Counting fuzzy subgroups of a special class of non-abelian groups of order p^3

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Abstract

In this paper, we calculate the number of fuzzy subgroups of a special class of non-abelian groups of order p^3.

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1. Introduction and Preliminaries

In [1], Murali and Makamba studied equivalence classes of fuzzy subgroups of a given group under a suitable equivalence relation. They characterize the number of fuzzy subgroups of finite abelian groups; in particular the fuzzy subgroups of $p-$groups. One of the most important problems of fuzzy group theory is concerned with classifying the fuzzy subgroups of a finite group. In [2] the number of distinct fuzzy subgroups of a finite cyclic group of square-free order is determined. Also, recall here the paper [3], where a recurrence relation is indicated which can successfully be used to count the number of distinct fuzzy subgroups for two classes of finite abelian groups: finite cyclic groups and finite elementary abelian $p-$groups.

Starting point for our discussion is given by the paper [4], where a recurrence relation is indicated which can successfully be used to count the number of distinct fuzzy subgroups for dihedral groups. In the present paper we extend the above study to a special class of non-abelian groups of order p^3. Without any equivalence on fuzzy subsets of a set, the number of fuzzy subgroups of even the trivial group is infinite. So, we shall define an equivalence relation on the set of fuzzy subsets of a given set. We use the natural equivalence relation introduced in [3, 4] and we shall determine the number of fuzzy subgroups of G with respect to this equivalence. In our case the corresponding equivalence classes of fuzzy subgroups are closely connected to the chains of subgroups in G. Note that an essential role in solving our counting problem will be played by the Inclusion-Exclusion Principle. In many situations this leads us to some recurrence relations, whose solutions can be easily found. We shall exemplify our method for a special class of finite non-abelian groups of order p^3.

Let G be a group and μ be a fuzzy subset of G. Then μ is called a *fuzzy subgroup* of G [6] if it satisfies the following conditions:

(1) $\min\{\mu(x), \mu(y)\} \leq \mu(xy)$, for all $x, y \in G$;

(2) $\mu(x) \leq \mu\left(x^{-1}\right)$, for all $x \in G$.

For each $t \in [0,1]$, we define the *level subset* $U(\mu, t) = \{x \in G \mid \mu(x) \geq t\}$. Let G be a group and μ be a fuzzy subset of G. Then μ is a fuzzy subgroup of G if and only if its non-empty level subsets are subgroups of G [6]. Fuzzy subgroups of G can be classified up to some natural equivalence relations on the set consisting of all fuzzy subsets of G. Let μ and η be two fuzzy subsets of G. Then we define $\mu \sim \eta$ if and only if $[\mu(x) > \mu(y) \Longleftrightarrow \eta(x) > \eta(y)$, for all $x, y \in G]$, and two fuzzy subgroups μ and η of G is called *distinct* if $\mu \nsim \eta$.

Let G be a finite group and μ be a fuzzy subgroup of G. Suppose that $Im\mu = \{t_1, t_2, \cdots, t_r\}$ such that $t_1 > t_2 > \cdots > t_r$. Then μ determines the following chain of subgroups of G which ends in G:

$$U(\mu, t_1) \subset U(\mu, t_2) \subset \cdots \subset U(\mu, t_r) = G.$$

Moreover, for any $x \in G$ and $i = 1, \cdots, r$ we have

$$\mu(x) = t_i \Longleftrightarrow i = \max\{j \mid x \in U(\mu, t_j)\} \Longleftrightarrow x \in U(\mu, t_i) \backslash U(\mu, t_{i-1}),$$

where by convention, we set $U(\mu, t_0) = \emptyset$.

THEOREM 1. [5]. Let G be a group and μ, η be two fuzzy subgroups of G. Then a necessary and sufficient condition for fuzzy subgroups μ, η of G to be equivalent with respect to $\sim$ is they determine the same chain of subgroups of G which end in G.

