Norm–Based Approximation in E-Convex Multi-objective Programming

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Abstract: This paper addresses the problem of capturing nondominated points on non-convex Pareto frontiers, which are encountered in E-convex multi-objective optimization problems. We define nondecreasing map T which transfer non-convex Pareto frontier to convex Pareto frontier. An algorithm to find a piecewise linear approximation of the nondominated set of convex Pareto frontier are applied. Finally, the inverse map of T is used to get non-convex Pareto frontier.

Keyword: Approximation, Nondominated points, E-convex multiobjective, Block norms.

1 Introduction

In multi-objective programming, several conflicting and non-commensurate objective functions have to be optimized over a feasible set determined by constraint functions. Due to the conflicting nature of the objectives, a unique feasible solution optimizing all the objectives does not exist. Based on the commonly used Pareto concept of optimality, one has to deal with a rather large or infinite number of efficient solutions.

Since there are infinitely many efficient solutions, an approximated description of the solution set becomes an appealing alternative. The approximation algorithms proposed in this paper follow earlier researches initiated by Schandl [3] and continued by Schandl et al. [4]. The approximation comes in the form of a polyhedral distance measure that is being constructed successively during the execution of the algorithm. The measure is being utilized both to evaluate the quality of the approximation and to generate additionally nondominated solutions. For convex problems, the approximation measure is defined as a polyhedral gauge. The concept of a gauge cannot be carried over to the non-convex case due to the lack of convexity, so that Schandl [4] use the Tchebycheff method to search for a candidate in the interior of the approximation which is unnecessary in the $R^k_{\geqq}$ -convex case because there cannot be a nondominated point in the interior of the approximation. But this method has disadvantage of performing some additional calculations, hence, we shall present an alternative method to the Tchebycheff method by defining a nondecreasing map T which transfer non-convex Pareto frontier to convex Pareto frontier and then applying the gauge method of $R^k_{\geqq}$ -convex case.

The outline of the paper is as follows: In the next section, we mention some mathematical preliminaries that have an important role not only in traditional programming but also in multi-objective optimization. In section 3, we state the E-convex multi-objective programming problem and extend E-convex functions to the so called cone E-convex functions and derive some results about it. The oblique norms are defined and a theoretical basis for the approximation algorithm is discussed in section 4. Approximation approach for $R^k_{\geqq}$ -convex set of feasible criterion vectors are presented in section 5. Finally, approximation approach developed in section 5 is applied to an E-convex multi-objective optimization problem [8] in section 6.

2 Mathematical Preliminaries

To facilitate further discussions, the following notation is used throughout thesis. Let$u, v \in R^n$ be two vectors.

1. We denote components of vectors by subscripts and enumerate vectors by superscripts.

2. $u < v$ denotes $u_i < v_i$ for all $i = 1, 2, ..., n$. $u \leq v$ denotes $u_i < v_i$ for all $i = 1, 2, ..., n$, but $u \neq v$. $u \leqq v$ allows equality. The symbols $<, \leq, \leqq$ are used accordingly.

3. Let $R^n_{\geqq} = \{x \in R^n : x \geqq 0\}$. If $S \subseteq R^n$, then $S_{\geqq} = S \bigcap R^n_{\geqq}$. The sets $R^n_{\geq}, R^n_{>}, S_{\geq}$ and $S_{>}$ are defined accordingly.

In the following, we recall some general definitions and notations.

Definition 1 *(Cone and convex cone) [2] A subset M of R^n is called a cone if $\lambda x \in M$ whenever $x \in M$ and $\lambda > 0$. Moreover, a cone M is said to be a convex cone when it is also convex.*

Definition 2 *(Cone Convexity) [2] Given a set M and a convex cone D in R^n, M is said to be D-convex if M+D is a convex set.*

Definition 3 *(Cone convex function) [2] Let M be a convex set in R^n, f be a function from M into R^k, and a convex cone in R^k, Then f is said to be D-convex if for any $x^1, x^2 \in M$ and for any $\lambda \in [0, 1]$,*

$$\lambda f(x^1) + (1 - \lambda)f(x^2) - f(\lambda x^1 + (1 - \lambda)x^2) \in D.$$

Definition 4 *(E-convex Set) [7] A set$M \subseteq R^n$ is said to be an E-convex set with respect to an operator$E : R^n \to R^n$ if $\lambda E(x) + (1 - \lambda)E(y) \in M$ for each$x, y \in M$, and $0 \leqq \lambda \leqq 1$.*

