The Minimal Limit Point of the Third Largest Laplacian Eigenvalues of Graphs

Yarong Wu[a,b]; Jinlong Shu[a,c]*; Yuan Hong

[a]Department of Mathematics, East China Normal University, shanghai, 200241, China
[b]Department of Mathematics, Shanghai Maritime University, Shanghai, 200135, China
[c]Key Laboratory of Geographic Information Science Ministry of Education, East China Normal University, Shanghai, 200241, China

Abstract

Let G be a simple connected graph with n vertices. Denoted by $L(G)$ the Laplacian matrix of G. In this paper, we present a sequence of graphs $\{G_n\}$ with $\lim_{n\to\infty} \mu_3(G_n) \doteq 1.5550$ by investigating the eigenvalues of the line graphs of $\{G_n\}$. Moreover, we prove that the limit is the minimal limit point of the third largest Laplacian eigenvalues of graphs.

AMS classification: 05C50 ,05C35
Key words: Laplacian spectra; limit point; forbidden subgraph; diameter.

1. Introduction

In this paper, all graphs are undirected connected graphs without loops or multiple edges.

Let $G = (V, E)$ denote a simple graph with n vertices and d_u denote the degree of the vertex u. The matrix $A = A(G)$ is the adjacent matrix of G. Its eigenvalues will be called the eigenvalues of graph G. They will be denoted by $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$, always enumerated in nonincreasing

*Corresponding author, who was partially supported by the National Natural Science Foundation of China (No. 11071078 and No. 11075057), Open Research Funding Program of LGISEM and Shanghai Leading Academic Discipline Project (B407)
E-mail address: wuyarongl @ yahoo.com.cn (Yarong Wu); jlshu @ math.ecnu.edu.cn (Jinlong Shu)

order. We shall use $\lambda_k(G)$ to denote the kth largest eigenvalue of graph G, and use $P(G;\lambda)$ to denote the characteristic polynomial of $A(G)$.

Let $D = D(G)$ denote the diagonal matrix of vertex degrees. Then $L(G) = D(G) - A(G)$ is called the Laplacian matrix of G (for example, see [8]). Obviously, $L(G)$ is a positive semidefinite matrix. Its eigenvalues will be called the Laplacian eigenvalues of graph G. Similarly, they will be denoted by $\mu_1(G) \geq \mu_2(G) \geq \cdots \geq \mu_n(G) = 0$, always enumerated in nonincreasing order. We shall use $\mu_k(G)$ to denote the kth largest Laplacian eigenvalue of graph G, and use $\Phi(G;\mu)$ to denote the Laplacian characteristic polynomial of $L(G)$.

The study of the limit points of the eigenvalues of a sequence of graphs was initiated by Alan Hoffman in [6]. Many important results about the limit points of the adjacency matrix $A(G)$ of graphs can be found in ([6],[3],[12]). Since the algebraic properties of the Laplacian matrix are useful in researching the structural properties of a graph G, the properties of the corresponding Laplacian matrix are very important (see [8],[9]). Thereby, people began to consider the limit points of the Laplacian eigenvalues ([5] etc.) and study their minimal or second minimal limit points. Particularly, Petrović et al. [10] characterized all connected bipartite graphs with $\mu_3(G) \leq 2$ and mentioned the result can be of interest in the investigation on the photoelectron spectroscopy of organic compounds. So, we study the minimal limit points of the third Laplacian eigenvalues of graphs in this paper. We present a sequence of graphs $\{G_n\}$ with $\lim\limits_{n\to\infty} \mu_3(G_n) \doteq 1.5550$ by investigating the eigenvalues of the line graphs of $\{G_n\}$ in Section 2. Moreover, we prove that the limit is the minimal limit point of the third largest Laplacian eigenvalues of graphs in Section 3. Now, we present the definition of the limit point as follows.

A real number r is said to be a limit point of the kth largest Laplacian eigenvalues of graphs if there exists a sequence of graphs $\{G_n\}$ such that $\mu_k(G_i) \neq \mu_k(G_j)$ $(i \neq j)$ and $\lim\limits_{n\to\infty} \mu_k(G_n) = r$.