There exists a bijection between the equivalence classes of fuzzy subgroups of G and the set of chains of subgroups of G which ends in G. The largest class of groups for which it was completely solved is constituted by finite cyclic groups (Corollary 4 in [3]). A special class of abelian $p-$groups is constituted by elementary abelian $p-$groups. Such a group G has a direct decomposition of type $\mathbb{Z}_p^k = \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p$, where p is a prime and $k \in \mathbb{N}^*$. The number of fuzzy subgroups of G is equal to the number $a_{k,p}$ of all chains of subgroups of G terminated in G. For $r \in \{1, 2, \cdots, n_{k,1}\}$, let $\mathcal{A}_r$ be the set consisting of all matrices in $\mathcal{M}_{r\times k}(\mathbb{Z}_p)$ satisfying the property: for any $1 \leq u, v \leq r$ and any $A = (a_{ij})$, $A' = (a'_{ij}) \in \mathcal{A}_r$, there is no $\alpha \in \mathbb{Z}_p$ such that $a'_{vj} = \alpha a_{uj}$, $(\forall) 1 \leq j \leq k$. Also, for each $1 \leq s \leq \min\{r, k\}$, let us denote by $x_{r,s}$ the number of matrices contained in $\mathcal{A}_r$ whose rank is s.

THEOREM 2. [3] The number $a_{k,p}$ of all distinct fuzzy subgroups of the finite elementary abelian p−group $G \cong \mathbb{Z}_p^k$ satisfies the following recurrence relation, where $n_{k,1}$ denotes the number of minimal subgroups in ${\mathbb{Z}_p}^k$:

$$a_{k,p} = 2\sum_{r=1}^{n_{k,1}}(-1)^{r-1}\sum_{s=1}^{\min\{r,k\}} x_{r,s}a_{k-s,p}.$$

THEOREM 3. [4] Let G be a group and $M_1, M_2, \cdots, M_k$ be maximal subgroups of G. We denote by $h(G)$ the number of chains of subgroups of G ended in G. Then $h(G)$ is equal to:

$$2\left(\sum_{i=1}^{k} h(M_i) - \sum_{1\le i_1<i_2\le k} h(M_{i_1}\cap M_{i_2}) + \cdots + (-1)^{k-1}h\Big(\bigcap_{i=1}^{k} M_i\Big)\right).$$

If the maximal subgroup structure of G is known, in some cases, Theorem 3 will lead to recurrence relations that permit us to determine $h(G)$. This fact holds for some groups. A class of these is introduced in the next section.

There are two non-abelian groups of order p^3, where p is an odd prime number. The representation of one of them is as follows:

$$M(p^3) = \left\langle x, y \mid x^{p^2} = y^p = 1, y^{-1}xy = x^{p+1}\right\rangle.$$

THEOREM 4. The number of distinct fuzzy subgroups of the group $M(p^3)$ is $h(M(p^3)) = 8(p+1)$.

2. PROOF OF THEOREM 4

With respect to the representation of $M(p^3)$, we have:

$$x^i y = y x^{(p+1)i}, \text{ for all } 0 \le i \le p^2 - 1 \text{ and}$$
$$x y^j = y^j x^{(p+1)^j}, \text{ for all } 0 \le j \le p - 1.$$

Therefore,

$$x^i y^j = y^j x^{i(p+1)^j}, \text{ for all } 0 \le i \le p^2 - 1, 0 \le j \le p - 1.$$

According to the above equalities, the order of $(p-1) + p(p-1) = (p-1)(p+1)$ elements of $M(p^3)$ is equal to p. They are as follows:

$$\begin{array}{l}
x^p,\ x^{2p},\ x^{3p}, \cdots, x^{(p-1)p},\\
y,\ x^p y,\ x^{2p} y,\ x^{3p} y, \cdots, x^{(p-1)p} y,\\
y^2,\ x^p y^2,\ x^{2p} y^2,\ x^{3p} y^2, \cdots, x^{(p-1)p} y^2,\\
\vdots\\
y^{p-1},\ x^p y^{p-1},\ x^{2p} y^{p-1},\ x^{3p} y^{p-1}, \cdots, x^{(p-1)p} y^{p-1}.
\end{array}$$