Definition 5 *(E-convex Function) [7] A real valued function $f : M \subseteq R^n \to R$ is said to be an E-convex function, with to an operator$E : R^n \to R^n$ on M if M is an E-convex set and, for each$x, y \in M$, and $0 \leqq \lambda \leqq 1$,*

$$f(\lambda E(x) + (1 - \lambda)E(y)) \leqq \lambda f(Ex) + (1 - \lambda)f(Ey).$$

Definition 6 *(Domination structure) [2] For each $z \in Z \subset R^k$, we define the set of domination factors*

$$\mathrm{D(z)} = \{\mathrm{d} \in \mathrm{R^k}: \mathrm{z} \succ \mathrm{z} + \mathrm{d}\} \bigcup \{0\}.$$

This means that deviation $d \in D(z)$ from z is less preferred to the original z. Then the point to set map D from Z to R^k clearly represent the given preference order. We call D the domination structure.

Definition 7 *(Nondominated set) [2] Given a set Z inR^k and a and a domination structure D (.), the set of nondominated elements is defined by $N(Z, D) = \{\bar{z} \in Z : \text{ther is no } z \neq \bar{z} \in Z \text{ s.t. } \bar{z} \in z + D(z)\}$, and is called the nondominated set.*

3 Problem Formulation

Let $E : R^n \to R^n$ be a mapping, $f : R^n \to R^k$ and $g : R^n \to R^m$ are E-convex functions on R^n. A multi-objective E-convex programming problem is formulated as follows:

$$(P) \qquad \begin{array}{l} Min\ f(x), \\ subiect\ to \\ x \in M = \{x \in R^n : g(x) \leqq 0\}. \end{array}$$

In the objective space R^k, for problem (P), the set of all feasible criterion vectors is as follows: $Z = f(M) = \{z : z = f(x), x \in M\}$.

The most fundamental kind of efficient solution is obtained when $D = R^k_{\geqq}$ and is usually called a Pareto solution or Noninferior solution.

Definition 8 *(efficient solution) [2] A point $x^* \in M$ for (P) is be an efficient solution to the problem (P) if there is no$x \in M$ such that such that $f(x) \leq f(x^*)$. define the set of all nondominated points N and the set of all efficient points F of (P) as follows:*

$$N = \{z \in Z : \nexists\, z' \text{ s.t. } z' \leq z\},$$

and

$$F = \{x \in M : f(x) \in N\}.$$

Definition 9 *(Geoffrion's proper efficiency)[2] A point $\bar{z} \in N$ is called properly nondominated, if there exists a scalar $q > 0$ such that for each i, $i = 1, 2, ..., k$, and each $z \in Z$ satisfying $z^i > \bar{z}^i$, there exists at least one $j \neq i$ with $z^j > \bar{z}^j$ and $\frac{z^i - \bar{z}^i}{\bar{z}^j - z^j} \leq q$. Otherwise $\bar{z} \in N$ is called improperly nondominated. The set of all properly points is denoted byN_p.*

Now, we extend the concept of *E-convexity* to *the cone E-convexity* which enables us to deal with E-convex multi-objective optimization problems.

Definition 10 *(Cone E-convex Function) Let $E : R^n \to R^n$ be a M be an E-convex set in R^n, f be a function from M intoR^k, and D be a convex cone inR^k, Then f is said to be D-E-convex with respect to E if for any $x^1, x^2 \in M$ and for any$\lambda \in [0,1]$,*

$$\lambda f(Ex^1) + (1-\lambda) f(Ex^2) - f(\lambda Ex^1 + (1-\lambda)Ex^2) \in D.$$

Remark 1 *Let $E : R^n \to R^n$ be a mapping. A function f is E-convex with respect to E if and only if f is $R^k_{\geqq}$ -E-convex with respect to E.*

Proposition 1 *Let $E : R^n \to R^n$ be a mapping, $M \subseteq R^n$, $E(M) \subseteq M$ be a convex set, and D be a convex cone inR^k. If a function $f : R^n \to R^k$ is D-E-convex with respect to E, then the set $(f \circ E)M$ is D-convex set.*