Throughout this paper, We use $l_k(\lambda, G_n)$ to denote the limit point of the kth largest eigenvalues $\lambda_k(G_n)$ of graphs $\{G_n\}$, and use $l_k(\mu, G_n)$ to denote the limit point of the kth largest Laplacian eigenvalues $\mu_k(G_n)$ of graphs $\{G_n\}$.

The terminology and notations not defined are standard and can be found in [2].

2. The Limit Points of $\{G_n\}$

In this section, we focus on the limit point of $\{G_n\}$. We firstly present some well known results which will be used often in our proof.

Lemma 2.1 [11] *Let G be a simple graph, $v \in V(G)$, $C(v)$ be the set of all circuits including v. Then*

$$P(G;\lambda) = \lambda P(G-v;\lambda) - \sum_{u\sim v} P(G-u-v;\lambda) - 2\sum_{z\in C(v)} P(G-V(z);\lambda).$$

Where $P(G;\lambda)$ is the characteristic polynomial of the adjacent matrix of G in λ.

The *line graph* $l(G)$ of a graph G is the graph whose vertices correspond to the edges of G with two vertices being adjacent if and only if the corresponding edges in G have a vertex in common. Obviously, if a tree has n vertices and $n-1$ edges, its line graph has $n-1$ vertices.

Lemma 2.2 [8] *Let G be a bipartite graph. Then $L(G) = D(G) - A(G)$ and $2I + A(l(G))$ have the same non-zero eigenvalues.*

Lemma 2.3 [13] *Let $F(x;n)$ be a polynomial with variable x and parameter n and let $x_1(n)$ be the root of $F(x;n) = 0$. If $F(x;n) = g_1(x;n) + ng_2(x;n)$ and $\lim_{n\to\infty} x_1(n) = x_1$, then $g_2(x_1;n) = 0$.*

Lemma 2.4 [4] *Let G be a simple graph with order n. If H is a subgraph (not necessarily an induced subgraph) of G with order $m \le n$, then for $i = 1, 2, \cdots, m$, we have $\mu_i(G) \ge \mu_i(H)$.*

Lemma 2.5 [7] *Let $\Delta(G)$ denote the maximum degree of graph G, then $\lambda_1(G) \ge \sqrt{\Delta(G)}$.*

Let $\{G_n\}$ be a sequence of graphs with n vertices in Fig.1. It is easy to get that, for different number n, each graph G_n of $\{G_n\}$ is a different tree with n vertices. Let K_{n-2}^2 denote the graphs which can be obtained by joining one vertex of the complete graph K_{n-2} to another vertex of the path P_2 with a new edge. Obviously, the line graph of G_n is K_{n-3}^2 with order $n-1$.

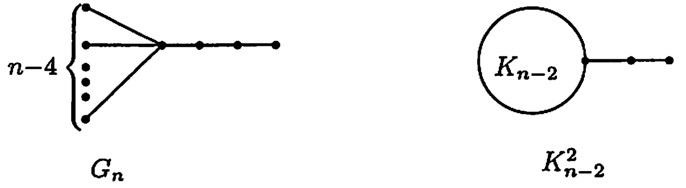


Fig.1.

By Lemma 2.2, the Laplacian eigenvalues $\mu_i(G_n)$ $(i = 1, 2, \cdots, n)$ are $\lambda_i(l(G_n)) + 2$ $(i = 1, 2, \cdots, n-1)$ and 0. So, we can investigate $\mu_i(G_n)$ of G_n by investigating $\lambda_i(l(G_n)) + 2$, that is $\lambda_i(K_{n-3}^2) + 2$.

