SOME RESULTS ON STRONG GENERALIZED NEIGHBORHOOD SYSTEMS

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ABSTRACT. The aim of our paper is to introduce generalized neighborhood bases and $gn-T_2$-spaces. (ψ,ψ')-continuity, sequentially (ψ,ψ')-continuity and ψ-convergency are investigated on strong generalized first countable spaces, and also two results about ψ-convergency on $gn-T_2$-spaces are given.

1. Introduction

Császár introduced the concepts of generalized topological spaces and generalized neighborhood systems in [2]. He also introduced continuous functions on both systems, and studied characterizations of such functions in [2]. Furthermore, he presented separation axioms T_0, T_1, T_2, S_1, S_2 by replacing open sets with more general ones in [3]. In [4], Min obtained some properties of generalized topological spaces and (g,g')-continuity by means of strong generalized interior operators. He also introduced the concept of (ψ,ψ')- open map, gn-continuity and gn-open map. He established strong generalized neighborhood systems, and obtained sg_ψ-open and sg_ψ-closed sets in [5]. Sequentially (ψ,ψ')-continuity and ψ-convergency are introduced in [1]. In this paper, we introduce the notions of generalized neighborhood bases and define strong generalized first countable spaces by means of generalized neighborhood bases. Also, we give the definition of $gn-T_2$-spaces by using generalized neighborhood systems. Then, we investigate ψ-convergency, (ψ,ψ')-continuity and sequentially (ψ,ψ')-continuity on strong generalized first countable spaces. Finally, we give two results about ψ-convergency on $gn-T_2$-spaces.

2. Preliminaries

We now recall some concepts and notations defined by Császár in [2].

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Let X be a nonempty set and g be a collection of subsets of X. Then g is called a generalized topology (briefly GT) on X if and only if $\emptyset \in g$ and $G_i \in g$ for $i \in I \neq \emptyset$ implies $G = \bigcup_{i \in I} G_i \in g$. The elements of g are called g-open sets and their complements are called g-closed sets.

Let X be a nonempty set and $\psi : X \to \wp(\wp(X))$ satisfy $x \in V$ for $V \in \psi(x)$. Then $V \in \psi(x)$ is called a generalized neighborhood (briefly GN) of $x \in X$ and ψ is called a generalized neighborhood system (briefly GNS) on X. The collection of all GNS's on X is denoted by $\Psi(X)$. If ψ is generalized neighborhood system on X and $A \subset X$, then

$$\imath_\psi(A) = \{x \in A : \text{ there exists } V \in \psi(x) \text{ such that } V \subset A\}$$

and

$$\gamma_\psi(A) = \{x \in X : V \cap A \neq \emptyset \text{ for all } V \in \psi(x)\}$$

Let ψ be a GNS on X and $G \in g$ iff $G \subset X$ satisfies: if $x \in G$ then there is $V \in \psi(x)$ such that $V \subset G$. This GT g is shown as $g = g_\psi$. If g is a GT on X, then there is a $\psi \in \Psi(X)$ satisfying $g = g_\psi$ and $V \in g$ for $V \in \psi(x), x \in X$. This GNS ψ is shown as $\psi = \psi_g$. If g is a GT on X and $A \subset X$, then $i_g A$ is the largest subset of A belonging to g and $c_g A$ is the smallest g-closed set containing A.

Let g and g' be generalized topologies on X and Y, respectively. Then a function $f : X \to Y$ is (g, g')-continuous iff $G' \in g'$ implies that $f^{-1}(G') \in g$. Let ψ and ψ' be generalized neighborhood systems on X and Y, respectively. Then a function $f : X \to Y$ is (ψ, ψ')-continuous iff given $x \in X$ and $V' \in \psi'(f(x))$, there is $V \in \psi(x)$ such that $f(V) \subset V'$.