Proof. Let x, y be any two points in the set$(f \circ E)M + D$, then there exist$z^1, z^2 \in M$, and$d^1, d^2 \in D$ such that

$$x = (f \circ E)z^1 + d^1, y = (f \circ E)z^2 + d^2.$$

For $\lambda \in [0,1]$, we have from, D-E-convexity of f and convexity of $E(M)$,

$$\begin{aligned}
\lambda x + (1-\lambda)y &= \lambda[(f \circ E)z^1 + d^1] + (1-\lambda)[(f \circ E)z^2 + d^2] \\
&= \lambda d^1 + (1-\lambda)d^2 + \lambda(f \circ E)z^1 + (1-\lambda)(f \circ E)z^2 \\
&+ f(\lambda Ez^1 + (1-\lambda)Ez^2) - f(\lambda Ez^1 + (1-\lambda)Ez^2)
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \lambda x + (1-\lambda)y &= f(\lambda Ez^1 + (1-\lambda)Ez^2 + \lambda(f \circ E)z^1) + \{\lambda d^1 + (1-\lambda)d^2 \\
&+ (1-\lambda)(f \circ E)z^2 - f(\lambda Ez^1 + (1-\lambda)Ez^2)\} \\
&\in f(EM) + D.
\end{aligned}$$

Therefore $(f \circ E)M + D$ is convex set and hence $(f \circ E)M$ is D-convex set.

Proposition 2 *Let $E : R^n \to R^n$ be a mapping, $M \subseteq R^n$, $E(M) \subseteq M$ be a convex set, and $f = (f_1, f_2, ..., f_k)$ be a vector function from M into R^k. The function f is $R^k_{\geqq}$ -E-convex with respect to E, if and only if each f_i is E-convex, and in this case the set $(f \circ E)M$ is $R^k_{\geqq}$ -convex set.*

Proof. Follows from Proposition 1.

We assume that the set Z is $R^k_{\geqq}$ -closed and that we can find $u \in R^k$ so that $u + Z \subseteq R^k_{\geqq}$. The point $\hat{z} \in R^k$ with $\hat{z}_i = \min\{f_i(x) : x \in M\} - \varepsilon_i$, $i = 1, 2, ..., k$ is called the ideal (utopia) criterion vector, where the components of $\varepsilon = (\varepsilon_1, \varepsilon_2, ..., \varepsilon_k) \in R^k$ are small positive numbers. The point $z^{\times} \in R^2$ with $z_i^{\times} = \min\{\bar{z}_i : \bar{z}_j = \min_{z \in Z} z_j, j \neq i\}$, $i = 1, 2$ is called the nadir point. This definition definition cannot be easily generalized to more than two dimensions.

Define the set $Z^* = \{\hat{z}^i, i = 1, 2, ..., k\}$. Then the *ith* component, $i = 1, 2, ..., k$ of the default reference point is defined as$z_i^{\times} = \max\{z_i : z \in Z^*\}$, $i = 1, 2, ..., k$.

This is a possible generalization of the nadir point concept, can be calculated relatively easily and also provides the utopia point

4 Oblique Norms in Multi-objective Programming

The concept of oblique norms was introduced in Schandl et al. [5] and Schandl [3]. Since oblique norms can be viewed as a special class of block norms, we first review some basic definitions about block norms and more general, polyhedral gauges. Then oblique norms are discussed in the context of multi-objective programming.

Definition 11 *[4] Let B be a convex and compact set in R^k containing the origin in its interior and let $x \in R^k$. The gauge γ of x with respect to B is then defined as*

$$\gamma(x) = \min\{\lambda \geq 0 : x \in \lambda B\}.$$

Given a gaugeγ, the set$B = \{x \in R^k : \gamma(x) \leq 1\}$ is called called *unit ball* or *sublevel set* of level 1.

Definition 12 *[4] If the set B is a convex polytope, then γ is called a polyhedral gauge and is sometimes referred to as γ_k. Let $ext(B) = \{v^1, v^2, ..., v^k\}$ be the set of extreme points of B; $v^1, v^2, ..., v^k$ are called fundamental vectors. The half-lines$d^1, d^2, ..., d^k$ starting at the origin and passing through the extreme points$v^1, v^2, ..., v^k$ are called are called fundamental directions.*

If B is symmetric with respect to the origin, then it is called a *block norm.*

Definition 13 *The fundamental vectors defined by the extreme points of a facet of B span a fundamental cone. The cone spanned by the fundamental vectors vi and vj are referred to as C (vi, vj).*

If z is in a fundamental cone C of polyhedral gauge γ then one needs to consider only the fundamental vectors generating this cone to calculate the gauge of z. This result was proven in Schandl [4] for the multi-objective case.