The order of other elements is equal to p^2. Obviously, the order of every maximal subgroups is p^2. So, the maximal subgroups of $M(p^3)$ are: $M_1 = \langle x\rangle$, $M_2 = \langle xy\rangle$, $M_3 = \langle xy^2\rangle$, $\cdots$, $M_p = \langle xy^{p-1}\rangle$, $M_{p+1} = \langle x^p, y\rangle$. Therefore, according to Theorem 3, $h(M(p^3))$ is equal to:

$$2\left(\sum_{i=1}^{p+1} h(M_i) - \sum_{1\leq i_1<i_2\leq p+1} h(M_{i_1}\cap M_{i_2}) + \cdots + (-1)^p h\Big(\bigcap_{i=1}^{p+1} M_i\Big)\right). \qquad (*)$$

For every $1 \leq i \leq p$, the maximal subgroup M_i is isomorphic to $\mathbb{Z}_{p^2}$ and the maximal subgroup M_{p+1} is isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$. Clearly, the intersection of every number of maximal subgroups of $M(p^3)$ is isomorphic to $< x^p >\cong \mathbb{Z}_p$. Hence, the relation $(*)$ can be rewritten as following:

$$\begin{aligned} h(M(p^3)) &= 2\Big(ph(\mathbb{Z}_{p^2}) + h(\mathbb{Z}_p\times\mathbb{Z}_p) - \tbinom{p+1}{2}h(\mathbb{Z}_p) + \tbinom{p+1}{3}h(\mathbb{Z}_p) - \\ &\quad \cdots + (-1)^p\tbinom{p+1}{p+1}h(\mathbb{Z}_p)\Big). \end{aligned}$$

Consequently,

$$\begin{aligned} h(M(p^3)) &= 2\Big(ph(\mathbb{Z}_{p^2}) + h(\mathbb{Z}_p\times\mathbb{Z}_p) \\ &\quad + \Big(-\tbinom{p+1}{2} + \tbinom{p+1}{3} - \cdots - \tbinom{p+1}{p+1}\Big)h(\mathbb{Z}_p)\Big) \\ &= 2\Big(ph(\mathbb{Z}_{p^2}) + h(\mathbb{Z}_p\times\mathbb{Z}_p) + \Big(\tbinom{p+1}{0} - \tbinom{p+1}{1}\Big)h(\mathbb{Z}_p)\Big). \qquad (**) \end{aligned}$$

Now, we have:

$$\begin{aligned} h(\mathbb{Z}_p\times\mathbb{Z}_p) &= a_{2,p} = 2\sum_{r=1}^{p+1}(-1)^{r-1}\sum_{s=1}^{\min\{r,2\}} x_{r,s}a_{2-s,p} \\ &= 2\Big(x_{1,1}a_{1,p} + \sum_{r=2}^{p+1}(-1)^{r-1}\Big(x_{r,1}a_{1,p} + x_{r,2}a_{0,p}\Big)\Big) \\ &= 2\Big((p+1)\times 2 + \sum_{r=2}^{p+1}(-1)^{r-1}x_{r,2}\Big) \\ &= 2\Big(2p+2 - \tbinom{p+1}{2} + \tbinom{p+1}{3} - \cdots + (-1)^p\tbinom{p+1}{p+1}\Big) \\ &= 2\Big(2p+2+\tbinom{p+1}{0} - \tbinom{p+1}{1}\Big) = 2(p+2) = 2p+4. \end{aligned}$$

Also, $h(\mathbb{Z}_p) = 2$ and $h(\mathbb{Z}_{p^2}) = 4$. So, the equality $(**)$ becomes:

$$h(M(p^3)) = 2\Big(4p + 2p + 4 + 2(1-p-1)\Big) = 8(p+1).$$

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