Let X be a non-empty set and $\psi \in \Psi(X)$. A sequence (x_n) is said to be ψ-converge to a point x in X [1] if (x_n) is eventually in every set V in $\psi(x)$ and denoted by $(x_n) \to^\psi x$. Let ψ and ψ' be generalized neighborhood systems on X and Y, respectively. Then a function $f : X \to Y$ is said to be sequentially (ψ, ψ')-continuous [1] if for each sequence (x_n) in X ψ-converging to x, then $(f(x_n))$ in Y ψ'-converges to $f(x)$. Also f is said to be gn-continuous [4] if $f^{-1}(A)$ is in $\psi(x)$ for every $A \in \psi'(f(x))$.

3. Some Results on (ψ, ψ')-continuity, Sequentially (ψ, ψ')- continuity and ψ-convergency

THEOREM 3.1. *Let ψ and ψ' be generalized neighborhood systems on X and Y, respectively. If a function $f : X \to Y$ is (ψ, ψ')-continuous, then it is also sequentially (ψ, ψ')- continuous.*

PROOF. Assume that f is (ψ, ψ')-continuous and $(x_n) \to^\psi x$ such that (x_n) is a sequence in X and $x \in X$. Given a generalized neighborhood V' of $f(x)$, then there exist $V \in \psi(x)$ such that $f(V) \subset V'$ by the hypothesis. Since (x_n) ψ-converges to x, there exist $n_0 \in \mathbb{N}$ such that each $n \geq n_0$ implies $(x_n) \in V$. Hence, $f(x_n)$ is eventually in V'. □

REMARK 3.1. Since every gn-continuous function is (ψ, ψ')-continuous [4], every gn-continuous function is sequentially (ψ, ψ')- continuous.

DEFINITION 3.1. Let $\varepsilon : X \to \wp(\wp(X))$ satisfy $x \in E$ for $E \in \varepsilon(x)$ and $\varepsilon(x) \subset \psi(x)$ where ψ is generalized neighborhood system on X. Then $\varepsilon(x)$ is called a generalized neighborhood base (briefly GNB) of $x \in X$ if for every $V \in \psi(x)$ there exists $E \in \varepsilon(x)$ such that $E \subset V$.

REMARK 3.2. Let g be GT on X and $\psi_g(x)$ be a GNS of $x \in X$ which is generated by g. Then, there exists $\varepsilon(x)$ such that $\psi_g(x) = \varepsilon(x)$.

DEFINITION 3.2. [5] Let $\psi : X \to \wp(\wp(X))$. Then ψ is called a strong generalized neighborhood system on X if it satisfies the following:

(1) $x \in V$ for $V \in \psi(x)$;
(2) for $U, V \in \psi(x)$, $V \cap U \in \psi(x)$.

Then the pair (X, ψ) is called a strong generalized neighborhood space (briefly SGNS) on X. Then $V \in \psi(x)$ is called a strong generalized neighborhood of $x \in X$.

$A \subset X$ is called an sg_ψ-open set if for each $x \in A$, there is $V \in \psi(x)$ such that $V \subset A$. The complements of sg_ψ-open sets are called sg_ψ-closed sets. Also, A is sg_ψ-open iff $\iota_\psi(A) = A$.

DEFINITION 3.3. Let ψ be a strong generalized neighborhood system on X. If for each point in X has countable GNB, then (X, ψ) is said to be strong generalized first countable space.

EXAMPLE 3.1. Let $X = \mathbb{R}$ and $\psi(x) = \{(a_i, \infty) | a_i \in \mathbb{R}\}$ for $x \in \mathbb{R}$ such that $a_i < x - \frac{1}{n}, n \in \mathbb{N}$. (X, ψ) is strong generalized first countable space since $\varepsilon(x) = \{(x - \frac{1}{n}, \infty) | n \in \mathbb{N}\}$ is countable generalized neighborhood base of $x \in \mathbb{R}$.