Theorem 1 *[4] Let γ be a polyhedral gauge with unit ball $B \subseteq R^k$. Let$\bar{z} \in C$ where C is the fundamental cone generated by $v^1, v^2, ..., v^l$, $l \geq k$. Let $\bar{z} = \sum_{i=1}^{l} \lambda_i v^i$ be a representation of $\bar{z}$ in terms of $v^1, v^2, ..., v^l$. Then $\gamma(\bar{z}) = \sum_{i=1}^{l} \lambda_i$.*

Not that all representations$\bar{z} = \sum_{i=1}^{l} \lambda_i v^i$ can be used to calculate$\gamma(\bar{z})$, even combinations where one or more$\lambda_i' s$ are negative which is only possible if$l > k$.

If C is generated by k fundamental vectors though, the representation of$\bar{z}$ in term $v^1, v^2, ..., v^k$ is unique and all corresponding$\lambda_i's$ all corresponding $\lambda_i's$ are nonnegative.

For the definition of oblique norm we additionally need the concept of sets. Let$u \in R^k$. The reflection set of u is defined as

$$R(u) = \{w \in R^k : |w_i| = |u_i| \quad \forall i = 1, 2, ..., k\}.$$

Definition 14 *A block norm γ with a unit ball B is called oblique if*
$(i)\gamma(w) = \gamma(u) \quad \forall\, w \in R(u), \quad u \in R^k$, *and*
$(ii)(z - R^k_{\geqq}) \bigcap R^k_{geqq} \bigcap \partial B = \{z\} \quad \forall z \in (\partial B)_{\geqq}$.

Observe that an oblique norm is a block norm where no facet of the unit ball is parallel to any coordinate axis. Moreover, the structure of the norm's unit ball is the same in each orthant of the coordinate system.This property is convenient for the generation of nondominated solutions of (P) since they may only occur in $\hat{z} + R^k_{\geqq}$. An example of an oblique norm in R^k is given in Figure 1.

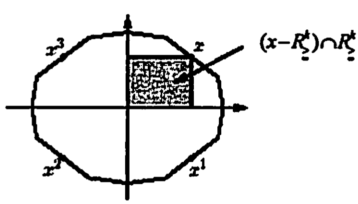

Figure 1:

Schandl et al. [4] show that oblique norms are effective tools to generate nondominated solutions of general multi-objective programs. In particular, he examines the relationship between properly nondominated solutions of problems with an $R^k_{\geqq}$ -convex feasible set Z, and optimal solutions of their scalarization by means of an oblique norm. The following two theorems justify the application of oblique norm for the generation of nondominated solutions.

Theorem 2 *[4] Assume without loss of generality that $0 \in Z + R^k_{\geqq}$. Let γ be an oblique norm with the unit ball B. If $\bar{z} \in R^k$ is a solution of*

$$max \;\; \gamma(z) \qquad s.t. \qquad z \in -R^k_{\geqq} \bigcap Z. \qquad\qquad (P_\gamma)$$

Then $\bar{z}$ is nondominated.

Unfortunately, we cannot guarantee to find all nondominated points using an oblique norm with its unit ball's center in $Z + R^k_{\geqq}$ in the general setting of

Theorem 2. Therefore the next Theorem applies only to problems with an $R^k_{\geqq}$ -convex feasible set Z.

Theorem 3 *[4] Let $Z \subseteq R^k$ be $R^k_{\geqq}$ -convex and assume without loss of generality that $0 \in Z + R^k_{\geqq}$. Let $\bar{z}$ be properly nondominated with $\bar{z} \in -R^k_{\geqq} \bigcap N_p$. Then there exists an oblique norm γ so that $\bar{z}$ solves the problem (P_γ)*

