9) and (2.10) we get $q^3(q-1)(q-3)+4q^2+6q+2\leq 0$. Since $q\geq 3$, we have a contradiction.

Now let us consider the case $q=2^h$ with $h\geq 2$.

Thus $mn\equiv 0 \pmod{2^{2h-1}}$.

We have that $m=a2^s$, $n=b2^t$, a odd, b odd, $0\leq s\leq 2h-1$, $0\leq t\leq 2h-1$.

Equation (2.5) becomes

(2.11) $$(2^{2h}+1)ab2^{s+t}=2^{2h-1}(2^h-1)^2[a2^s+b2^t-2^{h-1}(2^h-1)].$$

Let r denote the minimum between s and t.

If $r\leq h-1$, then from (2.11) we obtain

(2.12) $$(2^{2h}+1)ab2^{s+t}=2^{2h-1+r}(2^h-1)^2[a2^{s-r}+b2^{t-r}-2^{h-1-r}(2^h-1)].$$

So $s+t\geq 2h-1+r$.

If $r=t$ then $s\geq 2h-1$ and so $s=2h-1$. Thus we have that $m=a2^s=a2^{2h-1}=aq^2/2>q^2/2-q$, a contradiction.

If $r=s$ then $t\geq 2h-1$ and so $t=2h-1$. Thus we have that $n=b2^t=b2^{2h-1}=bq^2/2$. Since $n\leq q^2+q+1$, $q\geq 4$ and b is an odd integer it is easy to see that $b=1$, necessarily. So $n=q^2/2$. By (2.5) we get $4m=(q-1)^2$ which implies q odd, a contradiction.

If $r\geq h$, then from (2.11) we obtain

(2.13) $$(2^{2h}+1)ab2^{s+t}=2^{3h-2}(2^h-1)^2(a2^{s-h+1}+b2^{t-h+1}-2^h+1).$$

Since $r-h+1\geq 1$ we have that $s-h+1\geq 1$ and $t-h+1\geq 1$. So $a2^{s-h+1}+b2^{t-h+1}-2^h+1$ is an odd integer. Since $(2^{2h}+1)ab$ is an odd integer too, then from (2.13) we get $s+t=3h-2$.

So $mn=ab2^{3h-2}=abq^3/4$ and

(2.14) $$(2^{2h}+1)ab=(2^h-1)^2(a2^{s-h+1}+b2^{t-h+1}-2^h+1).$$

Equation (2.14) implies that $(2^{2h}+1)ab\equiv 0 \pmod{(2^h-1)^2}$.

Since $\mathrm{GCD}(2^{2h}+1, 2^h-1)=1$ we have that $ab\equiv 0 \pmod{(2^h-1)^2}$. Hence

(2.15) $$mn=abq^3/4\geq q^3(2^h-1)^2/4=q^3(q-1)^2/4.$$

By $n\leq q^2+q+1$ and (2.4), in the case q even, we obtain

(2.16) $$mn\leq(q^2+q+1)(q^2-2q)/2.$$

From (2.15) and (2.16) we have that $q^3(q-4)+3q^2+2q+2\leq 0$. Since $q\geq 4$ we have a contradiction.

Therefore $m=0$ and $n=q(q-1)/2$. Thus, K is a $\frac{(q^2-q)^2}{2}$–set of type $(0,\frac{q^2-q}{2})_2$ in H_5.