THEOREM 3.2. *If (X, ψ) be a strong generalized first countable space, then for each point in X has countable GNB as $\{V_n\}_{n \in \mathbb{N}}$ such that $V_1 \supset V_2 \supset \dots \supset V_n \supset \dots$.*

PROOF. Suppose that (X, ψ) is strong generalized first countable space. There is a countable GNB as $\varepsilon(x) = \{E_n\}_{n \in \mathbb{N}}$ for each $x \in X$. If we take $V_1 = E_1, V_2 = E_1 \cap E_2, \dots, V_n = E_1 \cap E_2 \cap \dots \cap E_n$, then $V_1 \supset V_2 \supset \dots \supset V_n \supset \dots$. Hence, we have $\{V_n\}_{n \in \mathbb{N}}$ is a countable GNB of $x \in X$. □

The following example shows that if (X, ψ) is not a strong generalized first countable space, then Theorem 3.2 is not always true.

EXAMPLE 3.2. Let $X = \{a, b, c\}$, $\psi(a) = \{X, \{a, b\}, \{a, c\}\}$, $\psi(b) = \{X, \{a, b\}, \{b, c\}\}$ and $\psi(c) = \{X, \{a, c\}, \{b, c\}\}$. (X, ψ) is not a strong generalized first countable space since ψ is not strong generalized neighborhood system on X. Then, we have $\varepsilon(a) = \{\{a, b\}, \{a, c\}\}$ and $\varepsilon(a) = \{X, \{a, b\}, \{a, c\}\}$. Hence, X does not have countable GNB as $\{E_n\}_{n \in \mathbb{N}}$ such that $E_1 \supset E_2 \supset \dots \supset E_n \supset \dots$ for a.

COROLLARY 3.1. *Let $f : X \to Y$ be a function between strong generalized first countable space (X, ψ) and (Y, ψ') where ψ' is generalized neighborhood system on Y. Then f is (ψ, ψ') -continuous function if and only if it is sequentially (ψ, ψ')- continuous.*

PROOF. Necessity. This is an immediate consequence of the Theorem 3.1.
Sufficiency. Assume that f is sequentially (ψ, ψ')-continuous but not (ψ, ψ')-continuous. We have for each $V \in \psi(x)$ and $x \in X$ there exist $V' \in \psi'(f(x))$ such that $V \not\subset f^{-1}(V')$. By the hypothesis, there exist $\varepsilon(x) = \{V_n\}_{n\in\mathbb{N}}$ countable GNB for $x \in X$ such that $V_1 \supset V_2 \supset ... \supset V_n \supset ....$ So, we can take $V = V_n$ and we obtain $V_n \not\subset f^{-1}(V')$. Hence, for every $n \in \mathbb{N}$, there exist $(x_n) \in V_n$ such that $(x_n) \notin f^{-1}(V')$ which implies $f(x_n) \notin V'$. Thus, (x_n) ψ-converges to x but $f(x_n)$ does not ψ'-converge to $f(x)$. This is a contradiction.

□

The following example shows that if (X, ψ) is not a strong generalized first countable space, then the converse of Corollary 3.1 is not always true.

EXAMPLE 3.3. Let $X = \{a, b, c\}$. Consider to GNS's ψ and ϕ on X defined as $\psi(a) = \{\{a, b\}, \{a, c\}\}$, $\psi(b) = \{\{a, b\}, \{b, c\}\}$, $\psi(c) = \{\{a, c\}, \{b, c\}\}$, $\phi(a) = \{\{a\}\}$, $\phi(b) = \{\{b\}\}$, $\phi(c) = \{\{c\}\}$. Let $f : (X, \psi) \to (X, \phi)$ be a function defined by $f(x) = x$, for $x \in X$. Hence f is sequentially (ψ, ϕ)-continuous but not (ψ, ϕ)-continuous.