$$max \quad \gamma(z) \qquad s.t. \qquad z \in -R^k_{\geqq} \bigcap Z.$$

5 Approximation Approach for $R^k_{\geqq}$ -Convex

For multi-objective programs with an $R^k_{\geqq}$ -convex feasible set Z, an approximation algorithm based on Theorem 2 and 3 can be designed that utilizes oblique norms for the generation of nondominated solutions. To keep explanations straight-forward, the general idea of this approach will be outlined using a bicriteria example problem. The approximation process is started by choosing a reference point $z^0 \in Z + R^k_{\geqq}$ and defining $z^0 - R^k_{\geqq}$ as the region in which the nondominated set N is approximated. This might be a currently implemented (not nondominated) solution or just a (not necessarily feasible) guess. A first approximation is obtained by exploring the feasible set along $l \geq k$ search directions $d^1, d^2, ..., d^l \in -R^k_{\geqq}$, specified by the decision maker. To obtain nondominated points along these search directions, an adaptation of the direction method introduced in Pascoletti and serafini [1] is modified in Schandl [3]. In the example given in Figure (2-a), the search directions are chosen as the negative unit vectors in R^2, $d^1 = (-1, 0)$ and $d^2 = (0, -1)$ yielding the points z^1 and z^2. These two points together with the reference point z^0 are used to define a cone and a first approximation, see Figure (2-b). Interpreting this approximation as the lower left part of the unit ball of an oblique norm γ (or, more general, of a polyhedral gauge) with z^0 as its center, this norm is then maximized in $Z \bigcap (z^0 + R^2_{\geqq})$. Consequently the next point z^3 in the problem) is found as a solution of problem (P_γ), where γ is an γ is an oblique norm (gauge), see Figure(2-c).

The point z^3 is added to the approximation by building the convex hull of the points generated so far and thus updating the approximation and the underlying norm (gauge) simultaneously as shown in Figure (2-d). Continuing this process, we get a finer approximation of nondominated set while generating of nondominated points and updating the unit ball of the oblique norm (gauge), see Figures (2-e) and (2-f). In each iteration, the point of maximal norm (gauge) is added. Since this point is "farthest away" from the approximation with respect to the current oblique norm (gauge), we always add the point of worst approximation with respect to this norm (gauge).

There are two possible stopping criteria; usually, at least one of them must given. The first one is an upper bound$\varepsilon > 0$ on the maximal deviation such that$dev(\bar{z}) = |\gamma(\bar{z}) - 1|$. As soon as we get $dev(\bar{z}) < \varepsilon$ for a point that should

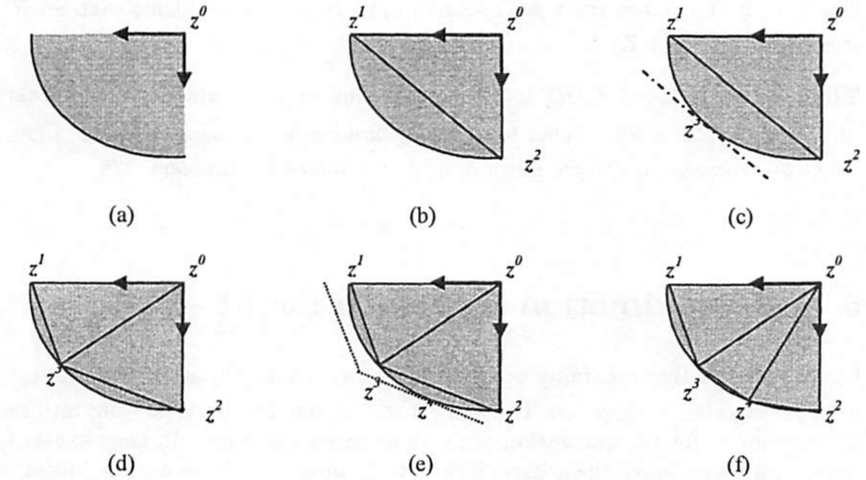

Figure 2:

be added next, the algorithm stops. The other possibility is to give an integer maxConeNo ≥ 1, which specified the maximum number of cones to be generated. The main loop of the algorithm continues until one of possible stopping criteria is satisfied. At the end of the loop the sorted list of r nondominated points is printed and can be used to visualize the approximated set.

Observe that in each iteration the maximization problem (P_γ) has to be has to be solved only in those cones whose facets were newly generated due to the addition of the last point. This includes new and modified cones. By updating the convex hull, the resulting approximation is always $R^k_{\geqq}$ -convex.

Schandl et al. [4] present the following Theorem which shows that the quality of the approximation improves with each new point if we assume that Z is $R^k_{\geqq}$ -convex.