Theorem 2.1 *Let $\{K_{n-2}^2\}$ be the sequence of graphs in Fig.1. Then* (i) $l_2(\lambda, K_{n-2}^2) \doteq 1.2470$, (ii) $l_3(\lambda, K_{n-2}^2) \doteq -0.4450$, (iii) $l_n(\lambda, K_{n-2}^2) \doteq -1.8019$, *which are the roots of the equation $-\lambda^3 - \lambda^2 + 2\lambda + 1 = 0$.*

Proof. By Lemma 2.1, we can get that

$$P(K_{n-2}^2;\lambda)=(\lambda+1)^{n-4}(\lambda^4+5\lambda^3+2\lambda^2-10\lambda-4+n(-\lambda^3-\lambda^2+2\lambda+1)) \quad (1)$$

As $\Delta(K_{n-2}^2)=n-2$, by Lemma 2.5, we have $\lambda_1(K_{n-2}^2)\geq\sqrt{n-2}$. (2)
Let $f(\lambda)=\lambda^4+5\lambda^3+2\lambda^2-10\lambda-4+n(-\lambda^3-\lambda^2+2\lambda+1)$. Then

$$\begin{cases} f(1)>0 & , \quad f(2)<0 \\ f(-0.4)>0 & , \quad f(-0.5)<0 \\ f(-2)>0 & , \quad f(-1.5)<0 \end{cases} \quad (3)$$

The above (3) implies that there are three real roots of $f(\lambda)=0$ in the intervals $(1,2)$, $(-0.5,-0.4)$ and $(-2,-1.5)$. From (1)(2)(3), we can get the distribution of the n spectra $\lambda_i(K_{n-2}^2)$ $(i=1,2,\cdots,n)$:

$$\lambda_1(K_{n-2}^2)\geq\sqrt{n-2};\qquad 1<\lambda_2(K_{n-2}^2)<2;\qquad -0.5<\lambda_3(K_{n-2}^2)<-0.4;$$

$$\lambda_4(K_{n-2}^2)=\cdots=\lambda_{n-1}(K_{n-1}^2)=-1;\qquad -2<\lambda_n(K_{n-2}^2)<-1.5.$$

If $i<j$, K_{i-2}^2 is a subgraph of K_{j-2}^2. We can get that $\lambda_2(K_{i-2}^2)<\lambda_2(K_{j-2}^2)$. Namely, $\lambda_2(K_{n-2}^2)$ is a strictly increasing function with regard to n and $\lambda_2(K_{n-2}^2)<2$. So, there must be a real number $l_2(\lambda)$ satisfying $\lim\limits_{n\to\infty}\lambda_2(K_{n-2}^2)=l_2(\lambda)$. By Lemma 2.3, $l_2(\lambda)$ is the largest root of $-\lambda^3-\lambda^2+2\lambda+1=0$. A more accurate calculation yields $l_2(\lambda)\doteq 1.2470$.

By a similar argument, if $i<j$, both $\lambda_3(K_{n-2}^2)$ and $\lambda_n(K_{n-2}^2)$ are strictly increasing functions with regard to n, too. They are satisfying $-0.5<\lambda_3(K_{i-2}^2)<\lambda_3(K_{j-2}^2)<-0.4$ and $-2<\lambda_n(K_{i-2}^2)<\lambda_n(K_{j-2}^2)<-1.5$. Suppose $\lim\limits_{n\to\infty}\lambda_3(K_{n-2}^2)=l_3(\lambda)$ and $\lim\limits_{n\to\infty}\lambda_n(K_{n-2}^2)=l_n(\lambda)$. By Lemma 2.3, we have that $l_3(\lambda)$ and $l_n(\lambda)$ are the second largest and the minimal roots of $-\lambda^3-\lambda^2+2\lambda+1=0$, respectively. An accurate calculation yields $l_3(\lambda)\doteq-0.4450$ and $l_n(\lambda)\doteq-1.8019$.

That completes the proof. □

Theorem 2.2 *Let $\{G_n\}$ be the sequence of graphs in Fig.1. Then* (i) $l_2(\mu,G_n)\doteq 3.2470$, (ii) $l_3(\mu,G_n)\doteq 1.5550$, (iii) $l_{n-1}(\mu,G_n)\doteq 0.1981$, *which are the roots of the equation* $-\mu^3+5\mu^2-6\mu+1=0$.