THEOREM 3.3. *Let ψ be a generalized neighborhood system on X, $A \subset X$ and $x \in X$. If the sequence (x_n) contained in A ψ-converges to x, then $x \in \gamma_\psi A$.*

PROOF. It is obvious. □

LEMMA 3.1. [2] *If $\psi \in \Psi_g(X)$ for GT $g = g_\psi$ on X, then $\gamma_\psi = c_\psi$.*

The following Corollary 3.2 follows from Theorem 3.3 and Lemma 3.1.

COROLLARY 3.2. *Let $\psi \in \Psi_g(X)$ for GT $g = g_\psi$ on X. If $(x_n) \to^\psi x$ and $(x_n) \subset A$, then $x \in c_\psi A$.*

COROLLARY 3.3. *Let (X, ψ) is a strong generalized first countable space, $A \subset X$ and $x \in X$. Then, the sequence (x_n) contained in A ψ-converges to x if and only if $x \in \gamma_\psi A$.*

PROOF. Necessity. This is an immediate consequence of the Theorem 3.3.
Sufficiency. Let $x \in \gamma_\psi A$, $A \subset X$ and $x \in X$. We have $V \cap A \neq \emptyset$ for all $V \in \psi(x)$. Since (X, ψ) is a strong generalized first countable space, there exist $\varepsilon(x) = \{E_n\}_{n\in\mathbb{N}}$ countable GNB for $x \in X$ such that $E_1 \supset E_2 \supset ... \supset E_n \supset ....$ We have $E_n \cap A \neq \emptyset$ for each $n \in \mathbb{N}$, so we can pick $(x_n) \in E_n \cap A$. Hence, $(x_n) \in E_n \subset V$ and $(x_n) \in A$ for every $n \in \mathbb{N}$. Finally, $(x_n) \to^\psi x$. □

THEOREM 3.4. *Let (X, ψ) be a strong generalized first countable space. $A \subset X$ is sg_ψ-closed if and only if whenever there exists a sequence consisting of elements of A ψ-converging to x, then $x \in A$.*

PROOF. Necessity. Let $(x_n) \subset A$ and (x_n) ψ-converges to x for $x \in X$. By Theorem 3.3, we have $x \in \gamma_\psi A$. Since $A \subset X$ is sg_ψ-closed, we obtain $x \in \gamma_\psi A = A$.

Sufficiency. Suppose that $x \in \gamma_\psi A$. We have $(x_n) \subset A$ and (x_n) ψ-converges to x from Corollary 3.3. By the hypothesis, we obtain $x \in A$. Thus, we have $\gamma_\psi A \subset A$. Since $A \subset \gamma_\psi A$, A is sg_ψ-closed. □

THEOREM 3.5. *Let (X, ψ) be a strong generalized first countable space. $A \subset X$ is sg_ψ-open if and only if each sequence which ψ-converges to x in A is eventually in A.*

PROOF. Necessity. It is obvious from the definition of sg_ψ-open sets and ψ-convergency.

Sufficiency. Assume that $(x_n) \to^\psi x$, $x \in A$ and (x_n) is eventually in A but A is not sg_ψ-open. We have $X - A$ is not sg_ψ-closed. Hence, there exist a point x such that $x \in \gamma_\psi(X - A)$ but $x \notin X - A$. Thus, we obtain $V \cap (X - A) \neq \emptyset$ for all $V \in \psi(x)$. Since (X, ψ) is strong generalized first countable space, we have $V_n \cap (X - A) \neq \emptyset$ for countable GNB $\varepsilon(x) = \{V_n\}_{n \in \mathbb{N}}$ such that $V_1 \supset V_2 \supset \ldots \supset V_n \supset \ldots$.. We can construct the sequence (x_n) in $V_n \cap (X - A) \neq \emptyset$ for each $n \in \mathbb{N}$, then we have $(x_n) \in (X - A)$. This is a contradiction. □

DEFINITION 3.4. [3] Assume $\mu \subset \wp(X)$. (T_2) $x, y \in X, x \neq y$ imply the existence of $K, K' \in \mu$ such that $x \in K, y \in K'$ and $K \cap K' = \emptyset$. Then we will call (X, μ) is T_2- space.