Theorem 4 *[4] Let $Z \subseteq R^k$ be $R^k_{\geqq}$ -convex and γ^q be an approximating oblique norm (oblique gauge) constructed from q nondominated points, or points on the boundary of Z. Let $\bar{z}$ be the solution of*

$$max \quad \gamma^q(z) \qquad s.t. \qquad z \in Z \bigcap (z^0 - R^k_{\geqq}).$$

Let γ^{q+1} be the updated norm (gauge) including the new point z^*. Then

$$\gamma^{q+1}(z) \leq \gamma^q(z) \qquad \forall z \in Z \bigcap (z^0 - R^k_{\geqq}).$$

6 Oblique Norms in E-convex Multi-objective Programming

In E-convex multi-objective optimization problems, Pareto frontier may be non-convex so that we shall handle this case by transferring its non-convex Pareto frontier to convex Pareto frontier, and then applying above algorithm.

Schandl generalized the approximation algorithm described above to $R^k_{\geqq}$ -nonconvex problem by combining the maximization problem (P_γ) with methods particularly designed to handle nonconvexity variables which need a more details analysis. He used the Tchebycheff method for the $R^k_{\geqq}$ -nonconvex case to search for a candidate in the interior of the approximation which has disadvantage of performing some additional calculations, so that we shall present an alternative to using the Tchebycheff method for the nonconvex areas by defining a nondecreasing map T which transfer non-convex Pareto frontier to convex Pareto frontier and then applying the gauge method described above and use inverse map of T to get non-convex Pareto frontier.

Now, we review Schandl's work and its disadvantage and then explain our work. Schandl show that in the case of an $R^k_{\geqq}$ -nonconvex problem the approximation algorithm given above generates an approximation of the convex hull of the nondominated set, see Figure 3 for an example. Note that the nondominated point $\bar{z}$ in Figure 3 cannot be found using the gauge method described above.

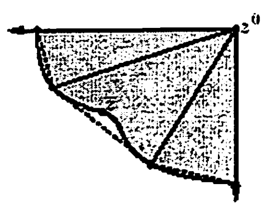

Figure 3:

To overcome this difficulty in the $R^k_{\geqq}$ -nonconvex case, Schandl switch to a different method, namely to the Tchebycheff method (see Steuer [6]) in those cones where no significant improvement can be made with the gauge method. To use the lexicographic Tchebycheff method, a local utopia point $\hat{z}$ and a nadir point $z^{\times}$ defining the weights of the norm are needed, see Figure 4 for an example.

The weights of the Tchebycheff norm are calculated as follows:

$$w_i = \frac{1}{z_i^{\times} - \hat{z}_i}, i = 1, 2, ..., k$$

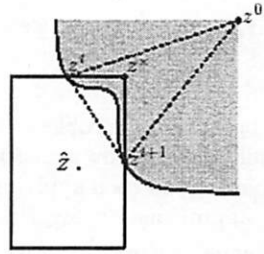

Figure 4:

and the lexicographic Tchebycheff method solved in the current cone is given by

$$lex\ min \quad (\|z - \hat{z}\|_{\infty}^{w}, \|z - \hat{z}\|_1),$$

$$s.t. \quad z = \sum_{i=1}^{k} \lambda_i z^i, \quad \lambda_i \geqq 0, \quad i = 1, 2, ..., k, \quad z \in Z. \qquad (Pl)$$

The norm (and therefore the deviation) of the candidate $\bar{z}$ is implicitly calculated because $\gamma(\bar{z}) = \sum_{i=1}^{k} \bar{\lambda}_i$ where $\bar{z} = \sum_{i=1}^{k} \bar{\lambda}_i z^i$ and and $(\bar{z}, \bar{\lambda})$ is an optimal solution of (Pl).

The approximation is not necessarily convex anymore but calculating something "similar to a norm" still yields the necessary information to evaluate the quality of the approximation in the considered cone. After identifying a candidate in the cone by the norm method or the Tchebycheff method, we only know that the point is locally nondominated in the current cone, but there might be a dominating point outside of this cone. Moreover, the candidate found using this two-stage procedure is not necessarily the already been found using program (P_γ), it is the point of worst found using program (P_γ), it is the point of worst approximation among all points "outside" the current approximation in this cone. Finding a point with the lexicographic Tchebycheff method (that is, in the second stage) does not imply anything about how well this point is currently approximated in comparison with other points. We can even have a case such as in Figure 5 where the method finds the point $\bar{z}$ with a very small deviation while the point $\tilde{z}$ with a deviation is missed. But unless the deviation of $\bar{z}$ is so small that the cone is not further considered, there is a good chance that $\tilde{z}$ will be found in a later iteration when the cone generated by $\bar{z}$ and z^{i+1} is by $\bar{z}$ and z^{i+1} is examined.