Proof. It is easy to see that, for different n, G_n is a different tree with n vertices and its corresponding line graph is K_{n-3}^2 with $n-1$ vertices. By Lemma 2.2, the Laplacian eigenvalues $\mu_i(G_n)$ of G_n $(i=1,2,\cdots,n)$ are $\lambda_i(l(G_n))+2=\lambda_i(K_{n-3}^2)+2$ $(i=1,2,\cdots,n-1)$ and 0. By Theorem 2.1, we can get the distribution of the n spectra $\mu_i(G_n)$ $(i=1,2,\cdots,n)$:

$$\mu_1(G_n)>\sqrt{n-3}+2;\qquad 3<\mu_2(G_n)<4;\qquad 1.5<\mu_3(G_n)<1.6;$$

$$\mu_4(G_n)=\cdots=\mu_{n-2}(G_n)=1;\qquad 0<\mu_{n-1}(G_n)<0.5;\qquad \mu_n(G_n)=0$$

If $i < j, G_i$ is a subgraph of G_j. We can get that $\mu_2(G_i) < \mu_2(G_j)$. Namely, $\mu_2(G_n)$ is a strictly increasing function with regard to n and $\mu_2(G_n) < 4$. So, there must be a real number $l_2(\mu)$ satisfying $\lim_{n\to\infty} \mu_2(G_n) = l_2(\mu)$. As $l_2(\lambda)$ is the largest root of $-\lambda^3 - \lambda^2 + 2\lambda + 1 = 0$, $l_2(\mu)$ is a root of the $-\mu^3 + 5\mu^2 - 6\mu + 1 = 0$. We can get that $\mu_2(G_n) \doteq 3.2470$

Similarly, we have that $1.5 < \mu_3(G_i) < \mu_3(G_j) < 1.6$ and $0 < \mu_{n-1}(G_i) < \mu_{n-1}(G_j) < 0.5, (i < j)$. Suppose $\lim_{n\to\infty} \mu_3(G_n) = l_3(\mu)$ and $\lim_{n\to\infty} \mu_{n-1}(G_n) = l_{n-1}(\mu)$.

Then $l_3(\mu)$ and $l_{n-1}(\mu)$ are the second largest and the minimal roots of $-\mu^3 + 5\mu^2 - 6\mu + 1 = 0$, respectively. An accurate calculation yields $l_3(\mu) \doteq 1.5550$ and $l_{n-1}(\mu) \doteq 0.1981$.

That completes the proof. □

3. The Minimal Limit Point of $\mu_3(G)$

Now, we will prove that $l_3(\mu) = \lim_{n\to\infty} \mu_3(G_n) \doteq 1.5550$ ($\{G_n\}$ in Fig.1) is the minimal limit point of the third Laplacian spectra by finding the forbidden subgraphs.

Suppose there exists a sequence of graphs $\{G'_n\}$ with the property:

$$\mu_3(G'_i) \neq \mu_3(G'_j) \quad (i \neq j) \quad \text{and} \quad \lim_{n\to\infty} \mu_3(G'_n) < l_3(\mu) \doteq 1.5550 \tag{4}$$

If the graph G'_n of $\{G'_n\}$ has some subgraph H with $\mu_3(H) > l_3(\mu)$, by Lemma 2.4, we have that $\mu_3(G'_n) \geq \mu_3(H) > l_3(\mu)$. It implies that $\lim_{n\to\infty} \mu_3(G'_n) > l_3(\mu)$. We can say that such subgraph H violate the property (4) and call H the forbidden subgraph in G'_n or $\{G'_n\}$.

By a direct calculation, we have the following results.

Lemma 3.1 *The following graphs in Fig.2 are forbidden subgraphs in* $\{G'_n\}$.

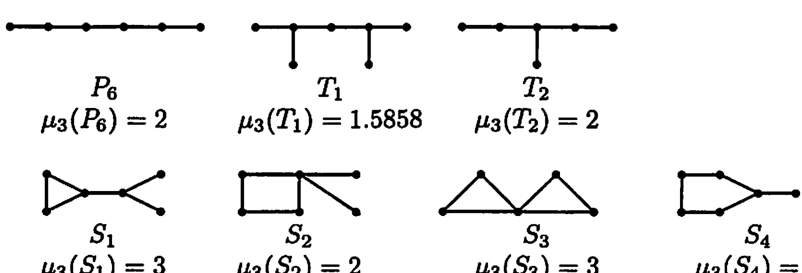
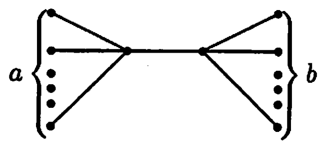
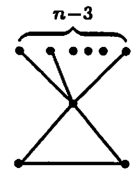


Fig.2.