DEFINITION 3.5. Let ψ be a GNS on X. Then (X, ψ) is said to be gn-T_2-space if each $x, y \in X$, $x \neq y$ imply the existence of $V \in \psi(x)$ and $V' \in \psi(y)$ such that $V \cap V' = \emptyset$.

REMARK 3.3. Let ψ be a GNS on X and $\bigcup_{x \in X} \psi(x) \subset \mu \subset \wp(X)$. If (X, ψ) is gn-T_2-space, then (X, μ) is T_2-space.

If we don't take $\bigcup_{x \in X} \psi(x) \subset \mu \subset \wp(X)$, then Remark 3.3 is not true, in general. We can easily see that the following example.

EXAMPLE 3.4.

a) Let $X = \{a, b, c, d\}$, $\psi(a) = \{\{a, b\}, \{a, c\}\}$, $\psi(b) = \{\{b, d\}, \{b, c\}\}$, $\psi(c) = \{\{c\}\}$, $\psi(d) = \{\{d\}\}$ and $\mu = \{\{a\}, \{a, c\}, \{a, d\}\}$. For $x, y \in X, x \neq y$ imply the existence of $V \in \psi(x)$ and $V' \in \psi(y)$ such that $V \cap V' = \emptyset$. Thus, (X, ψ) is gn-T_2-space but (X, μ) is not T_2-space.
b) Let $X = \mathbb{R}$ and for $x \in \mathbb{R}$, $V_x = (x - \varepsilon, x + \varepsilon)$ where $\psi(x)$ is composed of all sets V_x such that $x \in V_x$ for $x \in \mathbb{R}$. Consider the $\mu = \{(n, +\infty) | n \in \mathbb{N}\}$. We have $\bigcup_{x \in X} \psi(x) \not\subset \mu$. Also there exist $V \in \psi(x)$ and $V' \in \psi(y)$ for $x, y \in X$, $x \neq y$ such that $V \cap V' = \emptyset$. Thus, (X, ψ) is gn-T_2-space but (X, μ) is not T_2-space.

THEOREM 3.6. *Let ψ be a GNS on X and $(x_n) \subset X$. If (X, ψ) be a gn-T_2-space, then (x_n) ψ-converges to one point in X.*

PROOF. Assume that (X,ψ) is a a gn-T_2-space and (x_n) ψ-converges to both x and y. Since (x_n) ψ-converges to x, it is eventually in every set U in $\psi(x)$ and since (x_n) ψ-converges to y, it is eventually in every set V in $\psi(y)$. This implies (x_n) is eventually in $U \cap V$. Hence, we have $U \cap V \neq \emptyset$. This is a contradiction. □

COROLLARY 3.4. *Let (X,ψ) be a strong generalized first countable space and $(x_n) \subset X$. (X,ψ) is gn-T_2-space if and only if (x_n) ψ-converges to one point in X.*

PROOF. Necessity. This is immediate consequence of Theorem 3.6.

Sufficiency. Assume that (x_n) ψ-converges to one point in X and (X,ψ) is not a gn-T_2-space. Then, there exist $x, y \in X$, $x \neq y$ for every $V \in \psi(x)$ and $V' \in \psi(y)$ such that $V \cap V' \neq \emptyset$. Since (X,ψ) is a strong generalized first countable space, there are two countable generalized neighborhood bases $\varepsilon(x) = \{E_n\}_{n\in\mathbb{N}}$ and $\varepsilon(y) = \{E'_n\}_{n\in\mathbb{N}}$ such that $E_1 \supset E_2 \supset ... \supset E_n \supset ...$ and $E'_1 \supset E'_2 \supset ... \supset E'_n \supset ...$ for x and y, respectively. Hence, we can pick $(x_n) \in E_n \cap E'_n$. Consequently, (x_n) ψ-converges to both x and y. This is a contradiction. □

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