Now suppose that K is a k–set of class $[0, a, b]_1$ in H_5. Let ℓ denote a line of H_5. By counting in double way the total number of lines, of incident point–planes pairs (P, ℓ) with $P\in K\cap\ell$, and triples (P, Q, ℓ) with $P,Q\in K\cap\ell$, we have what are referred to as the *standard equations* on the integers $t_0=t_0^1(K)$, $t_a=t_a^1(K)$ and $t_b=t_b^1(K)$, see [9],

$$(2.17)\qquad \begin{cases} t_0+t_a+t_b=(q^3+q^2+q+1)(q^2+q+1) \\ at_a+bt_b=k(q+1)^2 \\ a(a-1)t_a+b(b-1)t_b=k(k-1)-2\tau \end{cases},$$

where τ denotes the number of pairs of non collinear points. For $k=\frac{(q^2-q)^2}{2}$ and $\tau=\frac{(q-1)^2q^3(q^3-3q^2+q+3)}{8}$ the system of equations (2.17) becomes

$$(2.18)\qquad \begin{cases} t_0+t_a+t_b=(q+1)(q^2+1)(q^2+q+1) \\ at_a+bt_b=\frac{(q^2-q)^2}{2}(q+1)^2 \\ a(a-1)t_a+b(b-1)t_b=\frac{(q^2-q)^2}{2}\frac{q+1}{2}(q^2-q-2) \end{cases}.$$

The $\frac{(q^2-q)^2}{2}$–set K has at least two character, with respect to lines of H_5, different from zero because in H_5 one character k–set with respect to lines is either the empty set or H_5, see [10]. We claim

If the set K is a two character set with respect to lines of the Klein quadric H_5, then the order q is even. In this case K represents the set of lines external to a hyperbolic quadric in PG(3,q).

Indeed, if K is a two character set with respect to lines of H_5, then K is of type $(0, a)_1$ because it is of type $(0, \frac{q^2-q}{2})_2$ in H_5. The system of equations (2.18) becomes

$$(2.19) \qquad \begin{cases} t_0 + t_a = (q+1)(q^2+1)(q^2+q+1) \\ at_a = \frac{(q^2-q)^2}{2}(q+1)^2 \\ a(a-1)t_a = \frac{(q^2-q)^2}{2}\frac{q+1}{2}(q^2-q-2) \end{cases}.$$

From the last two equations of (2.19), we get

$$a-1 = \frac{1}{2}\frac{q^2-q-2}{q+1} = \frac{q-2}{2} = \frac{q}{2}-1,$$

which implies that the order q is even and $a=\frac{q}{2}$. Therefore K is a $\frac{(q^2-q)^2}{2}$-set is of type $(0, \frac{q}{2})_1$ in H_5 and the assertion follows taking into account the Result 2.

Now suppose that the order q is odd, then K is a three character set with respect to lines of H_5.
Since $t_0=(q+1)^2(2q^2+1)$, equations (2.18) become

$$(2.20) \qquad \begin{cases} t_a + t_b = q^3(q-1)(q+1) \\ at_a + bt_b = q^2(q-1)^2(q+1)^2/2 \\ a(a-1)t_a + b(b-1)t_b = q^2(q-1)^2(q+1)^2(q-2)/4 \end{cases}.$$

From equations (2.20) we obtain

$$(2.21) \qquad 4abq=(q-1)(q+1)(2a+2b-q),$$

and also

$$(2.22) \qquad (2b-q-1)(q^2-1-2aq)+(q+1)(q-1-2a)=0.$$

Since $a\le(q-1)/2$ implies $(q^2-1-2aq)\ge(q+1)>0$, we have that equality (2.22) holds if and only if

$$2b-q-1=0 \text{ and } q-1-2a=0.$$

Hence we get $b=(q+1)/2$ and $a=(q-1)/2$, necessarily.

So, K is a $\frac{(q^2-q)^2}{2}$-set of type $(0, \frac{q-1}{2}, \frac{q+1}{2})_1$ and of type $(0, \frac{q^2-q}{2})_2$ in H_5.
Then, by the Result 1, K represents the family of external lines to a hyperbolic quadric of PG(3,q).

Thus, the Theorem is completely proved.

3. Conclusion

In this paper we give a characterization of the point-subset of the Klein quadric H_5 which represents the set of lines external to a hyperbolic quadric in PG(3,q) by incidence properties with respect to the subspaces of H_5. The arguments leading to these results are combinatorial arguments based largely on the integrality of the parameters at stake.

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