Let $T(a, b)$ denote the tree having exactly two non-pendant vertices which are adjacent, with one of these two vertices connected to a pendants (vertices with degree 1) and the other one to b pendants. In particular, the order of $T(a,b)$ is $n = a+b+2$. Let $K_{1,n-1}$ denote a star graph with n vertices. Let S_n^3 denote the graph which is obtained by joining one vertex of the cycle C_3 to other $(n-3)$ isolated vertices. $T(a,b)$ and S_n^3 are showed in Fig.3. Then we will prove that $\{T(a,b)\}, \{K_{1,n-1}\}$ and $\{S_n^3\}$ are three sequences of graphs with $\mu_3(G_i) = \mu_3(G_j) = 1, (i \neq j)$.



$T(a,b)$



S_n^3

Fig.3.

Lemma 3.2 $\mu_3(T(a,b)) = \mu_3(K_{1,n-1}) = \mu_3(S_n^3) = 1$.

Proof. (1) The Laplacian characteristic polynomial of $T(a,b)$ is $\Phi(T(a,b);\mu) = \mid \mu I - L(T(a,b)) \mid = \mu(\mu-1)^{n-4}(\mu^3-(n+2)\mu^2+(2n+ab+1)\mu-n)$ Let $f(\mu) = \mu^3-(n+2)\mu^2+(2n+ab+1)\mu-n$, we have

$$\begin{cases} f(0) = -n < 0 \quad , \quad f(1) = ab > 0 \\ f(a+2) = -b < 0 \quad , \quad f(2) = a(b-1)+b(a-1) > 0 \end{cases}$$

Without loss of generality, we suppose that $b \geq a$. As n is sufficiently large, it is easy to get that $b = n-a-2$ is sufficiently large too. Namely, $\mu_1(T(a,b)) \geq \Delta_1 + 1 \geq b+2$ (see in [1]). So the distribution of the n Laplacian eigenvalues of $T(a,b)$ are: $\mu_1 \geq b+2$, $\mu_2 \in (2, a+2)$, $\mu_3 = \cdots = \mu_{n-2} = 1$, $\mu_{n-1} \in (0,1)$, $\mu_n = 0$.

(2) The Laplacian characteristic polynomial of $K_{1,n-1}$ is $\Phi(K_{1,n-1};\mu) = \mid \mu I - L(K_{1,n-1}) \mid = \mu(\mu-1)^{n-2}(\mu-n)$. Obviously, $\mu_3(K_{1,n-1}) = 1$.

(3) The Laplacian characteristic polynomial of S_n^3 is $\Phi(S_n^3;\mu) = \mid \mu I - L(S_n^3) \mid = \mu(\mu-1)^{n-3}(\mu-3)(\mu-n)$. Then $\mu_1 = n, \mu_2 = 3, \mu_3 = \cdots = \mu_{n-1} = 1$ and $\mu_n = 0$.

That completes the proof. □

Let $d(u,v)$ denote the distance of the two vertices u and v of graph G and let $diam(G) = max\{d(u,v)|u,v \in V(G)\}$ denote the diameter of G.

Theorem 3.1 *Let $\{G_n\}$ be a sequence of graphs with n vertices in Fig.1. Then $l_3(\mu) = \lim_{n\to\infty} \mu_3(G_n) \doteq 1.5550$ is the minimal limit point of the third Laplacian spectra.*

Proof. Suppose there exists a sequence of graphs $\{G'_n\}$ such that $\mu_3(G'_i) \neq \mu_3(G'_j)$ $(i \neq j)$ and $\lim_{n\to\infty} \mu_3(G'_n) = l_3(\mu, G'_n) \leq l_3(\mu, G_n) \doteq 1.5550$. We denote the diameter of the graph G'_n of $\{G'_n\}$ by $d(G'_n)$. By Lemma 3.1, P_6 is a forbidden graph in $\{G'_n\}$. We can get that $d(G'_n) \leq 4$. Otherwise, P_6

is a subgraph of G'_n. So, we can only consider the following four cases with $d(G'_n) \leq 4$.