Now, we shall present an alternative to using the Tchebycheff method for the nonconvex areas. Define a nondecreasing map $T : Z \to Z$ such that $(T \circ f)x = (f \circ f)x = (f \circ E)x, \ \forall x \in M$. So that the image of objective space Z, for problem (P), can be defined as follows:

$$T(\mathrm{Z}) = \{\mathrm{y} : \mathrm{y} = T(\mathrm{z}), \mathrm{z} \in \mathrm{Z}\}.$$

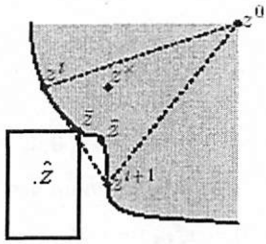

Figure 5:

We define the set of all nondominated criterion vectors N_T and the efficient points F_T of $T(Z)$ as follows:

$$N_T = \{\mathrm{y}^* \in T(\mathrm{Z}) : \nexists\, \mathrm{y} \in T(\mathrm{Z}) \mathrm{s.t.} y \leq y^*\},$$

$$F_T = \{\mathrm{x} \in \mathrm{M} : (\mathrm{T} \circ f)(\mathrm{x}) \in \mathrm{N_T}\}.$$

Our goal, now, is to find the nondominated solutions for $T(Z)$ and conclude the nondominated solutions for Z, by applying inverse map of T.

Proposition 3 *If $y^* \in N_T$, then there is at least one element $z^* \in Z$, $y^* = T(z^*)$ and $z^* \in N$.*

Proof. Let $y^* \in N_T$, then, there is $z^* \in Z$, such that $y^* = T(z^*)$. Let $z^* \notin N$, then there is $z \in Z$ such that $z \leq z^*$. Since T is a nondecreasing map then $T(y) \leq T(y^*)$ i.e. $y \leq y^*$ which contradicts $y^* \in N_T$. Hence $z^* \in N$.

Proposition 4 *$T(z^*) \in N_T$ for each $z^* \in N$.*

Proof. Let $z^* \in N$ and $T(z^*) \notin N_T$, then there is $z \in Z$ such that $T(z) \leq T(z^*)$. Since T is a nondecreasing map then $z \leq z^*$ which contradicts $z^* \in N$. Hence $T(z^*) \in N_T$.

Remark 2 *From Propositions 3 and 4 we obtain that $T(N) = N_T$.*

Corollary 1 *Let $E : R^n \to R^n$ be a mapping, $E(M) \subseteq M$ be a convex set in R^n and D be a convex cone in R^k. If the function $f : R^n \to R^k$ is said to be D-E-convex with respect to E, then the set $T(Z)$ is D-convex set.*

Proof. Follows from Proposition 1.

Corollary 2 *Let $E : R^n \to R^n$ be a mapping, $E(M) \subseteq M$ be a convex set in R^n and $f = (f_1, f_2, ..., f_k)$ be a function from M into R^k. The function f is $R^k_{\geqq}$ -E-convex with respect to E, if and only if each f_i is E-convex, and in this case the set $T(Z)$ is $R^k_{\geqq}$ -convex set.*

Proof. Follows from Proposition 2.

Example 1 *Let $E : R^2 \to R^2$ be defined as $E(x,y) = (\frac{x}{2}, y)$ and let M be M be given by*

$$M = \{(x,y) \in R^2 : \ x + y - 3 \leq 0, \ 3 \geq y \geq 1, \ x \geq 0\}.$$

Consider the bicriteria E-convex programming problem

$$\begin{aligned} &min \ \ f_1(x,y) = x^3, \ \ min \ \ f_2(x,y) = (y-x)^3, \\ &subject \ \ to \ \ (x,y) \in M. \qquad\qquad\qquad (P) \end{aligned}$$