Case 1. $d(G'_n) = 4$.

By deleting some edges of the graphs G'_n of $\{G'_n\}$, G'_n has at least one spanning tree T'_n as its subgraph. Obviously, $d(T'_n) \geq d(G'_n)$. If $d(T'_n) > 4$, there is a forbidden graph P_6 in T'_n. So, in this case, we only need to prove that $\lim_{n\to\infty} \mu_3(G_n) \doteq 1.5550$ is the minimal limit point of the third Laplacian spectra of all the sequences of trees $\{T'_n\}$ with $d(T'_n) = 4$.

We first construct a sequence of graphs denoted by $\{T(a_1, a_2, \cdots, a_c)\}$ (showed in Fig.4). $\{T(a_1, a_2, \cdots, a_c)\}$ can be obtained by joining all the non-pendant vertex v_i of the star graph K_{1,a_i} $(i = 1, 2, \cdots, c)$ to another common isolated vertex v_0 with c new edges v_iv_0 $(i = 1, 2, \cdots, c)$, and $a_i = 0, 1, \cdots; c = 0, 1, \cdots$. The order of $\{T(a_1, a_2, \cdots, a_c)\}$ is $n = \sum_{i=1}^{c} a_i + c + 1$.

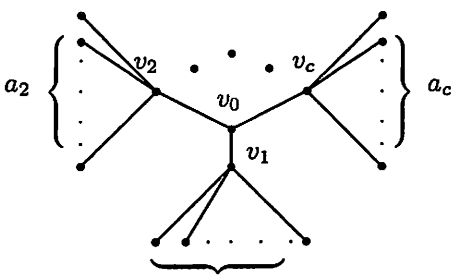


Fig.4: $T(a_1, a_2, \cdots, a_c)$

Obviously, all the trees T'_n with $d(T'_n) = 4$, must be one of the subgraphs of $T(a_1, a_2, \cdots, a_c)$ with some certain different value a_i and c. Now, we prove that there does not exist $\{T'_n\}$ with $d(T'_n) = 4$ satisfying: $\lim_{n\to\infty} \mu_3(T'_n) < l_3(\mu, G_n) \doteq 1.5550$ by discussing the parameters a_i and c of $\{T(a_1, a_2, \cdots, a_c)\}$.

Firstly, we can get that $c \leq 2$. If $c \geq 3$, then each graph of $\{T(a_1, a_2, \cdots, a_c)\}$ contains the forbidden subgraph T_2 which is showed in Fig.2. It is contradictory to $\lim_{n\to\infty} \mu_3(T_n) < l_3(\mu) \doteq 1.5550$.

Secondly, there is at most one $a_i \geq 2$. If there are two $a_i \geq 2$, then the trees $\{T(a_1, a_2, \cdots, a_c)\}$ have forbidden subgraph T_1 in Fig.2. It is a contradiction to $\lim_{n\to\infty} \mu_3(T'_n) < l_3(\mu) \doteq 1.5550$, too. Without loss of generality, we suppose that $a_1 \geq 2$.

By the two discussions above, we can conclude that T'_n with $d(T'_n) = 4$ must be the subgraphs of $T(a_1, a_2, \cdots, a_c)$ with $c = 2$ and $a_1 \geq 2$, $a_2 = 1$. Namely, $\{T'_n\}$ is $\{G_n\}$ (in Fig.1). So, $\lim_{n\to\infty} \mu_3(G_n) \doteq 1.5550$ is the minimal limit point of the third Laplacian spectra of all the sequences of graphs $\{G'_n\}$ with $d(G'_n) = 4$.

Case 2. $d(G'_n) = 3$.

If $\{G'_n\}$ is a sequence of trees with $d(G'_n) = 3$, by Lemma 3.2, $\mu_3(G'_n) = 1$. It is contradictory to the definition of the limit point. If $\{G'_n\}$ with $d(G'_n) = 3$ is not a sequence of trees. Since P_6 is a forbidden graph in $\{G'_n\}$, the length of the cycle in G'_n must be less than 6. So, there is at least one of the forbidden subgraphs $\{S_1, S_2, S_4\}$ (showed in in Fig.2) in G'_n or G'_n have subgraphs G_n. By Lemma 2.4 and 3.1, there does not exist $\{G'_n\}$ with $d(G'_n) = 3$ satisfying $\lim_{n\to\infty} \mu_3(G'_n) < l_3(\mu, G_n) \doteq 1.5550$.

Case 3. $d(G'_n) = 2$.

By a similar argument as the one in Case 2, the length of the cycle in G'_n must less than 6. If $G'_n \cong S_n^3$ or $K_{1,n-1}$, by Lemma 3.2, $\mu_3(G'_n) = 1$. It is contradictory to the definition of the limit point. If $G'_n \ncong S_n^3$ and $G'_n \ncong K_{1,n-1}$, then there must be at least one of the forbidden graphs $\{S_2, S_3\}$ (showed in Fig.2) in G'_n. By Lemma 2.4, there does not exist $\{G'_n\}$ with $d(G'_n) = 2$ satisfying $\lim_{n\to\infty} \mu_3(G'_n) < l_3(\mu) \doteq 1.5550$.

Case 4. $d(G'_n) = 1$.

In this case, as $d(G'_n) = 1$, we have $G'_n \cong K_n$. It is easy to see that G'_n contains the forbidden graph P_6 for $n \geq 6$. By Lemma 3.1, there does not exist $\{G'_n\}$ with $d(G'_n) = 1$ satisfying $\lim_{n\to\infty} \mu_3(G'_n) < l_3(\mu) \doteq 1.5550$.

From the dicussion of Cases 1-4, we can conclude that $\lim_{n\to\infty} \mu_3(G_n) = l_3(\mu) \doteq 1.5550$ is the minimal limit point of the third Laplacian eigenvalues ($\{G_n\}$ in Fig.1).

That completes the proof. □

Acknowledgements

The authors are much indebted to the referees for their careful reading the manuscript and many valuable comments to improve this paper.

References

[1] W. N. Anderson and T.D. Morley , Eigenvalues of the Laplacian of a graph, Linear and Multilinear Algebra, 18 (1985) 141-145.

[2] J. A. Bondy and U. S. R. Murty , Graph Theory with Applications. The Macmillan Press LTD (1976).

[3] M. Doob, The limit points of eigenvalues of graphs, Linear Algebra and its Application, 114/115 (1989) 659-662.

[4] R. Grone, R. Merris and V. Sunder, The Laplacian spetrum of a graph, SIAM J.Matrix Anal. Appl. 11(1990) 218-238.

[5] J. M. Guo, The limit points of Laplacian spectra of graphs, Linear Algebra and its Application,362 (2003) 121-128.

[6] A. J. Hoffman, On limit points on spectral radii of non-negative symmetric integral matrices, in:Y.Alavi(Ed.)et al., Lecture notes in mathematics, vol.303, Springer-Verlag, Berlin (1972) 165-172.

[7] L. Lovász and J.Pelinkán, On the eigenvalues of tress, Periodica Math. Hung. 3(1973) 175-182.

[8] R.Merris, Laplacian matrices of graphs:a surver, Linear Algebra and its Application, 197/198 (1994) 143-176.

[9] B. Mohar, Some applications of Laplace eigenvalues of graphs, IN : G.Hahn, G. Sabidussi (Eds), Graph symmetry, Kluwer Academic Press, Dordrecht (1997) 225-275.

[10] M. Petrović, I. Gutman and M. Lepovtić, On bipartite graphs with small number of Laplacian eigenvalues greater than two and three, Linear and Multilinear Algebra, 47 (2000) 205-215.

[11] A. Schwenk, Computing the characteristic polynomial of a graph, IN:Graphs and Combinatorics (R.Bary, F.Harary eds.), Lecture notes in mathematics, Berlin,Springer Verlag, 406 (1974) 27-261.

[12] J.B.Shearer, On the distribution of the maximum eigenvalue of graphs, Linear Algebra and its Application,114/115 (1984) 17-20.

[13] J. L. Shu, Bounds on Eigenvalues of Graphs (in Chinese), *Doctoral thesis*, East China Normal University (1999).