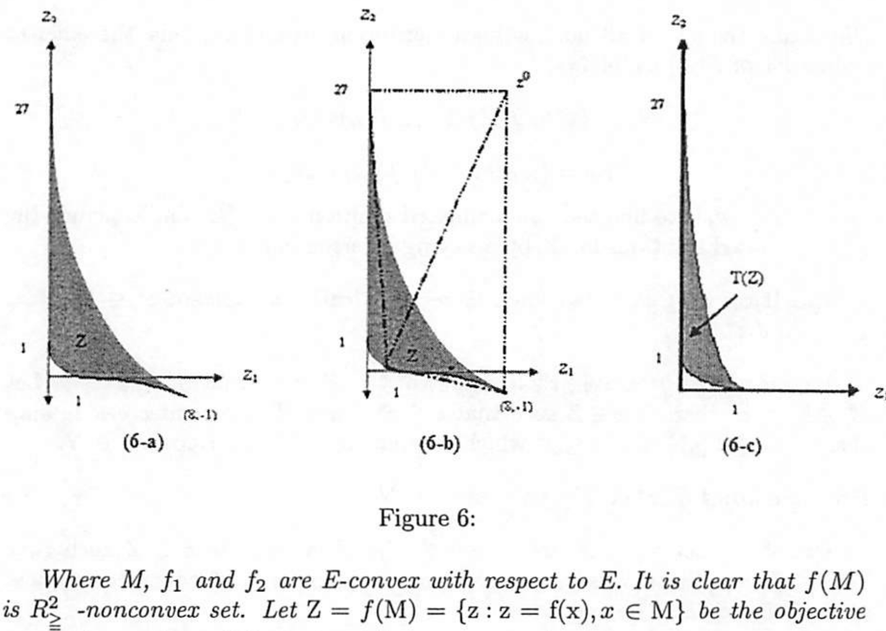

Figure 6:

Where M, f_1 and f_2 are E-convex with respect to E. It is clear that $f(M)$ is $R^2_{\geqq}$ -nonconvex set. Let $\mathrm{Z} = f(\mathrm{M}) = \{\mathrm{z} : \mathrm{z} = \mathrm{f}(\mathrm{x}), x \in \mathrm{M}\}$ *be the objective space, for problems (P) which is $R^2_{\geqq}$ -nonconvex (see Figure 6-a).*

In this case of this $R^2_{\geqq}$ -nonconvex problem the approximation algorithm given above generates an approximation of the convex hull of the nondominated set, see Figure (6-b). Note that the nondominated point $z^* = (\frac{27}{8}, \frac{-1}{8})$ in Figure (6-b) cannot be found using the gauge method described above. To overcome this difficulty we define the nondecreasing map

$$T(z_1, z_2) = \frac{1}{8}(z_1, 8z_2 + z_1 + 12\sqrt[3]{z_2^2 z_1} + 6\sqrt[3]{z_2 z_1^2}), \quad \forall\, z_1, z_2 \in Z.$$

such that $(T \circ f)x = (f \circ E)x, \ \ \forall x \in M$. So that the image of objective space Z, for problem (P), can be defined as follows: $T(\mathrm{Z}) = \{\mathrm{y} : \mathrm{y} = T(\mathrm{z}), \mathrm{z} \in \mathrm{Z}\}$ *(see Figure 6-c).*

It is clear that $T(Z)$ is $R^k_{\geqq}$-convex set, according to Corollary 2. Since $T(Z)$ is $R^k_{\geqq}$ -convex set then if we use the gauge method on $T(Z)$ we get the approximation of the nondominated set for $T(Z)$, namely

$$N_T = \{(y_1^*, y_2^*) \in T(Z) : 7y_2^* = 48(y_1^*)^2 - 55y_1^* + 7; \quad 0 \leqq y_1^* \leqq 1$$
$$\textit{and } y_1^* = 0; \quad 1 \leqq y_2^* \leqq 27\},$$

and hence the efficient set for it is

$$F_T = \{(x_1, 1) : 0 \leqq x_1 \leqq 1, \text{ and } (0, x_2) : 1 \leqq x_2 \leqq 3\},$$

and by using the inverse map we get the efficient set for problem (P), namely

$$F = \{(x_1, 1) : 0 \leqq x_1 \leqq 2, \text{ and } (0, x_2) : 1 \leqq x_2 \leqq 3\}.$$

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